Problem 1

Consider a flow that is defined by its stream function:

\[ \psi = 3e^{\alpha x} \sin(2y) + 2\alpha xy, \quad (1) \]

where \( \alpha \) is a free parameter.

1. Calculate the velocity field of the flow.
2. Calculate the vorticity field of the flow.
3. Find the value(s) of \( \alpha \) so that the flow possesses a potential function, if any.
4. Explain part (3).
5. Find the potential function of this flow.

Problem 2

Consider a paddle that is moving in water. The wake trailing the paddle can be modeled by a pair of counter-rotating vortices, as shown in Fig 2-1. Given the streamfunction \( \psi \) as

\[ \psi = \frac{\Gamma}{4\pi} \left[ \ln \left( \frac{(x - a)^2 + y^2}{(x + a)^2 + y^2} \right) \right]. \quad (2) \]

Assume the vortices are fixed at their current positions. Based on this given equation,

1. Draw the streamlines of this flow.
2. Calculate the volume flow rate per unit depth passing through the point \( x = \frac{a}{2} \) and \( x = -\frac{a}{2} \) on the x-axis.
3. Calculate the corresponding velocity field in Cartesian coordinates.
4. Calculate the circulation along path A.
5. Calculate the circulation along path B.
6. Interpret your results from part (4) and part (5). Do they make sense?
7. Now we set the vortices free, how will they move? Quantitatively describe the translational velocity at the center of each vortex.
Problem 3

Two sources and a sink with different intensities are placed on the x-axis as shown in Fig 3-1. Determine

1. The streamfunction of the flow. Apply the compatibility relations to the stream function and calculate the potential function.

2. The velocity of the flow.

3. The stagnation point(s), if any, and their locations.

4. Plot the streamlines of the flow. Clearly indicate the dividing streamlines if any.

Extra Credit Problem

Remember the 96’ movie Twister where Helen Hunt and Bill Paxton were chasing tornados? There was a scene where a cow is sucked into a fast-rotating tornado. By studying the aerodynamics of this cow, we want you to predict the radius of rotation of the cow and its aerodynamic efficiency.

Model the tornado as a sink (volume flow rate per unit length $Q = 3600 \, (m^2/s)$) plus a vortex (circulation $\Gamma = -36 \, (m^2/s)$) centered at the origin, and the cow having a spherical shape (yes, the mythical spherical cow) with radius $r = 1 \, (m)$ and density $\rho_{cow} = 1200 \, (kg/m^3)$. The drag coefficient of the cow is known to be $c_d = 0.67$ and the air density of the current situation is $\rho_a = 1.2 \, (kg/m^3)$. Calculate
1. The radius $R$ of the circular trajectory of the flying cow.

2. The aerodynamic efficiency $\frac{L}{D}$ of the cow.

Note: The centrifugal force of a mass $m$ rotating on a circular trajectory with radius $R$ is $F_c = m\frac{v^2}{R}$.

Figure E-1: The illustration of a cow in the tornado.