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The mechanics of belt friction revisited

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Abstract A new insight into the mechanics of belt friction is given. A conceptual and methodological drawback in the presentation of the classical derivation of the force required to pull the belt over a fixed drum against the hold-force and the friction between the belt and a drum is pointed out, corrected and discussed. The total forces due to pressure and friction (P and F) are evaluated in magnitude and direction. It is shown that not only the local friction force is proportional to the local pressure ($f = \mu p$), but also their resultants ($F = \mu P$), where μ is the coefficient of static friction. The magnitude of the pressure force is $P = F_R/(1 + \mu^2)^{1/2}$, where F_R is the magnitude of the resultant of the pull- and hold-forces applied at two ends of the belt. Different methodological approaches to the analysis are presented, which are particularly appealing from the educational point of view. These include the local and integral equilibrium considerations, the virtual work approach, and the dimensional analysis.

Keywords belt friction; contact angle; dimensional analysis; friction force; impending slip; pressure

Introduction

Flexible belts, cables, and ropes have wide applications in engineering, where they are used as belt drives for power transmission between rotating shafts, band breaks to reduce angular speed of rotating machine parts in automobile and other industries, hoist devices for lifting or lowering loads in construction industry, conveyors, magnetic tape drives, etc. In all cases, they operate by friction between the belt and the surface of a drum or a pulley. Because of its wide importance, the mechanics of belt friction and Euler's formula[1] relating the pull-force to the hold-force applied at two ends of the belt are discussed in every undergraduate textbook of engineering mechanics.[2–8]

Figure 1a shows a flat belt of negligible weight wrapped around a fixed circular disk or cylindrical drum with the contact (wrap) angle θ. The hold-force on the left end (low-tension side) is T_1, and the pull-force on the right end (high-tension side) at the instant of impending slip (incipient sliding) of the belt is T_2. The coefficient of static friction between the belt and the cylinder is μ. Figure 1b shows a free-body diagram of a differential element of the belt, subtended by the angle $\mathrm{d}\varphi$. The total normal force from the disk to the belt over the length $R\mathrm{d}\varphi$ is commonly denoted in the mechanics textbooks[4,5,7] by $\mathrm{d}N$. The total tangential force due to friction is then $\mu\mathrm{d}N$. The increment of the force in the belt from the left to the right of the infinitesimal element is $\mathrm{d}T$. The equilibrium conditions for the forces in the normal *(n)* and tangential *(t)* direction yield

$$\mathrm{d}N = T\,\mathrm{d}\varphi\ , \qquad \mathrm{d}T = \mu\,\mathrm{d}N\ , \tag{1}$$

neglecting the second-order products of the involved differentials. The elimination of $\mathrm{d}N$ between the aforementioned two expressions gives $\mathrm{d}T = \mu T\mathrm{d}\varphi$, and the integration

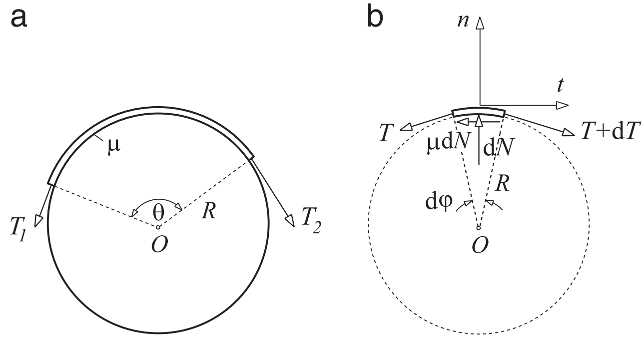

Fig. 1 *(a) A flat belt wrapped around a fixed circular cylinder of radius R. The contact angle between the belt and the cylinder is θ. The hold-force at the left end is T_1, and the pull-force at the right end of the belt at the instant of its impending slip is T_2. The coefficient of static friction between the contacting surfaces is μ. (b) An infinitesimal belt segment subtended by the angle* dφ. *The tension in the belt at the left end of the segment is T and at the right end T+*d*T. The total normal force exerted by the cylinder on the belt along the contact length R*dφ *is designated by d*N. *The total friction force at the instant of impending slip is* μd*N. The tangential and normal directions at the considered point of the belt are designated by t and n.*

over φ from 0 to the entire contact angle θ gives the Euler formula,[17] or the belt friction (capstan) equation,

$$T_2 = T_1 \exp(\mu\theta) \; . \tag{2}$$

From the methodological and pedagogical points of view, there are two drawbacks in the presented derivation. First, while the force increment dT is well-defined physically, representing an increment of the magnitude of the tensile force T in the belt with its arc length Rdφ, the force increment dN is designated poorly, because the notation suggests that it represents an increment of the force N, which is not clearly defined and which lacks a definite physical meaning. For example, it does not represent a vector quantity, because there is no direction that can be associated with N. Indeed, if $\mathbf{P} = \mathbf{P}(\varphi)$ is the total normal force exerted by the cylinder on the belt in the angle range $[0,\varphi]$, then its increment is d$\mathbf{P} = (pR\mathrm{d}\varphi)\mathbf{n}$, where p is the local pressure between the belt and the cylinder, and $\mathbf{n}$ is the unit vector orthogonal to the belt at the considered point of the contact. The magnitude of this force increment is $|\mathrm{d}\mathbf{P}| = pR\,\mathrm{d}\varphi \equiv \mathrm{d}N$. However, the so-defined dN is not equal to the increment of the magnitude of the physical force $\mathbf{P}(\mathrm{d}N \neq \mathrm{d}|\mathbf{P}|)$, because for the forces with the varying direction the magnitude of the force increment is not equal to the increment of its magnitude. Indeed,

$$\begin{aligned} \mathrm{d}P = \mathrm{d}|\mathbf{P}| &= |\mathbf{P} + \mathrm{d}\mathbf{P}| - |\mathbf{P}| = \cos(\varphi - \phi)pR\,\mathrm{d}\varphi = \cos(\varphi - \phi)\ |\mathrm{d}\mathbf{P}| \\ &= \cos(\varphi - \phi)\ \mathrm{d}N \; , \end{aligned} \tag{3}$$

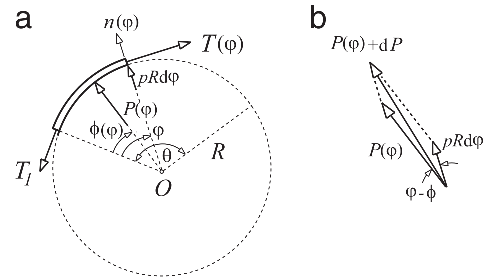

Fig. 2 *(a) A finite segment of the belt wrapped around a circular cylinder along the contact angle θ. The considered belt segment is subtended by the angle φ. The belt tension at the left end of the segment is T_1 and on the right end $T(\varphi)$. The total normal force acting on the belt segment from the pressure exerted by the cylinder on the belt along the contact angle φ is $\mathbf{P}(\varphi)$. Its direction is specified by the angle $\phi(\varphi)$. The increment of this force $\mathrm{d}\mathbf{P} = (pR\mathrm{d}\varphi)\mathbf{n}$ is due to pressure acting along an additional infinitesimal length of the belt $R\mathrm{d}\varphi$, where n is the unit vector orthogonal to the belt at the point specified by the angle φ. (b) The vector addition of the forces $\mathbf{P}(\varphi)$ and $\mathrm{d}\mathbf{P} = (pR\mathrm{d}\varphi)\mathbf{n}$, producing the force $\mathbf{P}(\varphi) + \mathrm{d}\mathbf{P}$.*

where ϕ is the angle specifying the direction of the normal force $\mathbf{P}$ relative to the direction $\varphi = 0$ (Fig. 2a).

The second drawback in the presented derivation is that it does not offer an opportunity to discuss more closely the independence of the force T_2 (required to slip the belt) of the radius R, evident from the absence of R in the expression (2). This independence is to some extent counterintuitive, if one considers the fact that the frictional resistance between the belt and the cylinder acts over the contact length $R\theta$, which is proportional to R. The objective of this paper is to present a derivation which does not suffer from either of these drawbacks and which offers additional insight into the mechanics of belt friction. Alternative approaches to the analysis are presented, which include the local and integral equilibrium considerations, the virtual work approach, and the dimensional analysis.

Euler's formula

The following derivation is based on the explicit use of the local pressure and the local friction force acting along the contact length between a belt and a cylinder, as originally pursued by Euler[1] and as commonly adopted in the literature on the mechanism and machine theory.[9,10] A belt wrapped around the fixed cylinder and pulled against it by the forces T_1 and T_2 builds a pressure $p = p(\varphi)$ over the entire contact angle θ, as well as the (local) friction force $f = f(\varphi)$ (both per unit length of the belt). The free-body diagram of the belt segment of length $R\mathrm{d}\varphi$ is shown in Fig. 3. The equilibrium conditions, before or at the instant of impending slip, give

$$\sum F_n = 0: \quad T = pR\,, \qquad \sum F_t = 0: \quad \mathrm{d}T = fR\,\mathrm{d}\varphi\,. \tag{4}$$

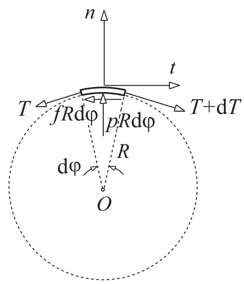

Fig. 3 *An infinitesimal belt segment subtended by the angle* dφ. *The tension in the belt at the left end of the segment is T and on the right end T* + d*T*. *The total force from the pressure p acting along the contact length R*dφ *is p*(φ)*R*dφ *(neglecting the small quantities of higher order). Likewise, the total force from the local friction f is f*(φ)*R*dφ. *The tangential and normal directions at the considered point of the belt are designated by t and n.*

Thus, the force in the belt at any point in the contact region is equal to the product of the radius of the cylinder and the pressure at that point i.e. $T(\varphi) = Rp(\varphi)$.[18] The two equilibrium conditions in (4) involve three unknown quantities, the force in the belt $T = T(\varphi)$, the pressure $p = p(\varphi)$, and the friction $f = f(\varphi)$. The problem is thus statically indeterminate, unless the pull-force is sufficiently increased to produce the state of impending gross slip of the belt. In the latter case, adopting the Amontons–Coulomb law of dry friction[12], the local friction force is $f = \mu p$ and the system of equations (4) becomes statically determinate. The increment of the force in the belt is then $\mathrm{d}T = \mu T\mathrm{d}\varphi$, and the integration gives $T_2 = T_1\exp(\mu\theta)$. If a weightless belt is actually slipping over the cylindrical surface, the coefficient of kinetic friction μ_k, rather than the coefficient of static friction μ, should be used in the aforementioned derivation. In either case, the coefficient of friction is assumed to be constant along the contact.

One could also proceed with the analysis by deriving from (4) the differential equation for the pressure, rather than the force in the belt. In this case, from the first of (4), the increment of the force can be expressed as $\mathrm{d}T = R\mathrm{d}p$, and the substitution into the second of (4) gives $\mathrm{d}p = f\mathrm{d}\varphi$. In the state of impending slip this becomes $dp = \mu p\mathrm{d}\varphi$, and the integration gives $p(\varphi) = p(0)\exp(\mu\varphi)$, with $p(0) = T_1/R$ being the pressure at the contact point $\varphi = 0$.

The derivation based on the free body diagram shown in Fig. 3, with the explicitly introduced and utilized local pressure p and local friction force f, is conceptually and pedagogically more appealing than the derivation presented in the introductory section (Fig. 1b), because it delivers a physically clear, fundamental expression between the force in the belt and the pressure ($T = R$p), rather than its masked[19] version $\mathrm{d}N = T\mathrm{d}\varphi$ appearing in (1). The expression $T = R$p shows that the contact pressure between the belt and the cylinder decreases with the increase of the radius of the cylinder, as physically expected, and it does so inversely proportional to the radius ($p = T/R$). The force in the belt cannot depend on R, because there is no length scale in the problem other than R, and thus by the dimensional argument $T = T(\varphi)$.[20] The local friction force is also inversely

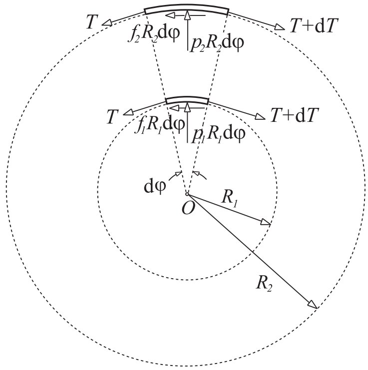

Fig. 4 *An infinitesimal belt segment taken from two belt/cylinder configurations with the same total contact angle θ, and the same hold- and pull-forces T_1 and T_2. In the first case, the radius of the cylinder is R_1 and in the second case R_2. The total pressure forces acting on two infinitesimal segments subtended by the angle* $\mathrm{d}\varphi$ *are the same* ($p_1R_1\,\mathrm{d}\varphi = p_2R_2\,\mathrm{d}\varphi$), *because* $T = p_1R_1 = p_2R_2$. *Likewise, the corresponding total friction forces are the same* ($f_1R_1\,\mathrm{d}\varphi = f_2R_2\,\mathrm{d}\varphi$), *where* $f_1 = \mu p_1$ *and* $f_2 = \mu p_2$. *It is assumed that the same coefficient of friction* μ *characterizes both contact surfaces.*

proportional to R, both before and at the instant of impending slip, because from the second of (4) it follows that $f = R^{-1}\,\mathrm{d}T/\mathrm{d}\varphi$ before impending slip, and $f = \mu T/R$ at the instant of impending slip.

For the same coefficient of friction (μ) and the same contact angle (θ), the force T_2 required to cause the slip and overcome the frictional resistance between the belt and the cylinder is the same for a large (R_2) and small (R_1) radius of the cylinder, because the local pressure and the local friction force are smaller but act over a longer contact length in the case of a larger cylinder, and vice versa for a smaller cylinder. This is sketched in Fig. 4. Students in classroom commonly express the opinion that the frictional resistance over a longer contact length should be greater, because more resistive force accumulate along a longer distance, forgetting that the

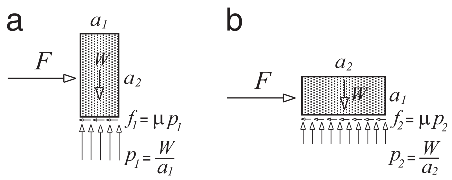

Fig. 5 *A rectangular block of unit thickness and weight W, having the sides a_1 and a_2, pushed against a rough horizontal substrate when it is placed with (a) shorter and (b) longer side in contact with it. The average pressure exerted by the substrate is $p_1 = W/a_1$ in case (a), and $p_2 = W/a_2$ in case (b). The corresponding average local friction forces (per unit contact length), at the instant of impending slipping, are $f_1 = \mu p_1$* and *$f_2 = \mu p_2$, where μ is the coefficient of static friction. The moment equilibrium condition is satisfied by the actual nonuniform pressure distribution (not shown in the figure).*

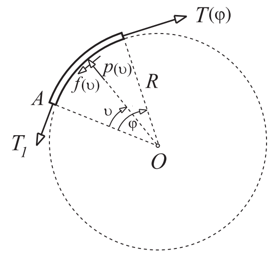

Fig. 6 *A free-body diagram of the finite segment of the belt wrapped around the cylinder along the contact angle θ. The shown segment is subtended by the angle φ. The force in the belt at the left end is T_1 and at the right end $T(\varphi)$. The local pressure and friction forces (per unit length) at an arbitrary angle $\vartheta \in [0,\varphi]$ are $p(\vartheta)$ and $f(\vartheta)$.*

increase of the contact length is accompanied by the decrease of the local pressure and thus the decrease of the local friction force.[21] This situation is analogous to the effect of the contact area on the sliding of a rectangular block over a rough flat surface (Fig. 5). The total frictional resistance is the same in cases (a) and (b), because the local pressure and the local friction force are larger in the first, and smaller in the second case. Since the contact area is smaller in the first case and larger in the second case, the total friction force is the same in both cases. The use of this elementary example is pedagogically very effective in the classroom discussion of the Euler's belt friction formula and the explanation of its independence of the radius R.

Integral equilibrium considerations

Figure 6 shows a free-body diagram of a finite segment of the belt wrapped around a cylinder over the contact angle θ. The shown segment is subtended by the angle φ. The force in the belt at the right end is denoted by $T(\varphi)$. For the equilibrium of the entire belt segment, the moment of all forces for the point A has to be zero, which gives

$$T(\varphi)R(1-\cos\varphi) - R\int_0^{\varphi} [(\sin\vartheta)p(\vartheta) + (1-\cos\vartheta)f(\vartheta)]R\,\mathrm{d}\vartheta = 0\ . \tag{5}$$

Upon taking the derivative $\mathrm{d}/\mathrm{d}\varphi$, (5) reduces to

$$\left[\frac{\mathrm{d}T}{\mathrm{d}\varphi} - Rf(\varphi)\right](1-\cos\varphi) + [T(\varphi) - Rp(\varphi)]\sin\varphi = 0\ . \tag{6}$$

Since this must hold for any φ, there follows

$$\frac{\mathrm{d}T}{\mathrm{d}\varphi} = Rf(\varphi) \ , \quad T(\varphi) = Rp(\varphi) \ . \tag{7}$$

Physically, applying the derivative to the integral condition of equilibrium (5) in effect reproduces the local conditions of equilibrium for an infinitesimally small belt element.

Virtual work consideration

The Euler belt friction formula can be derived by the virtual work consideration as follows. Suppose that a belt is given a uniform virtual displacement in the radial direction δu_n. The virtual work for the belt segment $R\varphi$ is

$$\delta W = \int_0^{\varphi} p(\vartheta)R \ \mathrm{d}\vartheta \ \delta u_n = N(\varphi)\delta u_n \ , \quad N(\varphi) = R\int_0^{\varphi} p(\vartheta) \ \mathrm{d}\vartheta \ . \tag{8}$$

The force $N(\varphi)$ can here be interpreted as a generalized force in the sense that $\delta W = N(\varphi)\delta u_n$. By the principle of virtual work, this virtual work must be equal to the virtual strain energy associated with the virtual hoop (circumferential) strain due to radial belt expansion ($\delta u_n/R$), which is

$$\delta U = \int_0^{\varphi} T \ \frac{\delta u_n}{R} \ R \ \mathrm{d}\vartheta = \int_0^{\varphi} T \ \delta u_n \ \mathrm{d}\vartheta \ . \tag{9}$$

Thus, by equating (8) and (9),

$$N(\varphi) = \int_0^{\varphi} T \ \mathrm{d}\vartheta \ . \tag{10}$$

Suppose next that the belt is giving a uniform virtual displacement δu_t tangential to the belt (uniform slip displacement throughout the contact length). The corresponding virtual work for the belt segment $R\varphi$ is

$$\delta W = [T(\varphi) - T_1]\delta u_t - \int_0^{\varphi} f(\vartheta)R \ \mathrm{d}\vartheta \ \delta u_t = 0 \ . \tag{11}$$

Since $f = \mu p$, this gives

$$T(\varphi) - T_1 = \mu N(\varphi) \ . \tag{12}$$

By combining (10) and (12), we obtain

$$T(\varphi) - T_1 = \mu \int_0^{\varphi} T \, \mathrm{d}\vartheta \ , \tag{13}$$

which has the solution $T(\varphi) = T_1\exp(\mu\varphi)$, in agreement with the Euler's formula.

Dimensional consideration

Almost entire expression for the force T_2 required to pull the belt over a rough circular cylinder against the hold-force T_1 can be deduced by the dimensional arguments. Since there is no length scale other than the radius of the circular cylinder, the force T_2 must scale with T_1 and be a function of the nondimensional coefficient of friction μ and the contact angle θ, i.e. $T_2 = T_1 g(\mu, \theta)$. The function $g(\mu, \theta)$ must have the property

$$g(\mu, \theta_1 + \theta_2) = g(\mu, \theta_1) g(\mu, \theta_2) \ , \tag{14}$$

because physically the force in the belt must obey the transitivity property (Fig. 7)

$$T_2(\mu, \theta_1 + \theta_2) = T_1 g(\mu, \theta_1 + \theta_2) = T_{1,2} g(\mu, \theta_2) = [T_1 g(\mu, \theta_1)] g(\mu, \theta_2) \ . \tag{15}$$

The function $g(\mu, \theta)$ satisfying the property (14) is an exponential function, $g(\mu, \theta) = \exp[c(\mu)\theta]$. Thus, for any angle $\varphi \in [0, \theta]$, we can write

$$T(\mu, \varphi) = T_1 \exp[c(\mu)\varphi] \ , \quad \mathrm{d}T = c(\mu) T \, \mathrm{d}\varphi \ . \tag{16}$$

It remains to specify the function $c(\mu)$. The analysis cannot proceed further on the basis of the dimensional analysis alone, and one has to invoke the equilibrium

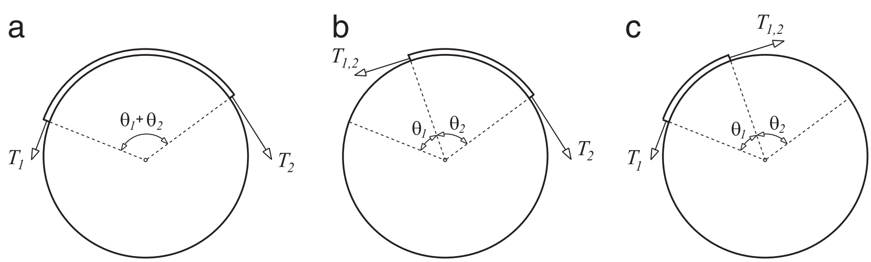

Fig. 7 *(a) A flat belt wrapped around a fixed cylinder along the contact angle $\theta = \theta_1 + \theta_2$. The hold-force at the left end is T_1, and the pull-force at the right end of the belt at the instant of impending slip is T_2. (b) The belt segment from part (a) subtended by the angle θ_2. The force at the left end is designated by $T_{1,2}$, while the force at the right end is T_2. (c) The belt segment from part (a) subtended by the angle θ_1. The force at the left end is T_1, and at the right end $T_{1,2}$.*

conditions. The increment of the force in the belt at any point within the contact angle is physically due to the local friction force, so that $\mathrm{d}T = fR\,\mathrm{d}\varphi = \mu pR\,\mathrm{d}\varphi$. Since the circumferential force in the belt is supported by the pressure in the belt ($T = Rp$), the increment of the force becomes $\mathrm{d}T = \mu T\,\mathrm{d}\varphi$, and the comparison with (16) establishes $c(\mu) = \mu$.

Total interaction forces

By comparing (1) and (4), we can write $\mathrm{d}N = pR\,\mathrm{d}\varphi$, and by integrating from $\varphi = 0$ to an arbitrary $\varphi \in (0, \theta]$, there follows

$$N(\varphi) = R\int_0^{\varphi} p(\vartheta)\,\mathrm{d}\vartheta = \int_0^{s} p(s)\,\mathrm{d}s\,, \tag{17}$$

in agreement with (8). Thus, the force $N(\varphi)$ at an arbitrary point of the belt represents the integral of pressure $p = p(s)$ over the length $s = R\varphi$ of the belt, regardless of the direction of the local pressure p along the belt within the contact region. Such force quantity (N) has no physical meaning in the considered problem, apart from the fact that its increment represents the magnitude of the normal force acting on the arc length $\mathrm{d}s$ of the belt due to the contact pressure (p) along that length. Nevertheless, we proceed to evaluate $N = N(\varphi)$ to later compare it with the physical force P from section 3.1. Since $Rp(\varphi) = T(\varphi) = T_1\exp(\mu\varphi)$, the integration in (17) gives

$$N(\varphi) = \frac{1}{\mu}[T(\varphi) - T_1] = \frac{T_1}{\mu}[\exp(\mu\varphi) - 1]. \tag{18}$$

In particular $N_1 = N(0) = 0$ and $N_2 = N(\theta) = (T_2 - T_1)/\mu$. The difference $T_2 - T_1$ can be interpreted as the line integral of the friction force $f(\varphi) = \mu p(\varphi)$, while N_2 is the line integral of the pressure $p(\varphi)$ over the entire contact length $R\theta$. Also, the gradient of N with respect to φ is the force in the belt $\mathrm{d}N/\mathrm{d}\varphi = T(\varphi)$, in accord with (1).

Alternatively, the expression (18) can be deduced by imposing the equilibrium condition for the vanishing moment of all forces acting in Fig. 6 for the point O. This gives

$$[T(\varphi) - T_1]R = R\int_0^{\varphi} f(\vartheta)R\,\mathrm{d}\vartheta\,. \tag{19}$$

At the instant of impending slip, $f(\vartheta) = \mu p(\vartheta)$ and (19) gives $T(\varphi) = T_1 + \mu N(\varphi)$.

Total forces due to pressure and friction

It is appealing to evaluate the total pressure and friction forces acting between the belt and a cylinder. The total force due to pressure is

$$\mathbf{P}(\theta) = \int_0^\theta \mathbf{n}(\varphi)\; p(\varphi) R \,\mathrm{d}\varphi \;, \qquad \mathrm{d}\mathbf{P} = p(\varphi) R \,\mathrm{d}\varphi \;\mathbf{n}(\varphi) \;, \tag{20}$$

where $\mathbf{n}(\varphi) = -\cos(\theta_0 + \varphi)\mathbf{i} + \sin(\theta_0 + \varphi)\mathbf{j}$ is the unit vector orthogonal to the cylinder at an arbitrary contact point (Fig. 8a). By substituting the pressure expression $p(\varphi) = (T_1/R)\exp(\mu\varphi)$ and by integrating, it follows that $\mathbf{P}(\theta) = P_x(\theta)\mathbf{i} + P_y(\theta)\mathbf{j}$, with

$$\begin{aligned} P_x(\theta) &= -\frac{1}{1+\mu^2}\{T_2[\mu\cos(\theta_0+\theta)+\sin(\theta_0+\theta)] - T_1(\mu\cos\theta_0 + \sin\theta_0)\}, \\ P_y(\theta) &= \frac{1}{1+\mu^2}\{T_2[\mu\sin(\theta_0+\theta)-\cos(\theta_0+\theta)] - T_1(\mu\sin\theta_0 - \cos\theta_0)\}. \end{aligned} \tag{21}$$

The direction of the force $\mathbf{P}$ passes through O at an angle α defined by $\tan\alpha = P_y/P_x$ (Fig. 8b).

Since each local friction force f is orthogonal and proportional to the corresponding pressure p, the total friction force $\mathbf{F}$ is orthogonal to $\mathbf{P}$ and equal to $\mathbf{F}(\theta) = F_x(\theta)\mathbf{i} + F_y(\theta)\mathbf{j}$, where

$$F_x(\theta) = -\mu P_y(\theta) \;, \quad F_y(\theta) = \mu P_x(\theta) \;. \tag{22}$$

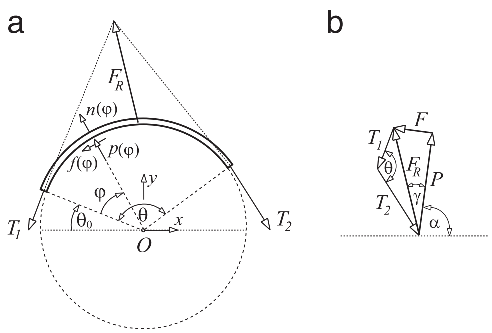

Fig. 8 *(a) A free-body diagram of the belt wrapped around a cylinder along the contact angle θ. The hold-force at the left end (defined by the angle θ_0) is T_1, and the pull-force at the right end is T_2. The local pressure and friction forces at an arbitrary angle $\varphi \in (0, \theta)$ are $p(\varphi)$ and $f(\varphi)$. The total reactive force from the cylinder, balancing the forces T_1 and T_2 is $\mathbf{F}_R = \mathbf{T}_1 + \mathbf{T}_2$. (b) The force-triangles showing the components of the total reactive force $\mathbf{F}_R = \mathbf{T}_1 + \mathbf{T}_2 = \mathbf{P} + \mathbf{F}$, where $\mathbf{P}$ is the total reactive force from the pressure, and $\mathbf{F}$ is the total reactive force from the friction. The angles α and γ are defined by $\tan\alpha = P_y/P_x$ and $\tan\gamma = F/P = \mu$.*

The magnitudes $F = \left(F_x^2 + F_y^2\right)^{1/2}$ and $P = \left(P_x^2 + P_y^2\right)^{1/2}$ are related by

$$F = \mu P \ , \quad P = \frac{1}{\sqrt{1+\mu^2}} \, F_R \ , \tag{23}$$

where F_R is the magnitude of the total reactive force from the cylinder to the belt, $\mathbf{F}_R = \mathbf{P} + \mathbf{F} = (P_x - \mu P_y)\mathbf{i} + (P_y + \mu P_x)\mathbf{j}$, which is (Fig. 8a)[22]

$$F_R(\theta) = \left[T_1^2 + T_2^2(\theta) - 2T_1T_2(\theta)\cos\theta\right]^{1/2} \ , \quad T_2(\theta) = T_1\exp(\mu\theta) \ . \tag{24}$$

Figure 9 shows the variation of the forces P, F, F_R, and T_2 (all scaled by T_1) with θ/π for two selected values of the friction coefficient ($\mu = 0.2$ and $\mu = 0.3$). Due to directional changes of the local pressure and friction forces, the total reactive force F_R as well as the total pressure and friction forces (P and F) can have ascending and descending portions along the axis of increasing contact angle θ. This is particularly pronounced for lower values of the friction coefficient μ.

The normal distance of the direction of the total friction force F from the point O can be calculated from the moment equilibrium condition for the point O (Fig. 10), which gives $h = R(T_2 - T_1)/F$, i.e.

$$\frac{h}{R} = \frac{\sqrt{1+\mu^2}}{\mu} \, \frac{T_2 - T_1}{\left(T_1^2 + T_2^2 - 2T_1T_2\cos\theta\right)^{1/2}} \ , \quad T_2 = T_1\exp(\mu\theta) \ . \tag{25}$$

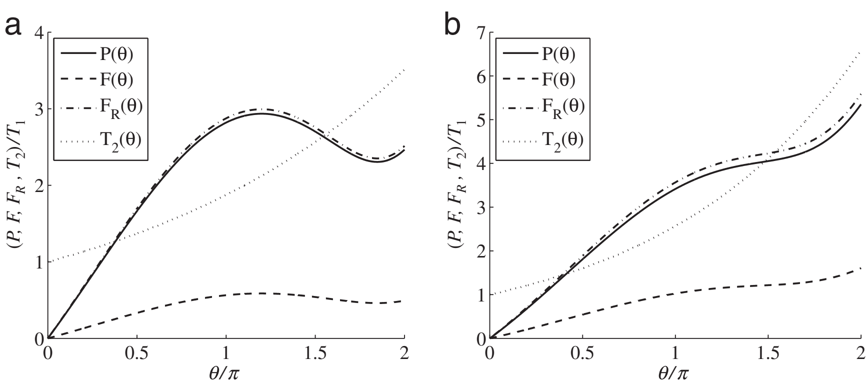

Fig. 9 *The variation of the forces P, F, F_R, and T_2 (all scaled by T_1) with the contact angle θ/π in the case: (a) $\mu = 0.2$ and (b) $\mu = 0.3$.*

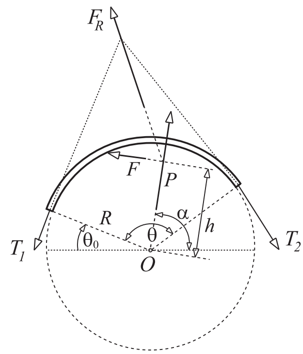

Fig. 10 *The total reactive force from the cylinder to the belt, due to applied forces T_1 and T_2, is F_R. The three forces intersect at the same point. The pressure component of F_R is P, which passes through the point O at the angle α =* arctan(P_y/P_x) *with respect to the horizontal direction. The normal distance of the direction of the total friction force F from the point O is h =* R($T_2 - T_1$)/F.

Finally, we compare the forces $N(\theta)$ and $P(\theta)$, which are defined by

$$N(\theta) = \frac{T_1}{\mu}\left[\exp(\mu\theta) - 1\right] ,$$

$$P(\theta) = \frac{T_1}{\sqrt{1+\mu^2}}\left[1 + \exp(2\mu\theta) - 2\cos\theta\exp(\mu\theta)\right]^{1/2} . \tag{26}$$

Their variation with the contact angle θ (scaled by the magnitude of the hold-force T_1), for the selected values of the friction coefficient μ, is shown in Fig. 11. While the direction of **P** passes through O for all θ, there is no direction associated with N (thus a vector **N** does not exist). As a consequence, N is monotonically increasing with θ, while the force P has the descending portions for the smaller values of μ, because it accounts for the directional changes of pressure along the contact between the cylinder and the belt. Being monotonically increasing, the gradient of $N(\varphi)$ with respect to φ gives a monotonically increasing force in the belt, $T(\varphi) = \mathrm{d}N/\mathrm{d}\varphi$, which is the most significant property of the otherwise unphysical force N.

Further remarks on the *R*-independence

Since the radius R does not enter the Euler's formula (2), one can conclude that (2) also applies to belts wrapped around smooth surfaces of any shape,[5] such as shown in Fig. 12a. Indeed, in this case the radius of the curvature changes along the arc length, $\rho = \rho(s)$, so that the equilibrium conditions applied to an infinitesimal belt segment shown in Fig. 12b (vanishing resulting force in the tangential and normal direction) give $T(s) = p(s)\rho(s)$ and $\mathrm{d}T(s) = f(s)\mathrm{d}s$, where $\mathrm{d}s = \rho(s)\mathrm{d}\varphi$.[23] In the state

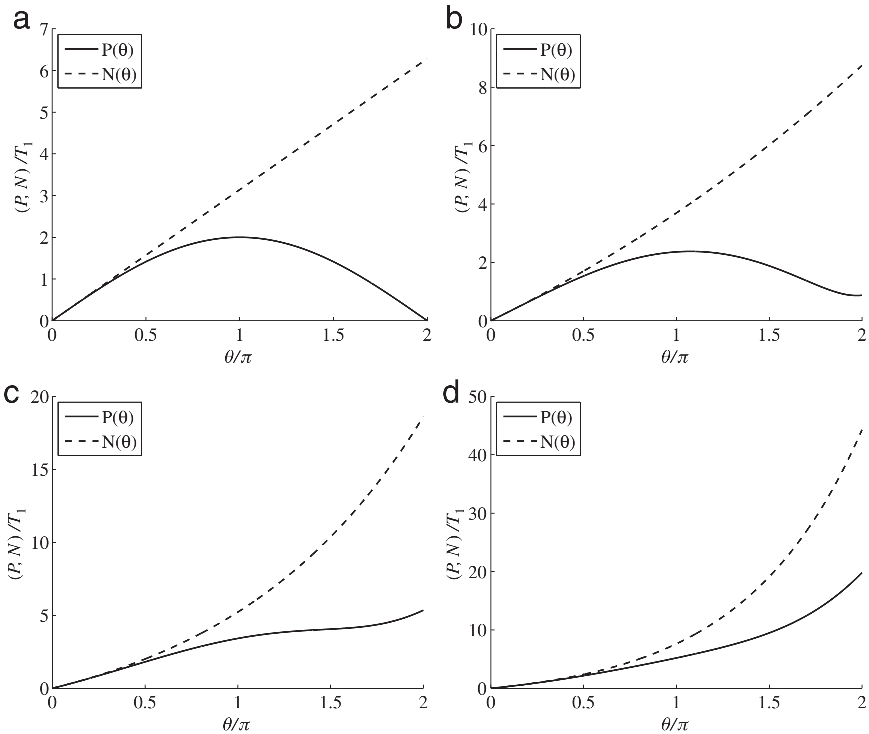

Fig. 11 *The variations of P and N (scaled by T_1) with the angle θ (scaled by π) in the cases: (a) $\mu = 0$, (b) $\mu = 0.1$, (c) $\mu = 0.3$, and (d) $\mu = 0.5$.*

of impending slip $f(s) = \mu p(s)$, and the integration gives $T_2 = T_1\exp(\mu\theta)$, where θ is the angle between the normals to the contact surface at the end points of the contact. The force in the belt and the contact pressure at an arbitrary point within the contact angle are

$$T(\varphi) = T_0\exp(\mu\varphi)\ , \quad p(\varphi) = \frac{\rho_0 p_0}{\rho(\varphi)}\ \exp(\mu\varphi)\ , \tag{27}$$

where p_0 is the pressure and ρ_0 the radius of curvature at the contact point $\varphi = 0$.

An independent derivation of the aforementioned results, in the spirit of an approach used to derive the expressions for the tangential and normal components of acceleration,[3,5] or that used in the mechanics of curved beams and thin shells,[11,14] is instructive. For equilibrium, the vector sum of the forces acting on the segment of the belt shown in Fig. 12b must be equal to zero. This gives

$$\mathrm{d}\mathbf{T} = f\rho\ \mathrm{d}\varphi\ \mathbf{t} - p\rho\ \mathrm{d}\varphi\ \mathbf{n}\ , \tag{28}$$

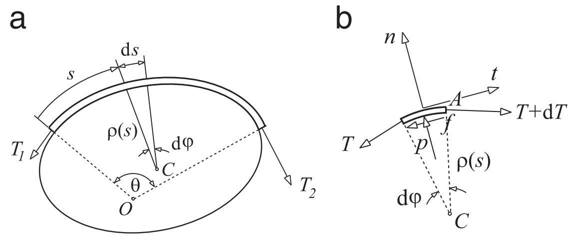

Fig. 12 *(a) A flat belt wrapped around a noncircular cylinder. The contact angle θ is defined by the intersection of the directions orthogonal to the cylindrical surface at the end contact points. (b) A free-body diagram of an infinitesimal segment of the belt. The local radius of the curvature is $\rho(s)$, where s is the arc length along the belt.*

where **n** and **t** are the unit vectors orthogonal and tangential to the cylinder at the considered contact point. On the other hand, by writing $\mathbf{T} = T\mathbf{t}$, its increment is $\mathrm{d}\mathbf{T} = dT\mathbf{t} + T\mathrm{d}\mathbf{t}$. Since $\mathrm{d}\mathbf{t} = -\mathbf{n}\mathrm{d}\varphi$, the increment of the belt force becomes

$$\mathrm{d}\mathbf{T} = \mathrm{d}T\ \mathbf{t} - T\ \mathrm{d}\varphi\ \mathbf{n}\ . \tag{29}$$

The comparison of (28) and (29) then yields $T = p\rho$ and $\mathrm{d}T = f\rho\mathrm{d}\varphi$, in agreement with the earlier more elementary considerations. A related discussion can be found in[15], which also offers an analysis of the belt friction in the case of the three-dimensional contact geometry.

Conclusions

We have pointed out and discussed a methodological and conceptual drawback in the derivation of the force required to pull a thin flexible belt over a fixed drum, present in all undergraduate mechanics textbooks which utilize an increment of a poorly defined and unphysical force (N). This drawback is corrected by employing in the derivation to local pressure and local friction force between the belt and the drum. The explicit use of the local pressure and friction forces also provides an opportunity to better explain physically the independence of the pull-force required to slip the belt of the radius of the drum. The latter is to some extent counterintuitive, if one considers the fact that the frictional resistance between the belt and the drum acts over the entire contact length between the two. Dimensional arguments are provided to shed additional light to this independence. Although the difference between the pull- and hold-force at the two ends of the bell equals the integral of the shear forces along the contact length, we evaluate the total forces due to pressure and friction alone (P and F) to examine their contributions in carrying the resultant (F_{R}) of the forces applied at two ends of the belt. It is shown that, in the state of impending slip, not only the local friction force is proportional to the local pressure, but also their resultants ($F = \mu P$), where μ is the coefficient of static friction. The magnitude of the pressure force is $P = F_{\mathrm{R}}/(1 + \mu^2)^{1/2}$. We quantify the difference between the physical force P and a nonphysical force quantity N by providing their variations with the contact angle for different values of the coefficient of friction. The presented analysis may be useful to the university instructors of engineering mechanics in their discussion of the belt friction, the derivation of the Euler's formula, and the explanation of its independence of the radius of the cylinder supporting the belt. The analysis of the belt force before the state of impending slip has been reached is presented separately.[16]

Acknowledgments

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17 Also known as the Euler–Eytelwein formula. Johann Albert Eytelwein (1764–1848) was a German engineer and university teacher, who included Euler's work on belt friction[1] in the second volume of his book "Handbuch der Statik fester Koerper" (1808).
18 This relation is reminiscent of the relation giving the circumferential (hoop) force in a thin ring under uniform internal pressure, although it differs from it by the angle dependence of the force and

the pressure. It is also reminiscent to the relationship between the circumferential force and the radial pressure in nonuniformly loaded thin cylindrical shells.[11]

19 Of course, one could divide $\mathrm{d}N = T\mathrm{d}\varphi$ by the arc length $\mathrm{d}s = R\mathrm{d}\varphi$, define the pressure by $p = \mathrm{d}N/\mathrm{d}s$, and recover $T = Rp$, but this is not commonly done in the textbooks on the subject.[2–8] Also, the problem with the definition of the force $\mathrm{d}N$ as the increment of an unclearly defined quantity N remains if such approach is taken, as further discussed in section 3.

20 If T is to depend on R, there would have to be another length scale (say L), independent of R, to cancel the length dimension of R through the ratio R/L. Since there is no such length scale in the considered problem, the force in the belt depends only on φ (and, of course, T_1 and μ).

21 The counterintuitive outcome of the capstan equation that the pull-force in the cable is independent of the radius of the cylinder is also discussed in Ref 13.

22 By the theorem of three forces, the direction of the force $\mathbf{F}_\mathrm{R}$ passes through the point of the intersection of $\mathbf{T}_1$ and $\mathbf{T}_2$, making them a set of three concurrent forces. The horizontal and vertical component of $\mathbf{F}_\mathrm{R}$ are $F_{Rx}(\theta) = T_1 \sin\theta_0 - T_2(\theta)\sin(\theta_0 + \theta)$, $F_{Ry}(\theta) = T_1\cos\theta_0 - T_2(\theta)\cos(\theta_0 + \theta)$.

23 Alternatively, this follows from the moment equilibrium conditions: the vanishing moment for point C gives $\mathrm{d}T(s) = f(s)\mathrm{d}s$, while the vanishing moment for the point A gives $T(s) = p(s)\rho(s)$.