MAE 107
Extra Study Problems with Solutions

1. Consider \( f(x) = \exp(x^2) \). With \( x_0 = 1 \) and \( h = 0.001 \), compute \( D_h(x_0) \) and \( \hat{D}_h(x_0) \). What are the actual errors in these derivative estimates? Recall

\[
D_h(x_0) = \frac{f(x_0 + h) - f(x_0)}{h}, \quad \hat{D}_h(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}.
\]

2. Consider \( f(x) = 4x^3 + x \). With \( x_0 = 1 \) and \( h = 1/4 \), compute \( D_h(x_0) \) and \( \hat{D}_h(x_0) \). What are the actual errors in these derivative estimates? Obtain bounds on the errors in \( D_h(x_0) \) and \( \hat{D}_h(x_0) \) using the estimates from class.
Solutions

Problem 1.

Using the given formulae, we have:

\[ f(x_0) = f(1) = e \approx 2.718281828459, \]
\[ f(x_0 + h) = e^{(1+0.001)^2} \approx 2.723726556031, \]
\[ f(x_0 - h) = e^{(1-0.001)^2} \approx 2.712853410595, \]
\[ D_h(x_0) \approx 5.444727571957, \]
\[ \hat{D}_h(x_0) \approx 5.436572717864, \]
\[ f'(x_0) = 2x_0 e^{x_0^2} \approx 5.436563656918, \]

and consequently, the actual errors are

\[ e_h(x_0) = |D_h(x_0) - f'(x_0)| \approx 8.16 \times 10^{-3}, \]
\[ \hat{e}_h(x_0) = |\hat{D}_h(x_0) - f'(x_0)| \approx 9.06 \times 10^{-6}. \]

Problem 2.

Using the given formulae, we have:

\[ f(x_0) = f(5/4) = 9.0625, \]
\[ f(x_0 - h) = f(3/4) = 2.4375, \]
\[ D_h(x_0) = 16.25, \]
\[ \hat{D}_h(x_0) = 13.25, \]
\[ f'(x_0) = 12x^2 + 1 = 13, \]

and consequently, the actual errors are

\[ e_h(x_0) = |D_h(x_0) - f'(x_0)| = 3.25, \]
\[ \hat{e}_h(x_0) = |\hat{D}_h(x_0) - f'(x_0)| = 0.25. \]

For the bounds in the derivative approximation errors, we use

\[ e_h(x_0) \leq \frac{\max_{\xi \in [x_0, x_0+h]} |f^{(2)}(\xi)|}{2} h, \]

and

\[ \hat{e}_h(x_0) \leq \frac{\max_{\xi \in [x_0-h, x_0+h]} |f^{(3)}(\xi)|}{6} h^2. \]
(A potentially slightly tighter bound is
\[ \hat{e}_h(x_0) \leq \left[ \frac{\max_{\xi \in [x_0, x_0+h]} |f^{(3)}(\xi)|}{12} + \frac{\max_{\xi \in [x_0-h, x_0]} |f^{(3)}(\xi)|}{12} \right] h^2, \]
but the difference should be irrelevant for small \( h \).) Noting that
\[ f^{(2)}(\xi) = 24\xi \quad \text{and} \quad f^{(3)}(\xi) = 24, \]
we find
\[ e_h(x_0) \leq \frac{24(1 + 1/4)}{2}(1/4) = 3.75, \]
\[ \hat{e}_h(x_0) \leq \frac{24}{6}(1/4)^2 = 0.25. \]
As expected, the actual errors are no greater than the bounds.