Guidelines

Please turn in a neat homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Required plots should be generated using computer software such as Matlab or Excel. Remember to specify all the units of your results.

Problem 1

Consider a flow defined by the stream function:

$$\psi(x, y) = 5 \sin(\alpha x) \cos(\alpha y) + 2x^2y,$$

where \(\alpha\) is a free parameter.

1. Calculate the velocity field: \(u(x, y)\) and \(v(x, y)\).
2. Calculate the vorticity field. What is the direction of the vorticity vector?
3. Find the value(s) of \(\alpha\) so that the flow possesses a potential function, if any.
4. Explain the previous point.
5. Find the potential function of this flow.
Problem 2

Consider an incompressible flow around an equilateral triangular airfoil with chord length $c$ (figure 1). The freestream velocity is $U_\infty$. The pressure coefficients on sides 1 and 2 can be directly determined using inviscid flow theory, resulting:

$$C_{p1}(x) = C_{p2}(x) = C_{p\infty} + (C_{p0} - C_{p\infty}) \frac{x}{c},$$

where $C_{p0}$ and $C_{p\infty}$ are known constants. The flow, however, separates at the rear part making impossible to evaluate $C_{p3}$. For this reason a wind tunnel test is performed. We have velocity measurements upstream of the airfoil $U_\infty$ and far downstream:

$$U(y) = U_\infty \frac{|y|}{\sqrt{2}c}, \quad |y| < \frac{\sqrt{2}c}{2},$$

$$U(y) = U_\infty, \quad |y| > \frac{\sqrt{2}c}{2}$$

Figure 1: Triangular airfoil in the wind tunnel.

The pressure is constant and equal to $p_\infty$ far from the airfoil. Assuming that it is constant, determine $C_{p3}$ following these steps:

1. Write an integral relation between the pressure coefficients $(C_{p1}(x), C_{p2}(x), C_{p3})$ around the airfoil and the drag coefficient $c_d$.

2. Solve the previous integral to find $c_d$ as a function of the three constants: $C_{p0}$, $C_{p3}$ and $C_{p\infty}$.

3. Determine the drag force per unit span $D'$ acting on the airfoil using the momentum balance equation. Obtain $c_d$.

4. Equate the two expressions obtained in 2. and 3. and obtain $C_{p3}$ in case we have $C_{p0} = C_{p\infty} = 1/2.$
Problem 3

A vortex cannon impulsively accelerates a mass of air through a circular hole generating a vortex ring (figure 2).

Let us model a cross-section of the vortex ring as two counter-rotating vortices (figure 3). The streamfunction

\[ \psi(x, y) = \frac{\Gamma}{4\pi} \ln \left( \frac{(x - a)^2 + y^2}{(x + a)^2 + y^2} \right) \]

defines the flow generated by these two counter rotating vortices of intensity \( \Gamma \) located at \( (x = \pm a) \).

Assume the vortices are fixed at their current positions. Based on the given equation:

1. Draw the streamlines of this flow
2. Calculate the volume flow rate per unit depth passing through the point \( x = \frac{a}{2} \) and \( x = -\frac{a}{2} \) on the x-axis.
3. Calculate the corresponding velocity field in Cartesian coordinates.
4. Calculate the circulation along path A.
5. Calculate the circulation along path B.
6. Interpret your results from part (4) and part (5). Do they make sense?
7. Now we set the vortices free, how will they move? Quantitatively describe the translational velocity at the center of each vortex.