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Energies of circular inclusions: sliding versus bonded interfaces

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The elastic strain energies of circular inclusions with a sliding and bonded interfaces are compared. It is shown that the energy in the inclusion with sliding interface due to uniform eigenstrain is greater than the energy in the inclusion with a bonded interface if the Poisson ratio of the material is less than $\frac{1}{6}$, and smaller if it is greater than $\frac{1}{6}$. The total energy in the inclusion and the matrix due to uniform eigenstrain is always smaller in the case of a sliding inclusion. The opposite is true for the inclusion under remote uniform loading at infinity. The relationships between the energies of sliding and bonded inhomogeneities are also derived.

Keywords: eigenstrain; Eshelby inclusions; Papkovich–Neuber potentials; sliding interface; strain energy

1. Introduction

A study of elastic inclusions with sliding interfaces is important for modelling certain features of material behaviour, such as the grain-boundary sliding in polycrystalline materials. In his pioneering work on the subject, Zener (1941) derived a solution for the circular sliding inclusion under remote uniaxial loading by an adequate superposition of three known elasticity solutions. Ghahremani (1980) extended his results to three dimensions, and obtained a solution for the sliding spherical inclusion by using the Papkovich–Neuber displacement potentials. In a series of papers, Mura and his collaborators studied the stress fields of ellipsoidal inclusions with sliding interfaces under various types of eigenstrain or remote loading (Mura & Furuhashi 1984; Mura *et al*. 1985; Tsuchida *et al*. 1986; Jasiuk *et al*. 1987; Furuhashi *et al*. 1992). The Papkovich–Neuber potentials in the form of infinite series were commonly used. Lubarda & Markenscoff (1998) examined an unusual nature of the stress field in sliding ellipsoidal inclusions due to uniform shear eigenstrain, and proposed a method to calculate this field for nearly circular or spherical inclusions. An energy study of sliding circular inclusions is presented in this paper. Simple relationships between the energies of inclusions with sliding and bonded interfaces are derived. It is shown that the energy in the sliding inclusion with uniform eigenstrain can be either greater or smaller than the energy in the inclusion with a bonded interface, depending on the value of the Poisson ratio. The total energy in the inclusion and the matrix is always smaller in the case of a sliding inclusion. The opposite is found to be true for the inclusion under remote uniform loading at infinity. The relationship between the energies of circular inhomogeneities with sliding and bonded interfaces is also derived. These results are of interest for the evaluation of average elastic properties of composites with sliding interfaces and for other applications.

Since the analysis presented requires a knowledge of the stress and displacement fields, these are given for both the inclusion and the matrix in the appendices of the paper. A new superposition method is constructed to determine the stresses and displacements for sliding inclusions under remote loading, based on the knowledge of these fields in the corresponding eigenstrain problems. All expressions are given in compact form, which enables easy comparison between the effects of sliding and bonded interfaces. For example, it is shown that a discontinuity in the hoop stress across the interface of a sliding inhomogeneity under remote shear loading does not depend on the material properties, while it does in the case of bonded interface.

2. Energy expressions for sliding inclusion

The elastic strain energy in an inclusion, which has undergone a stress-free eigenstrain transformation, is equal to the work done on the inclusion to insert it back into the matrix. This is

$$E^{\mathrm{I}} = E^* + \frac{1}{2}\int_{\mathrm{S}} (t_i^{\mathrm{I}} - t_i^*) u_i^{\mathrm{I}} \,\mathrm{d}S + \int_{\mathrm{S}} t_i^* u_i^{\mathrm{I}} \,\mathrm{d}S, \tag{2.1}$$

where u_i^{I} is the displacement, and $t_i^{\mathrm{I}} = \sigma_{ij}^{\mathrm{I}} n_j$ is the traction on the boundary, S, of the inclusion when inserted back into the matrix; $\sigma_{ij}^{\mathrm{I}} = C_{ijkl}(\epsilon_{kl}^{\mathrm{I}} - \epsilon_{kl}^*)$ is the corresponding stress, and $\epsilon_{ij}^{\mathrm{I}}$ and ϵ_{ij}^* are the strain and eigenstrain in the inclusion, respectively. Defining $\sigma_{ij}^* = -C_{ijkl}\epsilon_{kl}^*$, we also have $t_i^* = \sigma_{ij}^* n_j$ and $E^* = -\frac{1}{2}\int_V \sigma_{ij}^*\epsilon_{ij}^* \,\mathrm{d}V$. Therefore, upon the application of the divergence theorem, equation (2.1) gives

$$E^{\mathrm{I}} = \frac{1}{2}\int_V \sigma_{ij}^{\mathrm{I}}(\epsilon_{ij}^{\mathrm{I}} - \epsilon_{ij}^*) \,\mathrm{d}V. \tag{2.2}$$

The strain energy in the matrix is

$$E^{\mathrm{M}} = \frac{1}{2}\int_{\mathrm{S}} t_i^{\mathrm{M}} u_i^{\mathrm{M}} \,\mathrm{d}S, \tag{2.3}$$

where u_i^{M} is the displacement of the points of the matrix along the surface S, and $t_i^{\mathrm{M}} = -t_i^{\mathrm{I}} = -\sigma_{ij}^{\mathrm{I}} n_j$ is the corresponding traction. Thus

$$E^{\mathrm{M}} = -\frac{1}{2}\int_{\mathrm{S}} \sigma_{ij}^{\mathrm{I}} u_i^{\mathrm{M}} n_j \,\mathrm{d}S. \tag{2.4}$$

The total strain energy, in the inclusion and the matrix, is $E^{\mathrm{T}} = E^{\mathrm{I}} + E^{\mathrm{M}}$, which gives

$$E^{\mathrm{T}} = -\frac{1}{2}\int_V \sigma_{ij}^{\mathrm{I}}\epsilon_{ij}^* \,\mathrm{d}V. \tag{2.5}$$

In the case of a bonded interface between the inclusion and the matrix, this follows because $u_i^{\mathrm{I}} = u_i^{\mathrm{M}}$ at the interface (Eshelby 1957). In the case of the sliding interface, equation (2.5) follows because $\sigma_{ij}^{\mathrm{I}} n_j (u_i^{\mathrm{I}} - u_i^{\mathrm{M}}) = 0$, the traction vector being normal to the slip vector $u_i^{\mathrm{I}} - u_i^{\mathrm{M}}$ at the interface.

A complete solution for the displacement and stress distribution for a circular inclusion with a sliding interface in an infinite matrix under different types of loading is given in the appendices of the paper. These results were used to determine the relevant energies in the problem, and to compare them with those of the Eshelby inclusion problem with a perfectly bonded interface. In calculations, it is convenient to first calculate the total energy and the energy in the matrix, from the formulae

$$E^{\mathrm{T}} = -\frac{1}{2}\int_0^a\int_0^{2\pi}(\sigma_r^{\mathrm{I}}\epsilon_r^* + \sigma_\theta^{\mathrm{I}}\epsilon_\theta^* + 2\sigma_{r\theta}^{\mathrm{I}}\epsilon_{r\theta}^*)r\,\mathrm{d}r\mathrm{d}\theta, \tag{2.6}$$

$$E^{\mathrm{M}} = -\frac{1}{2}\int_0^{2\pi}(\sigma_r^{\mathrm{I}}u_r^{\mathrm{M}} + \sigma_{r\theta}^{\mathrm{I}}u_\theta^{\mathrm{M}})a\,\mathrm{d}\theta, \tag{2.7}$$

and then to calculate the energy in the inclusion as $E^{\mathrm{I}} = E^{\mathrm{T}} - E^{\mathrm{M}}$. In the case of the sliding interface, $\sigma_{r\theta}^{\mathrm{I}} = 0$ in equation (2.7). The energy expressions for sliding inclusions and inhomogeneities under remote loading at infinity are given in §§ 4 and 5.

3. Energies due to eigenstrain

The stress field within the circular inclusion with a sliding interface due to biaxial eigenstrain ϵ_x^* and ϵ_y^* is given by equations (A 10)–(A 12) of Appendix A. The radial displacement at the interface is given by equation (A 8) or (A 13), with $r = a$. Expressing the eigenstrain $(\epsilon_x^*, \epsilon_y^*)$ in polar coordinates by usual transformation formulae, substitution into equations (2.6) and (2.7) and integration, gives the following expressions for the energies (per unit length in the z-direction):

$$E_{\mathrm{S}}^{\mathrm{T}} = \frac{\mu a^2\pi}{32(1-\nu)}[8(\epsilon_x^* + \epsilon_y^*)^2 + 3(\epsilon_x^* - \epsilon_y^*)^2], \tag{3.1}$$

$$E_{\mathrm{S}}^{\mathrm{M}} = \frac{\mu a^2\pi}{256(1-\nu)^2}[32(\epsilon_x^* + \epsilon_y^*)^2 + 3(5-6\nu)(\epsilon_x^* - \epsilon_y^*)^2], \tag{3.2}$$

$$E_{\mathrm{S}}^{\mathrm{I}} = \frac{\mu a^2\pi}{256(1-\nu)^2}[32(1-2\nu)(\epsilon_x^* + \epsilon_y^*)^2 + 3(3-2\nu)(\epsilon_x^* - \epsilon_y^*)^2]. \tag{3.3}$$

The subscript 'S' indicates the sliding interface. The energy in the matrix is greater than in the inclusion, more so the less compressible the material. Indeed,

$$E_{\mathrm{S}}^{\mathrm{M}} - E_{\mathrm{S}}^{\mathrm{I}} = \frac{\mu a^2\pi}{128(1-\nu)^2}[32\nu(\epsilon_x^* + \epsilon_y^*)^2 + 3(1-2\nu)(\epsilon_x^* - \epsilon_y^*)^2]. \tag{3.4}$$

It is helpful to express the stress in the inclusion with respect to rectangular coordinates as:

$$\sigma_x^{\mathrm{S}} = -\frac{\mu}{8(1-\nu)}\left[4(\epsilon_x^* + \epsilon_y^*) + 3(\epsilon_x^* - \epsilon_y^*)\left(1 - 2\frac{y^2}{a^2}\right)\right], \tag{3.5}$$

$$\sigma_y^{\mathrm{S}} = -\frac{\mu}{8(1-\nu)}\left[4(\epsilon_x^* + \epsilon_y^*) - 3(\epsilon_x^* - \epsilon_y^*)\left(1 - 2\frac{x^2}{a^2}\right)\right], \tag{3.6}$$

$$\sigma_{xy}^{\mathrm{S}} = 0, \tag{3.7}$$

for, then, it follows that the average stresses in the sliding inclusion are

$$\bar{\sigma}_x^{\mathrm{S}}, \bar{\sigma}_y^{\mathrm{S}} = -\frac{\mu}{16(1-\nu)}[8(\epsilon_x^* + \epsilon_y^*) \pm 3(\epsilon_x^* - \epsilon_y^*)], \qquad \bar{\sigma}_{xy} = 0. \tag{3.8}$$

Thus, comparing with equation (3.1), the total energy can also be expressed as

$$E_{\mathrm{S}}^{\mathrm{T}} = \frac{(1-\nu)a^2\pi}{24\mu}[3(\bar{\sigma}_x + \bar{\sigma}_y)^2 + 16(\bar{\sigma}_x - \bar{\sigma}_y)^2]. \tag{3.9}$$

The energy expressions in the case of a circular inclusion with a bonded interface can be obtained from Eshelby's (1957) formulae. These give

$$E_{\mathrm{B}}^{\mathrm{T}} = \frac{\mu a^2 \pi}{8(1-\nu)}[2(\epsilon_x^* + \epsilon_y^*)^2 + (\epsilon_x^* - \epsilon_y^*)^2], \tag{3.10}$$

$$E_{\mathrm{B}}^{\mathrm{M}} = \frac{\mu a^2 \pi}{32(1-\nu)^2}[4(\epsilon_x^* + \epsilon_y^*)^2 + (3-4\nu)(\epsilon_x^* - \epsilon_y^*)^2], \tag{3.11}$$

$$E_{\mathrm{B}}^{\mathrm{I}} = \frac{\mu a^2 \pi}{32(1-\nu)^2}[4(1-2\nu)(\epsilon_x^* + \epsilon_y^*)^2 + (\epsilon_x^* - \epsilon_y^*)^2]. \tag{3.12}$$

The energy in the matrix is greater or smaller than the energy in the inclusion, depending on the value of the Poisson ratio and the type of biaxial strain, since

$$E_{\mathrm{B}}^{\mathrm{M}} - E_{\mathrm{B}}^{\mathrm{I}} = -\frac{\mu a^2 \pi}{16(1-\nu)^2}[2(3-8\nu)(\epsilon_x^* + \epsilon_y^*)^2 - (1-2\nu)(\epsilon_x^* - \epsilon_y^*)^2]. \tag{3.13}$$

The most interesting result follows by comparing the energies stored in the inclusions with sliding and bonded interfaces. From equations (3.3) and (3.12), the difference in these energies is

$$E_{\mathrm{S}}^{\mathrm{I}} - E_{\mathrm{B}}^{\mathrm{I}} = \frac{1-6\nu}{256(1-\nu)^2}\mu a^2 \pi (\epsilon_x^* - \epsilon_y^*)^2. \tag{3.14}$$

Thus, we have a simple but appealing result: $E_{\mathrm{S}}^{\mathrm{I}} < E_{\mathrm{B}}^{\mathrm{I}}$ if $\nu > \frac{1}{6}$, and $E_{\mathrm{S}}^{\mathrm{I}} > E_{\mathrm{B}}^{\mathrm{I}}$ if $\nu < \frac{1}{6}$. Hence, the shear stress relaxation at the interface actually increases the strain energy in the inclusion for very compressible materials ($\nu < \frac{1}{6}$). It is interesting to note that the value of the Poisson ratio $\nu = \frac{1}{6}$ is the value for which the two Lamé constants of elasticity are related by $\mu = 2\lambda$. The total strain energy (in the inclusion and the matrix) is always smaller in the case of the sliding inclusion. In fact,

$$E_{\mathrm{S}}^{\mathrm{T}} - E_{\mathrm{B}}^{\mathrm{T}} = -\frac{\mu a^2 \pi}{32(1-\nu)}(\epsilon_x^* - \epsilon_y^*)^2. \tag{3.15}$$

The energy differences in equations (3.14) and (3.15) do not depend on the mean eigenstrain, $\frac{1}{2}(\epsilon_x^* + \epsilon_y^*)$, since the sliding and bonded inclusion respond equally to in-plane hydrostatic eigenstrain. Also, since the stresses in the bonded inclusion are

$$\sigma_x^{\mathrm{B}}, \sigma_y^{\mathrm{B}} = -\frac{\mu}{4(1-\nu)}[2(\epsilon_x^* + \epsilon_y^*) \pm (\epsilon_x^* - \epsilon_y^*)], \qquad \sigma_{xy}^{\mathrm{B}} = 0, \tag{3.16}$$

we have the useful connections

$$\bar{\sigma}_x^{\mathrm{S}} + \bar{\sigma}_y^{\mathrm{S}} = \sigma_x^{\mathrm{B}} + \sigma_y^{\mathrm{B}}, \qquad \bar{\sigma}_x^{\mathrm{S}} - \bar{\sigma}_y^{\mathrm{S}} = \tfrac{3}{4}(\sigma_x^{\mathrm{B}} - \sigma_y^{\mathrm{B}}). \tag{3.17}$$

The energy expressions listed for biaxial eigenstrain hold in the case of an arbitrary uniform eigenstrain $(\epsilon_x^*, \epsilon_y^*, \epsilon_{xy}^*)$, provided that the principal values of this strain state are substituted there for the two axial strain components. Thus, the conclusion that the energy in the sliding inclusion is greater than the energy in the bonded inclusion for $\nu < \frac{1}{6}$, and smaller for $\nu > \frac{1}{6}$, applies to any uniform eigenstrain. The total strain energy, in the inclusion and the matrix, is always smaller in the case of the sliding inclusion.

4. Energies due to remote loading

Consider next an infinite homogeneous block of material under plane strain conditions and biaxial loading σ_x^0 and σ_y^0 at infinity. The strain energy within a circular region of radius a around the origin is

$$E_{\mathrm{B}}^{\mathrm{I}} = \frac{1}{8\mu}[(1-2\nu)(\sigma_x^0+\sigma_y^0)^2 + (\sigma_x^0-\sigma_y^0)^2]a^2\pi. \tag{4.1}$$

What happens to this energy, i.e. does it increase or decrease when a cut is made along the circle of radius a around the origin and the shear stress is reduced to zero, preserving the continuity of the normal displacement and traction there? The problem is, thus, of a circular sliding inclusion under remote biaxial stress at infinity, for which the complete stress and displacement distributions are derived in Appendix B. The strain energy in the inclusion is

$$E_{\mathrm{S}}^{\mathrm{I}} = \frac{1}{2}\int_0^{2\pi} \sigma_r^{\mathrm{I}} u_r^{\mathrm{I}} a \,\mathrm{d}\theta, \tag{4.2}$$

where σ_r^{I} is given by equation (B 2) and u_r^{I} by equation (B 8), both evaluated at $r = a$. The integration gives

$$E_{\mathrm{S}}^{\mathrm{I}} = \frac{a^2\pi}{64\mu}[8(1-2\nu)(\sigma_x^0+\sigma_y^0)^2 + 3(3-2\nu)(\sigma_x^0-\sigma_y^0)^2]. \tag{4.3}$$

Therefore,

$$E_{\mathrm{S}}^{\mathrm{I}} - E_{\mathrm{B}}^{\mathrm{I}} = \frac{1-6\nu}{64\mu}(\sigma_x^0-\sigma_y^0)^2 a^2\pi, \tag{4.4}$$

which demonstrates that the strain energy within the inclusion itself can either increase or decrease, depending on the Poisson ratio. The energy in the sliding inclusion is decreased if $\nu > \frac{1}{6}$, and increased if $\nu < \frac{1}{6}$. This is opposite to the corresponding results for a sliding inclusion with biaxial eigenstrain. We also point out that the average stresses in the inclusion are related to applied remote stress by

$$\bar{\sigma}_x^{\mathrm{S}} + \bar{\sigma}_y^{\mathrm{S}} = \sigma_x^0 + \sigma_y^0 \quad \text{and} \quad \bar{\sigma}_x^{\mathrm{S}} - \bar{\sigma}_y^{\mathrm{S}} = \tfrac{3}{4}(\sigma_x^0 - \sigma_y^0),$$

so that

$$\bar{\sigma}_x^{\mathrm{S}} - \sigma_x^0 = -\tfrac{1}{8}(\sigma_x^0 - \sigma_y^0) \quad \text{and} \quad \bar{\sigma}_y^{\mathrm{S}} - \sigma_y^0 = \tfrac{1}{8}(\sigma_x^0 - \sigma_y^0).$$

Therefore, if $\sigma_x^0 > \sigma_y^0$, the average stress, $\bar{\sigma}_x^{\mathrm{S}}$, in the inclusion is smaller, and $\bar{\sigma}_y^{\mathrm{S}}$ greater than the corresponding stress in the body without a sliding interface.

Let $E_{\mathrm{B}}^{\mathrm{M}}$ denote the (infinite) strain energy in the matrix domain outside the circular region of radius a, before the cut was introduced and sliding took place. The total strain energy in the infinite medium is $E_{\mathrm{B}}^{\mathrm{T}} = E_{\mathrm{B}}^{\mathrm{I}} + E_{\mathrm{B}}^{\mathrm{M}}$. After the cut is introduced and shear stress at the interface relaxed to zero, the strain energy in the sliding inclusion becomes $E_{\mathrm{S}}^{\mathrm{I}}$ and in the surrounding matrix $E_{\mathrm{S}}^{\mathrm{M}}$. The corresponding total strain energy is $E_{\mathrm{S}}^{\mathrm{T}} = E_{\mathrm{S}}^{\mathrm{I}} + E_{\mathrm{S}}^{\mathrm{M}}$. The strain energy change, $\Delta E = E_{\mathrm{S}}^{\mathrm{T}} - E_{\mathrm{B}}^{\mathrm{T}}$, produced by the shear stress relaxation at the interface, being equal to the negative of the total potential energy change, is the work done by the shear stress,

$$\sigma_{r\theta}(a,\theta) = -\tfrac{1}{2}(\sigma_x^0 - \sigma_y^0)\sin 2\theta,$$

on the slip discontinuity at the interface,

$$\Delta u_\theta = -[(1-\nu)/2\mu](\sigma_x^0 - \sigma_y^0)a\sin 2\theta.$$

This gives

$$\Delta E = \frac{1}{2}\int_0^{2\pi} \sigma_{r\theta}(a,\theta)\Delta u_\theta a\,\mathrm{d}\theta = \frac{1-\nu}{8\mu}(\sigma_x^0 - \sigma_y^0)^2 a^2\pi. \tag{4.5}$$

The energy difference is positive, so that $E_\mathrm{S}^\mathrm{T} > E_\mathrm{B}^\mathrm{T}$. The total strain energy increases because the whole system becomes more compliant in the presence of a sliding inclusion, and for an applied remote load the average deformation, and thus the strain energy, both increase. Note that $\Delta E = -\Delta \varPi$, $\varPi$ being the total potential energy (the strain energy less the load potential at infinity). Thus, while for the strain energies $E_\mathrm{S}^\mathrm{T} > E_\mathrm{B}^\mathrm{T}$, the opposite inequality holds for the potential energies, $\varPi_\mathrm{S}^\mathrm{T} < \varPi_\mathrm{B}^\mathrm{T}$.

Since $E_\mathrm{S}^\mathrm{I} + E_\mathrm{S}^\mathrm{M} = E_\mathrm{B}^\mathrm{I} + E_\mathrm{B}^\mathrm{M} + \Delta E$, the change in the matrix energy produced by the shear stress relaxation at the interface is

$$E_\mathrm{S}^\mathrm{M} - E_\mathrm{B}^\mathrm{M} = E_\mathrm{B}^\mathrm{I} - E_\mathrm{S}^\mathrm{I} + \Delta E = \frac{7-2\nu}{64\mu}(\sigma_x^0 - \sigma_y^0)^2 a^2\pi, \tag{4.6}$$

by equations (4.4) and (4.5). Thus, the matrix energy increases by the shear stress relaxation at the interface.

The expressions for the energies listed for biaxial remote loading hold in the case of an arbitrary uniform loading at infinity $(\sigma_x^0, \sigma_y^0, \sigma_{xy}^0)$, provided that the principal values of this stress state are substituted for the biaxial stress components. Thus, the conclusion that the energy in the inclusion decreases by introduction of the sliding interface if $\nu < \frac{1}{6}$, and increases if $\nu > \frac{1}{6}$, applies to any remote uniform loading. The matrix energy is also increased by the introduction of the sliding interface.

(*a*) *Some comments on energy calculations*

It is instructive to calculate the total strain energy within a large circle of radius R around the sliding inclusion. This is

$$E_\mathrm{S}^R = \frac{1}{2}\int_0^{2\pi} (\sigma_r^\mathrm{M} u_r^\mathrm{M} + \sigma_{r\theta}^\mathrm{M} u_\theta^\mathrm{M}) R\,\mathrm{d}\theta, \tag{4.7}$$

where σ_r^M and $\sigma_{r\theta}^\mathrm{M}$ are given in Appendix B by equations (B 5) and (B 7), and u_r^M and u_θ^M are given by equations (B 10) and (B 11), with $r = R$. After integration, letting $R \to \infty$, it follows that

$$E_\mathrm{S}^R - E_\mathrm{B}^R = \frac{1-2\nu}{16\mu}(\sigma_x^0 - \sigma_y^0)^2 a^2\pi. \tag{4.8}$$

Evidently, this is different from the total strain energy change ΔE, given by equation (4.5). Physically, $E_\mathrm{S}^R - E_\mathrm{B}^R$ does not capture the total strain-energy change, because there is a difference between the strain energies left in the medium behind the radius R, even in the limit as $R \to \infty$. However, if we use a solution for the problem of an inclusion within a finite concentric annulus, loaded over its external boundary of radius R by tractions associated with uniform stress state, the difference $E_\mathrm{S}^R - E_\mathrm{B}^R$ is, in the limit as $R \to \infty$, the total energy change due to shear stress relaxation at the sliding interface. This was pointed out to us by Professor David Barnett of Stanford University, who also indicated an analogy with the calculations of the energy release for the Griffith crack, or the void in an infinite medium under remote loading (Sih & Liebowitz 1967). We performed the calculations and confirmed it.

Since there is no material behind the radius R in the case of an inclusion within a concentric annulus, the difference $E^R_{\mathrm{S}} - E^R_{\mathrm{B}}$ is, in the limit as $R \to \infty$, exactly the total energy change ΔE of equation (4.5).

To make further connections between equations (4.5) and (4.8), we may proceed as follows. The energy within a circle of large radius R would be E^R_{B} if the slip was not allowed at the interface. When the slip is introduced, a change in strain energy arises due to the work done by the relaxing shear stress at the interface on the slip discontinuity there. Furthermore, an energy contribution comes from the work done by already applied tractions on additional displacements produced at the remote boundary (R, θ) by the shear stress relaxation at the interface (a, θ). Thus,

$$E^R_{\mathrm{S}} - E^R_{\mathrm{B}} = -\frac{1}{2}\int_0^{2\pi} \sigma^{\mathrm{B}}_{r\theta}\Delta u_\theta a \, \mathrm{d}\theta + \int_0^{2\pi} (\sigma^{\mathrm{B}}_r \delta u_r + \sigma^{\mathrm{B}}_{r\theta}\delta u_\theta) R \, \mathrm{d}\theta, \tag{4.9}$$

where

$$\delta u_r = u^{\mathrm{S}}_r(R,\theta) - u^{\mathrm{B}}_r(R,\theta) \quad \text{and} \quad \delta u_\theta = u^{\mathrm{S}}_\theta(R,\theta) - u^{\mathrm{B}}_\theta(R,\theta).$$

During additional displacements δu_r and δu_θ at the boundary (R, θ), the tractions on it change by

$$\delta\sigma_r = \sigma^{\mathrm{S}}_r - \sigma^{\mathrm{B}}_r \quad \text{and} \quad \delta\sigma_{r\theta} = \sigma^{\mathrm{S}}_{r\theta} - \sigma^{\mathrm{B}}_{r\theta},$$

but the products $\delta\sigma_r \delta u_r$ and $\delta\sigma_{r\theta}\delta u_\theta$ go to zero in the limiting process. As shown in Appendix B, $\delta\sigma \sim a^2/R^2$ and $\delta u \sim a/R$. It can be easily verified that the second integral in equation (4.9) is $(\sigma^0_x - \sigma^0_y)a^2\pi(3-4\nu)/16\mu$, in the limit as $R \to \infty$. Since the first term in equation (4.9) is equal to $-(\sigma^0_x - \sigma^0_y)a^2\pi(1-\nu)/8\mu$, equation (4.9) reduces to equation (4.8).

On the other hand, the nature of the stress and displacement fields for the sliding inclusion within a concentric annulus is such that the second integral in equation (4.9) becomes $(\sigma^0_x - \sigma^0_y)a^2\pi(1-\nu)/4\mu$, in the limit as $R \to \infty$. Adding this to the first term, $-(\sigma^0_x - \sigma^0_y)a^2\pi(1-\nu)/8\mu$, gives the total strain energy change ΔE of equation (4.5). This explains the difference between equations (4.5) and (4.8). Finally, we observe that by moving the second integral in equation (4.9) to the left-hand side, and combining it with the strain energy difference $E^R_{\mathrm{S}} - E^R_{\mathrm{B}}$, we obtain a general expression for the potential energy difference, $\Pi^R_{\mathrm{S}} - \Pi^R_{\mathrm{B}}$, between the problems of sliding and bonded inclusion within a concentric annulus. In view of equation (4.5), this expression confirms the relationship $\Pi^R_{\mathrm{S}} - \Pi^R_{\mathrm{B}} = -(E^R_{\mathrm{S}} - E^R_{\mathrm{B}})$.

5. An inhomogeneity under remote loading

We end the analysis by comparing the energies of circular inhomogeneities with sliding and bonded interfaces. Since a sliding interface is passive under an in-plane hydrostatic load, for the evaluation of the energy difference it suffices to consider only the remote shear loading, σ^0_{xy}. The stress and displacement components for the inhomogeneities with sliding and bonded interfaces are listed in Appendix C. Thus, by substitution into equation (4.2), we obtain the following expression for the strain energy in the sliding inhomogeneity

$$E^{\mathrm{I}}_{\mathrm{S}} = \frac{12(1-\nu_2)^2(3-2\nu_1)}{[5-6\nu_2+(3-2\nu_1)\mu_2/\mu_1]^2}\frac{{\sigma^0_{xy}}^2}{\mu_1}a^2\pi. \tag{5.1}$$

If the inhomogeneity is bonded to the matrix, its energy is

$$E_{\mathrm{B}}^{\mathrm{I}} = \frac{8(1-\nu_2)^2}{[3-4\nu_2+\mu_2/\mu_1]^2} \frac{{\sigma_{xy}^0}^2}{\mu_1} a^2\pi, \tag{5.2}$$

which is independent of ν_1. The energy in the sliding inhomogeneity can be greater or smaller than the energy in the bonded inhomogeneity, depending on the magnitudes of the elastic constants of the two materials. The two energies are equal to each other if

$$(3-4\nu_1)\left(\frac{\mu_2}{\mu_1}\right)^2 + 2\frac{\mu_2}{\mu_1} + \left[\frac{2(5-6\nu_2)^2}{3-2\nu_1} - 3(3-4\nu_2)^2\right] = 0. \tag{5.3}$$

If the matrix and the inclusion are of the same material, i.e. $\mu_2/\mu_1 = 1$, equation (5.3) is satisfied for $\nu_1 = \nu_2 = \frac{1}{6}$, thus confirming the result from § 4. If two materials are both incompressible, the energy in the sliding inhomogeneity is always smaller than in the bonded inhomogeneity, since

$$E_{\mathrm{S}}^{\mathrm{I}} - E_{\mathrm{B}}^{\mathrm{I}} = -\frac{1}{2}\frac{1}{(1+\mu_2/\mu_1)^2} \frac{{\sigma_{xy}^0}^2}{\mu_1} a^2\pi. \tag{5.4}$$

If both materials have Poisson ratio equal to $\frac{1}{3}$, the energy in the sliding inhomogeneity is smaller than in the bonded inhomogeneity, provided that the shear modulus ratio, μ_2/μ_1, is greater than about 0.255.

The energy change $\Delta E = E_{\mathrm{S}}^{\mathrm{T}} - E_{\mathrm{B}}^{\mathrm{T}}$, associated with a transition from the bonded to the sliding inhomogeneity, is equal to the work done by the relaxing shear stress on the displacement discontinuity at the interface. By using the results from Appendix C, this gives

$$\Delta E = \frac{4(1-\nu_2)^2}{5-6\nu_2+(3-2\nu_1)\mu_2/\mu_1} \frac{3-4\nu_1+\mu_1/\mu_2}{3-4\nu_2+\mu_2/\mu_1} \frac{{\sigma_{xy}^0}^2}{\mu_1} a^2\pi. \tag{5.5}$$

If $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$, equation (5.5) reduces to equation (4.5) of § 4, provided that $\sigma_x^0 = -\sigma_y^0$.

The change in the matrix energy between the problems with sliding and bonded inhomogeneities is $E_{\mathrm{S}}^{\mathrm{M}} - E_{\mathrm{B}}^{\mathrm{M}} = E_{\mathrm{B}}^{\mathrm{I}} - E_{\mathrm{S}}^{\mathrm{I}} + \Delta E$. The resulting expression that follows by substitution of equations (5.1), (5.2) and (5.5) is somewhat lengthy, although in the case of incompressible materials it takes a simple form,

$$E_{\mathrm{S}}^{\mathrm{M}} - E_{\mathrm{B}}^{\mathrm{M}} = \frac{2+\mu_1/\mu_2}{(1+\mu_2/\mu_1)^2} \frac{{\sigma_{xy}^0}^2}{2\mu_1} a^2\pi. \tag{5.6}$$

For completeness, we also record the change in strain energy contained within a large circle of radius R around the sliding and bonded inhomogeneity. In the limit as $R \to \infty$, the result is

$$E_{\mathrm{S}}^{R} - E_{\mathrm{B}}^{R} = \frac{8(1-\nu_1)(1-\nu_2)(1-2\nu_2)}{[5-6\nu_2+(3-2\nu_1)\mu_2/\mu_1][3-4\nu_2+\mu_2/\mu_1]} \frac{{\sigma_{xy}^0}^2}{\mu_1} a^2\pi. \tag{5.7}$$

If $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$, equation (5.7) reduces to equation (4.8), in the special case when $\sigma_x^0 = -\sigma_y^0$. For an incompressible matrix, or a rigid inhomogeneity, $E_{\mathrm{S}}^{R} = E_{\mathrm{B}}^{R}$.

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Appendix A. Stress and displacement fields for a sliding circular inclusion under uniform eigenstrain

The Papkovich–Neuber potentials for the displacement in the sliding inclusion associated with the eigenstrain ϵ_x^* and ϵ_y^* can be taken as

$$\Phi_0 = A_1 r^2 \cos 2\theta, \qquad \Phi_1 = A_2 r^3 \cos 3\theta + A_3 r \cos\theta, \qquad \Phi_2 = A_2 r^3 \sin 3\theta + A_3 r \sin\theta, \tag{A 1}$$

where r and θ denote the polar coordinates. The displacement components are derived from

$$u_x = \frac{\partial}{\partial x}(\Phi_0 + x\Phi_1 + y\Phi_2) - 4(1-\nu)\Phi_1 \mid +\epsilon_x^* x, \tag{A 2}$$

$$u_y = \frac{\partial}{\partial y}(\Phi_0 + x\Phi_1 + y\Phi_2) - 4(1-\nu)\Phi_2 \mid +\epsilon_y^* y. \tag{A 3}$$

The terms $\epsilon_x^* x$ and $\epsilon_y^* y$, appearing to the right of the vertical (|) line, correspond to stress-free eigenstrain, and should not be taken into account when calculating the stresses.

The Papkovich–Neuber potentials for displacements in the matrix are

$$\Phi_0 = B_1 r^{-2} \cos 2\theta, \qquad \Phi_1 = B_2 r^{-1} \cos\theta, \qquad \Phi_2 = B_3 r^{-1} \sin\theta. \tag{A 4}$$

The corresponding displacement components are obtained from (A 2) and (A 3), excluding $\epsilon_x^* x$ and $\epsilon_y^* y$ terms on their right-hand side. The boundary conditions for the sliding inclusion are the vanishing of the shear traction at the interface between the inclusion and the matrix, and the continuity of normal traction and normal displacement at the interface. Thus, at $r = a$

$$\sigma_{r\theta}^{\mathrm{I}} = 0, \qquad \sigma_{r\theta}^{\mathrm{M}} = 0, \qquad \sigma_r^{\mathrm{I}} = \sigma_r^{\mathrm{M}}, \qquad u_r^{\mathrm{I}} = u_r^{\mathrm{M}}. \tag{A 5}$$

The superscript ‘I’ designates the inclusion and ‘M’ the matrix. Upon calculation, we obtained

$$A_1 = -\tfrac{3}{8}k(\epsilon_x^* - \epsilon_y^*), \qquad A_2 = \tfrac{1}{8}k(\epsilon_x^* - \epsilon_y^*)a^{-2}, \qquad A_3 = \tfrac{1}{2}k(\epsilon_x^* + \epsilon_y^*), \tag{A 6}$$

and

$$B_1 = \tfrac{1}{8}k(\epsilon_x^* - \epsilon_y^*)a^4, \qquad B_{2,3} = -\tfrac{1}{8}k[8k(\epsilon_x^* + \epsilon_y^*) \pm 3(\epsilon_x^* - \epsilon_y^*)]a^2, \tag{A 7}$$

where $k = 1/4(1-\nu)$. The following displacement and stress components result in polar coordinates. The displacements in the inclusion are

$$u_r = \tfrac{1}{4}k\left[4(\epsilon_x^* + \epsilon_y^*)r + (\epsilon_x^* - \epsilon_y^*)\left(5 - 8\nu + 2\nu\frac{r^2}{a^2}\right) r\cos 2\theta\right], \tag{A 8}$$

$$u_\theta = -\tfrac{1}{4}k(\epsilon_x^* - \epsilon_y^*)\left[5 - 8\nu + (3-2\nu)\frac{r^2}{a^2}\right] r\sin 2\theta, \tag{A 9}$$

and the stresses

$$\sigma_r = -\tfrac{1}{2}k\mu[4(\epsilon_x^* + \epsilon_y^*) + 3(\epsilon_x^* - \epsilon_y^*)\cos 2\theta], \tag{A 10}$$

$$\sigma_\theta = -\tfrac{1}{2}k\mu\left[4(\epsilon_x^* + \epsilon_y^*) - 3(\epsilon_x^* - \epsilon_y^*)\left(1 - 2\frac{r^2}{a^2}\right)\cos 2\theta\right], \tag{A 11}$$

$$\sigma_{r\theta} = \tfrac{3}{2}k\mu(\epsilon_x^* - \epsilon_y^*)\left(1 - \frac{r^2}{a^2}\right)\sin 2\theta. \tag{A 12}$$

The displacement components in the matrix are, similarly,

$$u_r = \tfrac{1}{4}k\frac{a}{r}\left\{4(\epsilon_x^* + \epsilon_y^*)a + (\epsilon_x^* - \epsilon_y^*)\left[6(1-\nu) - \frac{a^2}{r^2}\right]a\cos 2\theta\right\}, \tag{A 13}$$

$$u_\theta = -\tfrac{1}{4}k(\epsilon_x^* - \epsilon_y^*)\frac{a}{r}\left[3(1-2\nu) + \frac{a^2}{r^2}\right]a\sin 2\theta, \tag{A 14}$$

with the corresponding stresses

$$\sigma_r = -\tfrac{1}{2}k\mu\frac{a^2}{r^2}\left[4(\epsilon_x^* + \epsilon_y^*) + 3(\epsilon_x^* - \epsilon_y^*)\left(2 - \frac{a^2}{r^2}\right)\cos 2\theta\right], \tag{A 15}$$

$$\sigma_\theta = \tfrac{1}{2}k\mu\frac{a^2}{r^2}\left[4(\epsilon_x^* + \epsilon_y^*) - 3(\epsilon_x^* - \epsilon_y^*)\frac{a^2}{r^2}\cos 2\theta\right], \tag{A 16}$$

$$\sigma_{r\theta} = -\tfrac{3}{2}k\mu(\epsilon_x^* - \epsilon_y^*)\frac{a^2}{r^2}\left(1 - \frac{a^2}{r^2}\right)\sin 2\theta. \tag{A 17}$$

The stress state at all points of the inclusion at the interface $r = a$ is purely dilatational in the sense $\sigma_r = \sigma_\theta$. A discontinuity in the tangential displacement at the boundary of the inclusion is

$$\Delta u_\theta = u_\theta^{\mathrm{M}}(a,\theta) - u_\theta^{\mathrm{I}}(a,\theta) = \tfrac{1}{4}(\epsilon_x^* - \epsilon_y^*)a\sin 2\theta. \tag{A 18}$$

A discontinuity in the hoop stress across the interface of the sliding inclusion is constant and equal to $\Delta\sigma_\theta = 4k\mu(\epsilon_x^* + \epsilon_y^*)$. This is in contrast to the Eshelby inclusion (with bonded interface), where the hoop stress experiences a variable jump

$$\Delta\sigma_\theta = 4k\mu[\epsilon_x^* + \epsilon_y^* - (\epsilon_x^* - \epsilon_y^*)\cos 2\theta].$$

The difference in the normal stress for the sliding and bonded inclusion at the interface is

$$\sigma_r^{\mathrm{S}} - \sigma_r^{\mathrm{B}} = -(\tfrac{1}{2}k\mu)(\epsilon_x^* - \epsilon_y^*)\cos 2\theta.$$

The term proportional to $(\epsilon_x^* + \epsilon_y^*)$ does not appear because under dilatational eigenstrain, the inclusion with sliding interface behaves as an inclusion with bonded interface.

The stress and displacement components in the inclusion and the matrix, associated with the shear eigenstrain ϵ_{xy}^*, can be obtained from equations (A 8)–(A 17) by the substitution

$$\epsilon_x^* - \epsilon_y^* = 2\epsilon_{xy}^* \quad \text{and} \quad \epsilon_x^* + \epsilon_y^* = 0,$$

and with the replacement of 2θ by $2\theta - \pi/2$. The stress state at all points of the interface in both the inclusion and the matrix is purely dilatational, in the sense $\sigma_r = \sigma_\theta$. There is no discontinuity in the hoop stress, σ_θ, across the interface of the sliding inclusion, in contrast to the inclusion with bonded interface, where the hoop stress experiences a jump of the amount $8k\mu\epsilon_{xy}^*\sin 2\theta$. Also, the normal tractions at the interface are related by $\sigma_r^{\mathrm{S}} = \frac{3}{2}\sigma_r^{\mathrm{B}}$, so that upon removal of the shear traction at the interface of the bonded inclusion, the normal traction there increases by a factor of $\frac{3}{2}$, to preserve the continuity of normal displacement across the interface.

Appendix B. Stress and displacement fields for a sliding circular inclusion under remote uniform loading

The stress field in an infinite body with a sliding circular inclusion due to remote biaxial loading at infinity σ_x^0 and σ_y^0 can be obtained by a superposition consideration as

$$\sigma_{ij} = \sigma_{ij}^0 + (\sigma_{ij}^{\mathrm{S}} - \sigma_{ij}^{\mathrm{B}})_*. \tag{B 1}$$

Here, σ_{ij}^0 is the constant biaxial stress state throughout the infinite medium, while σ_{ij}^{S} and σ_{ij}^{B} denote the stresses from the sliding and bonded inclusion problem, considered in Appendix A, both evaluated by selecting the eigenstrain difference $(\epsilon_x^* - \epsilon_y^*)$, such that $\sigma_x^0 - \sigma_y^0 = -2k\mu(\epsilon_x^* - \epsilon_y^*)$. In this manner, we ensure that the shear traction at the interface is equal to zero, preserving the continuity of the normal displacement and traction there. Consequently, by using equation (B 1) and the results from Appendix A, the stresses in the inclusion are found to be

$$\sigma_r = \tfrac{1}{2}(\sigma_x^0 + \sigma_y^0) + \tfrac{3}{4}(\sigma_x^0 - \sigma_y^0)\cos 2\theta, \tag{B 2}$$

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x^0 + \sigma_y^0) - \tfrac{3}{4}(\sigma_x^0 - \sigma_y^0)\left(1 - 2\frac{r^2}{a^2}\right)\cos 2\theta, \tag{B 3}$$

$$\sigma_{r\theta} = -\tfrac{3}{4}(\sigma_x^0 - \sigma_y^0)\left(1 - \frac{r^2}{a^2}\right)\sin 2\theta. \tag{B 4}$$

The stresses in the surrounding matrix are likewise:

$$\sigma_r = \tfrac{1}{2}(\sigma_x^0 + \sigma_y^0) + \tfrac{1}{4}(\sigma_x^0 - \sigma_y^0)\left(2 - 2\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta, \tag{B 5}$$

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x^0 + \sigma_y^0) - \tfrac{1}{4}(\sigma_x^0 - \sigma_y^0)\left(2 + 3\frac{a^4}{r^4}\right)\cos 2\theta, \tag{B 6}$$

$$\sigma_{r\theta} = -\tfrac{1}{4}(\sigma_x^0 - \sigma_y^0)\left(2 + \frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta. \tag{B 7}$$

A discontinuity in the hoop stress, when crossing from the inclusion to the matrix, is $-2(\sigma_x^0 - \sigma_y^0)\cos 2\theta$.

The displacement components in the inclusion can also be obtained by superposition. They are

$$u_r = \frac{1}{8\mu}\left[2(1-2\nu)(\sigma_x^0 + \sigma_y^0)r + (\sigma_x^0 - \sigma_y^0)\left(3 - 2\nu\frac{r^2}{a^2}\right)r\cos 2\theta\right], \tag{B 8}$$

$$u_\theta = -\frac{1}{8\mu}(\sigma_x^0 - \sigma_y^0)\left[3 - (3-2\nu)\frac{r^2}{a^2}\right]r\sin 2\theta, \tag{B 9}$$

and in the matrix

$$u_r = \frac{1}{8\mu}\left\{2(1-2\nu)(\sigma_x^0 + \sigma_y^0)r + (\sigma_x^0 - \sigma_y^0)\left[2\frac{r}{a} + 2(1-\nu)\frac{a}{r} - \frac{a^3}{r^3}\right]a\cos 2\theta\right\}, \tag{B 10}$$

$$u_\theta = -\frac{1}{8\mu}(\sigma_x^0 - \sigma_y^0)\left[2\frac{r}{a} + (1-2\nu)\frac{a}{r} + \frac{a^3}{r^3}\right]a\sin 2\theta. \tag{B 11}$$

The discontinuity in the tangential displacement across the interface is

$$\Delta u_\theta = -\frac{1-\nu}{2\mu}(\sigma_x^0 - \sigma_y^0)a\sin 2\theta. \tag{B 12}$$

The problem of a sliding inclusion under remote biaxial loading at infinity can also be solved directly, without using a superposition procedure. The functions of complex variables and the corresponding complex potentials can be used, as given for uniaxial loading by Muskhelishvili (1953, p. 226), or again the Papkovich–Neuber displacement potentials. These are given by equations (A 1) and (A 4), with added contributions to potentials Φ_1 and Φ_2, given, for convenience, in both the inclusion and the matrix, by

$$\Phi_1 = -\frac{1}{4\mu(1-2\nu)}[\sigma_x^0 - \nu(\sigma_x^0 + \sigma_y^0)]x, \qquad \Phi_2 = -\frac{1}{4\mu(1-2\nu)}[\sigma_y^0 - \nu(\sigma_x^0 + \sigma_y^0)]y. \tag{B 13}$$

These contributions ensure the proper behaviour at infinity, i.e. the applied biaxial stress state. The boundary conditions for the remaining constants are the vanishing of the shear traction, and the continuity of normal traction and normal displacement at the interface, which gives

$$A_1 = \frac{1}{16\mu}(\sigma_x^0 - \sigma_y^0), \qquad A_2 = -\frac{1}{16\mu}(\sigma_x^0 - \sigma_y^0)a^{-2}, \qquad A_3 = 0, \tag{B 14}$$

and

$$B_1 = \frac{1}{16\mu}(\sigma_x^0 - \sigma_y^0)a^4, \qquad B_2 = -B_3 = -\frac{1}{16\mu}(\sigma_x^0 - \sigma_y^0)a^2. \tag{B 15}$$

The constants are independent of the in-plane remote mean stress $\frac{1}{2}(\sigma_x^0 + \sigma_y^0)$, since under in-plane hydrostatic loading, the material does not feel a sliding interface (passive interface). The straightforward calculations confirm the stress and displacement expressions, already derived by using a superposition procedure.

The stresses for the sliding inclusion due to shear loading at infinity σ_{xy}^0, can be derived from equations (B 2)–(B 11) by the substitution $\sigma_x^0 - \sigma_y^0 = 2\sigma_{xy}^0$, $\sigma_x^0 + \sigma_y^0 = 0$, and with the replacement of 2θ by $2\theta - \pi/2$. A discontinuity in the hoop stress, when crossing from the inclusion to the matrix, is $-4\sigma_{xy}^0 \sin 2\theta$, and is entirely due to the contribution from the Eshelby part of the solution.

Appendix C. Stress and displacement fields for a inhomogeneity under remote shear loading

The solution based on the complex potentials can be obtained by superposition from the uniaxial loading solution given by Muskhelishvili (1953). Alternatively, the Papkovich–Neuber potentials can be used, which, for the inhomogeneity problem, can be taken in the same form as for the inclusion problem, with appropriately adjusted constants. The following compact forms of the displacement and stress expressions are thus obtained. In the case of a sliding interface, the displacement components in the inhomogeneity are

$$u_r = \frac{2(1-\nu_2)}{\alpha}\frac{\sigma_{xy}^0}{\mu_1}\left(3 - 2\nu_1\frac{r^2}{a^2}\right)r\sin 2\theta, \tag{C 1}$$

$$u_\theta = \frac{2(1-\nu_2)}{\alpha}\frac{\sigma_{xy}^0}{\mu_1}\left[3 - (3-2\nu_1)\frac{r^2}{a^2}\right]r\cos 2\theta, \tag{C 2}$$

and the stresses are

$$\sigma_r = \frac{12(1-\nu_2)}{\alpha}\sigma^0_{xy}\sin 2\theta, \tag{C 3}$$

$$\sigma_\theta = -\frac{12(1-\nu_2)}{\alpha}\sigma^0_{xy}\left(1-2\frac{r^2}{a^2}\right)\sin 2\theta, \tag{C 4}$$

$$\sigma_{r\theta} = \frac{12(1-\nu_2)}{\alpha}\sigma^0_{xy}\left(1-\frac{r^2}{a^2}\right)\cos 2\theta. \tag{C 5}$$

The displacements in the surrounding matrix are likewise:

$$u_r = \frac{\sigma^0_{xy}}{2\mu_2}\left[\frac{r}{a} - 4(1-\nu_2)b\frac{a}{r} - 4c\frac{a^3}{r^3}\right]a\sin 2\theta, \tag{C 6}$$

$$u_\theta = \frac{\sigma^0_{xy}}{2\mu_2}\left[\frac{r}{a} - 2(1-2\nu_2)b\frac{a}{r} + 4c\frac{a^3}{r^3}\right]a\cos 2\theta, \tag{C 7}$$

with the corresponding stresses

$$\sigma_r = \sigma^0_{xy}\left[1+4\left(b\frac{a^2}{r^2}+3c\frac{a^4}{r^4}\right)\right]\sin 2\theta, \tag{C 8}$$

$$\sigma_\theta = -\sigma^0_{xy}\left(1+12c\frac{a^4}{r^4}\right)\sin 2\theta, \tag{C 9}$$

$$\sigma_{r\theta} = \sigma^0_{xy}\left[1-2\left(b\frac{a^2}{r^2}+6c\frac{a^4}{r^4}\right)\right]\cos 2\theta. \tag{C 10}$$

The subscript 1 indicates the inhomogeneity and 2 the matrix. The introduced constants are

$$b = 6\frac{1-\nu_2}{\alpha} - 1, \qquad c = \tfrac{1}{4} - \frac{1-\nu_2}{\alpha} \tag{C 11}$$

and

$$\alpha = 5 - 6\nu_2 + (3-2\nu_1)\frac{\mu_2}{\mu_1}. \tag{C 12}$$

A discontinuity in the hoop stress across the sliding interface is $-4\sigma^0_{xy}\sin 2\theta$, independent of the material properties. In particular, if the inhomogeneity is a void ($\mu_1 = 0$), the stress concentration factor of four is recovered.

The displacement components in the bonded inhomogeneity are

$$u_r = \tfrac{1}{2}(1+\beta)\frac{\sigma^0_{xy}}{\mu_1}r\sin 2\theta, \qquad u_\theta = \tfrac{1}{2}(1+\beta)\frac{\sigma^0_{xy}}{\mu_1}r\cos 2\theta, \tag{C 13}$$

and the stresses

$$\sigma_r = (1+\beta)\sigma^0_{xy}\sin 2\theta, \qquad \sigma_\theta = -(1+\beta)\sigma^0_{xy}\sin 2\theta, \qquad \sigma_{r\theta} = (1+\beta)\sigma^0_{xy}\cos 2\theta. \tag{C 14}$$

The displacements in the surrounding matrix are likewise:

$$u_r = \frac{\sigma^0_{xy}}{2\mu_2}\left\{\frac{r}{a} - \beta\left[4(1-\nu_2)\frac{a}{r} - \frac{a^3}{r^3}\right]\right\}a\sin 2\theta, \tag{C 15}$$

$$u_\theta = \frac{\sigma^0_{xy}}{2\mu_2}\left\{\frac{r}{a} - \beta\left[2(1-2\nu_2)\frac{a}{r} + \frac{a^3}{r^3}\right]\right\}a\cos 2\theta, \tag{C 16}$$

with the corresponding stresses

$$\sigma_r = \sigma^0_{xy}\left[1 + \beta\left(4\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\right]\sin 2\theta, \tag{C 17}$$

$$\sigma_\theta = -\sigma^0_{xy}\left(1 - 3\beta\frac{a^4}{r^4}\right)\sin 2\theta, \tag{C 18}$$

$$\sigma_{r\theta} = \sigma^0_{xy}\left[1 - \beta\left(2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\right]\cos 2\theta. \tag{C 19}$$

The parameter β is defined by

$$\beta = \frac{1 - \mu_2/\mu_1}{3 - 4\nu_2 + \mu_2/\mu_1}. \tag{C 20}$$

A discontinuity in the hoop stress across the bonded interface is $4\beta\sigma^0_{xy}\sin 2\theta$. If the inhomogeneity is a void, $\beta = -1$. For a rigid inhomogeneity, $\beta = (3 - 4\nu_2)^{-1}$.

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