Problem 1

A vortex sheet is defined as a distribution of vortices in the limit where we have infinite point vortices that are infinitesimally close to each other, and their strength becomes infinitesimally small. This concept can be understood following these steps:

1. Show that the velocity field generated by a vortex sheet of uniform intensity per unit length, $\gamma(s) = 2U$, located at $y = 0$, $-a \leq x \leq a$ (see figure 1) is given by:

   \[
   u = \frac{U}{\pi} \arctan \left( \frac{x + a}{y} \right) - \frac{U}{\pi} \arctan \left( \frac{x - a}{y} \right)
   
   v = \frac{U}{2\pi} \log \left[ \frac{(x - a)^2 + y^2}{(x + a)^2 + y^2} \right]
   \]

   Calculate the two velocity components at $(x, y) = (0, -\sqrt{3}a)$.

2. Consider the velocity field generated by two point vortices of intensity $\Gamma = Ua/2$ located at $(x, y) = (-a, 0)$ and $(x, y) = (a, 0)$. Calculate $(u, v)$ generated by this configuration at $(x, y) = (0, -\sqrt{3}a)$. 

Figure 1: Vortex sheet.
3. Consider the velocity field generated by three point vortices of intensity $\Gamma = Ua/3$ located at $(x, y) = (-a, 0)$, $(x, y) = (0, 0)$ and $(x, y) = (a, 0)$. Calculate $(u, v)$ generated by this configuration at $(x, y) = (0, -\sqrt{3}a)$.

4. Consider the velocity field generated by four point vortices of intensity $\Gamma = Ua/4$ located at $(x, y) = (-a, 0)$, $(x, y) = (-a/3, 0)$, $(x, y) = (a/3, 0)$ and $(x, y) = (a, 0)$. Calculate $(u, v)$ generated by this configuration at $(x, y) = (0, -\sqrt{3}a)$.

5. Consider the velocity field generated by five point vortices of intensity $\Gamma = Ua/5$ located at $(x, y) = (-a, 0)$, $(x, y) = (-a/2, 0)$, $(x, y) = (0, 0)$, $(x, y) = (a/2, 0)$ and $(x, y) = (a, 0)$. Calculate $(u, v)$ generated by this configuration at $(x, y) = (0, -\sqrt{3}a)$.

6. How does the velocity generated by the vortex sheet at $(x, y) = (0, -\sqrt{3}a)$ compare to the velocity generated by the point vortex distribution as we progressively increase the number of vortices and decrease their intensity? What do you expect to happen when the number of vortices $N \rightarrow \infty$ if their intensity $\Gamma$ goes to zero as $Ua/N$?

Problem 2

Two plane sources of intensity $m$ are located at the points $(x, y) = (-a, b)$ and $(x, y) = (a, b)$, near an impermeable wall located at $y = 0$, as shown in figure 2, where $2a$ is the distance between the sources and $b$ the distance with the wall. The sources interact between them and with the wall. When we set both sources free, they propel with a velocity that we want to calculate.

![Figure 2: Sources near a wall.](image)

1. Assuming that both sources are fixed, calculate the potential function of the flow by applying the Method of Images.

2. Using the previous result, calculate the velocity field $[u(x, y), v(x, y)]$.

3. Calculate the velocity at the surface of the wall. Is it compatible with the boundary conditions introduced by an impermeable wall?
4. Find the stagnation point(s) where \((u, v) = (0, 0)\). **Note:** there are either one, two or three stagnation points, depending on the value of the ratio \(b/a\). Due to the symmetry of the flow, you can guess the region where the stagnation point(s) should be. Simplify the equation for \(u\) and \(v\) for that region and then solve \((u, v) = (0, 0)\).

5. Now we set both sources free. Calculate their velocity.

6. Using the velocity obtained in the previous part, calculate the trajectories of the sources. **Note:** you can try to solve the equations for the trajectories analytically or numerically. If you choose to do it numerically, plot the trajectories for the three limit cases \(a = b\), \(b << a\) and \(b >> a\).

7. Discuss what happens when the vortices are far away from each other, \(a \to \infty\). What happens when they are far away from the wall, \(b \to \infty\)?