MAE 104, FALL 2010
HOMEWORK 3
Due Thursday 10-14-2010 in class

Guidelines:
Please turn in a neat homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Required plots should be generated using computer software such as Matlab or Excel.

Problem 1
Take a plane source of strength $m$ and located at the point $(x, y) = (-a, 0)$, a plane sink of equal strength and located at $(a, 0)$, and superimpose to them a uniform stream $U$ directed along the x-axis.

1. Calculate the stream function of the flow.

2. Applying the compatibility relations to the stream function determined in the previous step, calculate the potential function.

3. Calculate the velocity vector $(u, v)$.

4. Find the two stagnation points where $(u, v) = (0, 0)$.

5. Sketch (by hand) the streamlines and the lines of constant potential.

Problem 2
The propeller of an airplane flying at speed $U_1$ can be modeled as a permeable disk. In a reference frame in which the air is moving and the propeller is stationary, the effect of the propeller is to accelerate the fluid from the upstream value $U_1$ to the downstream value $U_2 > U_1$. Assuming incompressibility:

1. Apply Bernoulli equation from a section far upstream (with known pressure $p_\infty$) to a section immediately upstream of the propeller. Find the pressure at that section, $p_I$ as a function of the velocity at that section, $u_I$.

2. Apply Bernoulli equation from a section far downstream (with known pressure $p_\infty$) to a section immediately downstream of the propeller. Find the pressure at that section, $p_{II}$ as a function of the velocity at that section, $u_{II}$. 

3. Apply the mass conservation equation to the disk and find a relation between \( u_I \) and \( u_{II} \).

4. The propeller produces a traction \( F = \Delta p A \), where \( A \) is the area of the disk and \( \Delta p \) the pressure jump between both faces. Find an expression for \( F \) as a function of the density of air \( \rho \), \( A \), \( U_1 \) and \( U_2 \).

5. Apply the momentum balance to the disk. Introduce the expression for \( F \) and show that the velocity of the fluid at the plane of the propeller is the average value:

\[
  u_p = \frac{U_1 + U_2}{2}.
\]

**Problem 3**

A circular cylinder of radius \( a \) is rotating with angular velocity \( \Omega \) in the presence of a uniform stream. Far away from the cylinder, the velocity is \( U_\infty \) and the pressure is \( p_\infty \), as seen in figure 1. The density \( \rho \) and viscosity \( \mu \) are constant everywhere in the fluid.

![Figure 1: Rotating cylinder in the presence of an uniform flow.](image)

The flow around the cylinder can be modeled as the superposition of a doublet and a vortex in the origin and a uniform flow. The stream function for this flow is

\[
  \psi = \left( 1 - \frac{a^2}{r^2} \right) U_\infty r \sin(\theta) + a^2 \Omega \ln \left( \frac{r}{a} \right).
\]

We would like to calculate the lift coefficient of the cylinder. For this purpose:

1. Revisit the solution of problem 2 in Homework 2 and write \( c_L \) as a function of \( a \), \( U_\infty \) and \( \Omega \).
2. Calculate the velocity in polar coordinates.

3. Apply the compatibility relations to the stream function and obtain the potential function for the flow. In polar coordinates, these relations are

\[ v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}, \]
\[ v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta}. \]

4. Prove that the circulation \( \Gamma \) around any closed curve enclosing the center is the same by

   (a) Calculating \( \Gamma \) around a closed circle of radius \( a \) centered at the origin.
   (b) Calculating \( \Gamma \) around a closed circle of radius \( 2a \) centered at the origin.

5. The circulation \( \Gamma \) around any closed curve not enclosing the center is the same. Calculate \( \Gamma \) around the closed curve shown in figure 2.

![Figure 2: Control volume excluding the center.](image)

6. Apply the Kutta-Jukowsky theorem and write the lift as a function of \( \Gamma \).

7. Write the lift coefficient as a function of \( a, U_\infty \) and \( \Omega \). Show that it is the same as in part 1.
Problem 4

1. A vortex of intensity $\Gamma$ located at the origin induces a flow whose stream function is

$$\psi = \frac{\Gamma}{2\pi} \ln r,$$

where $r$ is the distance to the origin.

(a) Calculate the velocity vector in polar coordinates.
(b) Write the velocity in Cartesian coordinates.
(c) Calculate the circulation around a circle of radius $a$ centered at the origin.
(d) Calculate the circulation around the rectangular path shown in figure 3, which does not enclose the center of the vortex.

![Figure 3: Rectangular path for integration in part 1.d.](image)

Note: you may need to use the integral:

$$\int \frac{c}{x^2 + c^2} dx = \arctan \left( \frac{x}{c} \right)$$

and the expression:

$$\arctan \left( \frac{1}{x} \right) = \frac{\pi}{2} - \arctan(x)$$

when $x > 0$.

2. Two vortices of intensity $\Gamma$ and $-\Gamma$ are fixed at $(x, y) = (-a, 0)$ and $(a, 0)$.

(a) Calculate the stream function (use Cartesian coordinates).
(b) Write the velocity in Cartesian coordinates.
(c) Calculate the circulation around the path shown in figure 4, which does not enclose the center or any vortex.

(d) Calculate the circulation around the path shown in figure 5, which encloses the center or one vortex.

(e) Calculate the circulation around the path shown in figure 6, which encloses the center or both vortices.

(f) If we set both vortices free so that they move with the fluid, calculate the trajectories of their centers.
Figure 6: Rectangular path for integration in part 2.e.