On conservation integrals in micropolar elasticity

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ABSTRACT

Two conservation laws of nonlinear micropolar elasticity ($J_k = 0$ and $L_k = 0$) are derived within the framework of Noether's theorem on invariant variational principles, thereby extending the earlier authors' results from the couple stress elasticity. Two non-conserved M-type integrals of linear micropolar elasticity are then derived and their values discussed. The comparison with related work is also given.

§ 1. INTRODUCTION

Three conservation integrals of infinitesimal non-polar elasticity (J_k, L_k, and M) were derived by employing Noether's (1918) theorem on variational principles invariant under a group of infinitesimal transformations by Günther (1962), and Knowles and Sternberg (1972). When evaluated over a closed surface which does not embrace any singularity, these integrals give rise to conservation laws $J_k = 0$, $L_k = 0$, and $M = 0$. The law $J_k = 0$ applies to anisotropic nonlinear material, the law $L_k = 0$ to isotropic nonlinear material, and $M = 0$ to anisotropic linear material. If the surface embraces a singularity or inhomogeneity (defect), Eshelby (1951,1956) has shown that the value of J_k is not equal to zero but represents a configurational or energetic force on the embraced defect (vacancy, inclusion, dislocation). The path-independent J integral of plane fracture mechanics, independently introduced by Rice (1968), has proved to be of great practical importance in modern fracture mechanics, allowing the prediction of the behavior at the crack tip from the values of the remote field quantities.

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Budiansky and Rice (1973) interpreted the L_k and M integrals as the energetic forces (potential energy release rates) conjugate to rotation (by erosion/addition of material) and self-similar expansion (erosion) of the traction-free void. Freund (1978) used the M conservation law for certain plane elastic crack problems to calculate the elastic stress intensity factor without solving the corresponding boundary value problem. The reference to other related work can be found in the papers by Eshelby (1975) and Rice (1985). Noether's theorem was further applied by Fletcher (1975) to obtain a class of conservation laws for linear elastodynamics. Jarić (1979) and Vukobrat and Jarić (1981) studied the conservation laws in thermoelasticity and linear theory of elastic dialectrics, and Vukobrat (1989) and Vukobrat and Kuzmanović (1992) in micropolar and nonlocal elastodynamics. Yang and Batra (1995) and Huang and Batra (1996) used Noether's theorem to derive the conservation laws and energy-momentum tensors for piezoelectric materials and nonsimple dialectrics. Pucci and Saccomandi (1990) applied a version of Noether's theorem to deduce the conservation laws of micropolar elasticity, but their analysis was unnecessarily restricted to linear constitutive equations. Nikitin and and Zubov (1998) derived the conservation laws for the Cosserat continuum under finite deformations. Lubarda and Markencoff (2000) modified a non-polar analysis of Knowles and Sternberg (*op. cit.*) and derived the conservation laws $J_k = 0$ and $L_k = 0$ for the couple stress elasticity. The results are here extended to the more general framework of the nonlinear micropolar elasticity, in which the local rotation of material elements is not constrained as in the couple stress elasticity, but independent of the displacement field. The derived conservation laws correspond to infinitesimal invariance of the strain energy relative to translational and rotational transformations of the position coordinates, and the displacement and rotation fields. It is then shown that the quadratic strain energy is not infinitesimally invariant under a self-similar scale change, which prevents the existence of the M conservation law in linear micropolar elasticity ($M \neq 0$). Two non-conserved M-type integrals are derived and their values discussed. The comparison with related work is also given.

§ 2. Basic Equations of Micropolar Elasticity

In a micropolar continuum the deformation is described by the displacement vector and an independent rotation vector. The rotation vector specifies the orientation of the triad of director vectors attached to each material particle, which are thus geometrically characterized by both their position and orientation. An infinitesimal material surface element transmits a force and a couple vector, which give rise to non-symmetric stress and couple stress tensors. The former is related to a non-symmetric strain tensor, and the latter to a non-symmetric curvature tensor, defined as the gradient of the rotation vector. This type of the continuum mechanics was originally introduced by Voigt (1887) and the brothers Cosserat (1909), and later further developed by Günther (1958), Grioli (1960), Aero and Kuvshinskii (1960), Mindlin (1964), and Eringen and Suhubi (1964). An extensive list of additional contributions can be found in the review article by Dhaliwal and Singh (1987). The physical rational for the extension of the classical nonpolar to micropolar elasticity was that the classical theory could not predict the size effect experimentally observed in the problems in which there is a geometrical length scale comparable to material's microstructural length, such as the grain size in a polycrystalline or granular material. For example, the apparent strength of some materials with stress concentrators such as holes and notches is higher for smaller grain size; for a given volume fraction of dispersed hard particles, the strengthening of metals is greater for smaller particles; the bending and torsional strengths are higher for very thin beams and wires (Mindlin 1963, Muki and Sternberg 1965, Kaloni and Ariman 1967, Fleck, Muller, Ashby and Hutchinson 1994). The classical theory was also in disagreement with experiments for highfrequency ultra-short wave propagation problems, when the wave length becomes comparable to the material's microstructural length (Mindlin 1964, Brulin and Hsieh 1982). The research in micropolar and related non-local and strain-gradient theories of material response (both elastic and plastic) has intensified during the last decade, largely because of an increasing interest to describe the deformation mechanisms and manufacturing of micro and nanostructured materials and devices (Fleck and Hutchinson 1997, Valiev, Islamgaliev and Alexandrov 2000), and inelastic localization and instability phenomena (Zbib and Aifantis 1989, De Borst and Van der Giessen 1998).

A brief review of the governing equations of infinitesimal, geometrically linear micropolar elasticity is as follows. The deformation is described by the displacement vector u_i and an independent rotation vector φ_i, which are both functions of the position vector x_i. A surface element $\mathrm{d}S$ transmits a force vector $T_i\,\mathrm{d}S$ and a couple vector $M_i\,\mathrm{d}S$. The surface forces are in equilibrium with the non-symmetric Cauchy stress t_{ij}, and the surface couples are in equilibrium with the non-symmetric couple stress m_{ij}, such that

$$T_i = n_j t_{ji}\,, \quad M_i = n_j m_{ji}\,, \tag{1}$$

where n_j are the components of the unit vector orthogonal to the surface element under consideration. In the absence of body forces and body couples, the integral forms of the force and moment equilibrium conditions are

$$\int_S T_i\,\mathrm{d}S = 0\,, \quad \int_S (e_{ijk} x_j T_k + M_i)\,\mathrm{d}S = 0\,. \tag{2}$$

The skew-symmetric alternating tensor is e_{ijk}. Upon using Eq. (1) and the Gauss divergence theorem, Eq. (2) yields the differential equations of equilibrium

$$t_{ji,j} = 0\,, \quad m_{ji,j} + e_{ijk} t_{jk} = 0\,. \tag{3}$$

For elastic deformations of micropolar continuum, the increase of the strain energy is due to external work done by the surface forces and couples, i.e.,

$$\int_V \dot{W}\,\mathrm{d}V = \int_S (T_i\dot{u}_i + M_i\dot{\varphi}_i)\,\mathrm{d}S\,. \tag{4}$$

The strain energy per unit volume is W, and the superposed dot denotes the time derivative. Incorporating Eqs. (1) and (3) and using the divergence theorem gives

$$\dot{W} = t_{ij}\dot{\gamma}_{ij} + m_{ij}\dot{\kappa}_{ij}\,, \tag{5}$$

where

$$\gamma_{ij} = u_{j,i} - e_{ijk}\varphi_k\,, \quad \kappa_{ij} = \varphi_{j,i} \tag{6}$$

are the non-symmetric strain and curvature tensors, respectively (e.g., Mindlin 1964, Nowacki 1986). The symmetric and anti-symmetric parts of γ_{ij} are

$$\begin{aligned} \gamma_{(ij)} &= \epsilon_{ij} = \frac{1}{2}\,(u_{j,i} + u_{i,j})\,, \\ \gamma_{<ij>} &= \omega_{ij} - e_{ijk}\varphi_k\,, \quad \omega_{ij} = \frac{1}{2}\,(u_{j,i} - u_{i,j})\,. \end{aligned} \tag{7}$$

In general, both ϵ_{kk} and κ_{kk} are different from zero. In addition, there is an identity $\kappa_{ij,k} = \kappa_{kj,i} = \varphi_{j,ik}$. Assuming that the strain energy is a function of the strain and curvature tensors, $W = W(\gamma_{ij}, \kappa_{ij})$, the differentiation and the comparison with Eq. (5) establishes the constitutive relations of micropolar elasticity

$$t_{ij} = \frac{\partial W}{\partial \gamma_{ij}}\,, \quad m_{ij} = \frac{\partial W}{\partial \kappa_{ij}}\,. \tag{8}$$

In the case of material linearity, the strain energy is a quadratic function of the strain and curvature components

$$W = \frac{1}{2}\,C_{ijkl}\,\gamma_{ij}\gamma_{kl} + \frac{1}{2}\,K_{ijkl}\,\kappa_{ij}\kappa_{kl}\,. \tag{9}$$

The fourth-order tensors of micropolar elastic moduli are C_{ijkl} and K_{ijkl}. Dimensionally, $[K_{ijkl}] = l^2\,[C_{ijkl}]$, where l is a material length parameter, characteristic of a particular micropolar material. Since the strain and curvature tensors are not symmetric, only reciprocal symmetries hold $C_{ijkl} = C_{klij}$ and $K_{ijkl} = K_{klij}$. The stresses associated with Eq. (9) are $t_{ij} = C_{ijkl}\,\gamma_{kl}$ and $m_{ij} = K_{ijkl}\,\kappa_{kl}$. In the case of isotropic micropolar elasticity, we have

$$\begin{aligned} C_{ijkl} &= (\mu + \bar{\mu})\,\delta_{ik}\,\delta_{jl} + (\mu - \bar{\mu})\,\delta_{il}\,\delta_{jk} + \lambda\,\delta_{ij}\,\delta_{kl}\,, \\ K_{ijkl} &= (\alpha + \bar{\alpha})\,\delta_{ik}\,\delta_{jl} + (\alpha - \bar{\alpha})\,\delta_{il}\,\delta_{jk} + \beta\,\delta_{ij}\,\delta_{kl}\,, \end{aligned} \tag{10}$$

where $\mu, \bar{\mu}, \lambda$ and $\alpha, \bar{\alpha}, \beta$ are the Lamé-type constants of isotropic micropolar elasticity. The symmetric and anti-symmetric parts of the stress tensors are in this case

$$\begin{aligned} t_{(ij)} &= 2\mu\,\epsilon_{ij} + \lambda\,\epsilon_{kk}\,\delta_{ij}\,, \qquad t_{<ij>} = 2\bar{\mu}\,(\omega_{ij} - e_{ijk}\,\varphi_k)\,, \\ m_{(ij)} &= 2\alpha\,\kappa_{(ij)} + \beta\,\kappa_{kk}\,\delta_{ij}\,, \quad m_{<ij>} = 2\bar{\alpha}\,\kappa_{<ij>}\,. \end{aligned} \tag{11}$$

More generally, suppose that the elastic strain energy of a nonlinear isotropic material is given by

$$W = W\left(I_\gamma, II_\gamma, \bar{II}_\gamma, III_\gamma, I_\kappa, II_\kappa, \bar{II}_\kappa, III_\kappa\right), \tag{12}$$

where

$$I_\gamma = \gamma_{kk}\,, \quad II_\gamma = \gamma_{ij}\,\gamma_{ij}\,, \quad \bar{II}_\gamma = \gamma_{ij}\,\gamma_{ji}\,, \quad III_\gamma = \frac{1}{6}\,e_{ijk}\,e_{lmn}\,\gamma_{il}\,\gamma_{jm}\,\gamma_{kn}\,, \tag{13}$$

and similarly for the first, second, and third-order invariants of the curvature tensor κ_{ij}. It follows that

$$\begin{aligned} t_{ij} &= c_1\,\delta_{ij} + c_2\,\gamma_{ij} + \bar{c}_2\,\gamma_{ji} + c_3\,e_{ikl}\,e_{jmn}\,\gamma_{km}\,\gamma_{ln}\,, \\ m_{ij} &= k_1\,\delta_{ij} + k_2\,\kappa_{ij} + \bar{k}_2\,\kappa_{ji} + k_3\,e_{ikl}\,e_{jmn}\,\kappa_{km}\,\kappa_{ln}\,, \end{aligned} \tag{14}$$

with

$$\begin{aligned} c_1 &= \frac{\partial W}{\partial I_\gamma}\,, \quad c_2 = 2\,\frac{\partial W}{\partial II_\gamma}\,, \quad \bar{c}_2 = 2\,\frac{\partial W}{\partial \bar{II}_\gamma}\,, \quad c_3 = \frac{1}{2}\,\frac{\partial W}{\partial III_\gamma}\,, \\ k_1 &= \frac{\partial W}{\partial I_\kappa}\,, \quad k_2 = 2\,\frac{\partial W}{\partial II_\kappa}\,, \quad \bar{k}_2 = 2\,\frac{\partial W}{\partial \bar{II}_\kappa}\,, \quad k_3 = \frac{1}{2}\,\frac{\partial W}{\partial III_\kappa}\,. \end{aligned} \tag{15}$$

In a simplified micropolar theory, the so-called couple stress theory (Toupin 1962, Mindlin and Tiersten 1962), the rotation vector φ_i is not independent of the displacement vector u_i, but related to it through the classical elasticity expression

$$\varphi_i = \frac{1}{2}\, e_{ijk}\, \omega_{jk} = \frac{1}{2}\, e_{ijk}\, u_{k,j}\,, \quad \omega_{ij} = e_{ijk}\, \varphi_k\,. \tag{16}$$

In this case the strain tensor is a symmetric tensor $(\gamma_{ij} = \epsilon_{ij})$, and the curvature tensor is a deviatoric tensor $(\kappa_{kk} = 0)$. A spherical part of the couple stress m_{ij} does not appear in any of the basic field equations of couple stress theory, and without loss of physical generality it may be assumed to vanish (Koiter 1964). The antisymmetric part of the stress tensor $t_{<ij>}$ is indeterminate by the constitutive analysis, but from the moment equilibrium equation it is determined as $t_{<ij>} = -e_{ijk}\, m_{lk,l}/2$. An expose of the finite-deformation micropolar elasticity can be found in Stojanović (1970) and Eringen (1968,1999).

§ 3. NOETHER'S THEOREM OF MICROPOLAR ELASTICITY

Stimulated by the paper of Knowles and Sternberg (1972), there has been a significant interest in the application of Noether's theorem to a variety of solid mechanics problems (e.g., Fletcher 1975, Golebiewska–Herrmann 1981, Olver 1984, Bui and Proix 1984, Maugin 1990, Honein and Herrmann 1997, Aparicio 2000). The original Noether's (1918) theorem on invariant variational principles states in essence that there is a conservation law for the Euler–Lagrange differential equations associated with each infinitesimal symmetry group of the Lagrangian functional. A comprehensive treatment of the general and various restricted forms of Noether's theorem, with a historical outline, can be found in the book by Olver (1986). In the sequel, we apply the formulation of Knowles and Sternberg (*op. cit.*) from the non-polar elasticity, and Lubarda and Markenscoff (2000) from the couple stress elasticity to derive the conservation laws for the nonlinear micropolar elasticity. Consider a family of coordinate mappings defined by a vector-valued function

$$\hat{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \eta)\,, \quad \eta \in (-\eta_*, \eta_*)\,, \tag{17}$$

such that $\mathbf{f}(\mathbf{x}, 0) = \mathbf{x}$ for all position vectors $\mathbf{x}$. Consider also the families of the displacement and rotation mappings

$$\hat{\mathbf{u}} = \mathbf{g}(\mathbf{u}, \eta)\,, \quad \hat{\boldsymbol{\varphi}} = \mathbf{h}(\boldsymbol{\varphi}, \eta)\,, \tag{18}$$

such that $\mathbf{g}(\mathbf{u}, 0) = \mathbf{u}$ and $\mathbf{h}(\boldsymbol{\varphi}, 0) = \boldsymbol{\varphi}$ for all displacement and rotation vectors $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{x})$. Finally, introduce a one-parameter family of functionals

$$E_\eta = \int_{\hat{V}} W(\hat{\gamma}_{ij}, \hat{\kappa}_{ij})\,\mathrm{d}\hat{V}\,, \tag{19}$$

where

$$\hat{\gamma}_{ij} = \frac{\partial \hat{u}_j}{\partial \hat{x}_i} - e_{ijk}\,\hat{\varphi}_k\,, \quad \hat{\kappa}_{ij} = \frac{\partial \hat{\varphi}_j}{\partial \hat{x}_i}\,, \tag{20}$$

and

$$\mathrm{d}\hat{V} = \det\left(\frac{\partial \hat{x}_j}{\partial x_i}\right)\mathrm{d}V = \det(f_{j,i})\,\mathrm{d}V\,. \tag{21}$$

When the parameter η is equal to zero, we have

$$E_0 = E = \int_V W(\gamma_{ij}, , \kappa_{ij})\,\mathrm{d}V\,, \tag{22}$$

which is the total strain energy within the volume V. The family E_η is, therefore, the family of functionals induced from the functional E by the families of mappings **f**, **g**, and **h**.

Definition: The functional E is considered to be invariant at $(\mathbf{u}, \boldsymbol{\varphi})$ with respect to **f**, **g**, and **h**, if

$$E_\eta = E\,, \quad \eta \in (-\eta_*, \eta_*)\,, \tag{23}$$

and infinitesimally invariant if

$$\left(\frac{\partial E_\eta}{\partial \eta}\right)_{\eta=0} = 0\,. \tag{24}$$

Theorem: If $\mathbf{u}$ and $\boldsymbol{\varphi}$ satisfy the equilibrium equations

$$\frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial \gamma_{ji}}\right) = 0\,, \quad \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial \kappa_{ji}}\right) + e_{ijk}\, t_{jk} = 0\,, \tag{25}$$

for all $\mathbf{x}$ in V, then the total strain energy E is infinitesimally invariant at $(\mathbf{u}, \boldsymbol{\varphi})$ with respect to mappings $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$, if and only if

$$\frac{\partial}{\partial x_i}\left(a_i W + b_j t_{ij} + c_j m_{ij}\right) = 0\,, \tag{26}$$

where

$$a_i = f_i'(\mathbf{x}, 0)\,, \tag{27}$$

$$b_i = g_i'(\mathbf{u}, 0) - f_k'(\mathbf{x}, 0)\, u_{i,k}\,, \tag{28}$$

$$c_i = h_i'(\boldsymbol{\varphi}, 0) - f_k'(\mathbf{x}, 0)\, \varphi_{i,k}\,. \tag{29}$$

The prime designates the derivative with respect to the parameter η, such that

$$f_i'(\mathbf{x}, 0) = \left[\frac{\partial}{\partial \eta}\, f_i(\mathbf{x}, \eta)\right]_0\,. \tag{30}$$

For brevity, the subscript 0 is used to indicate that the quantity within the brackets is evaluated at $\eta = 0$. The condition (26) implies the conservation law in the integral form

$$\int_S (a_i n_i W + T_i b_i + M_i c_i)\, \mathrm{d}S = 0\,, \tag{31}$$

for every surface S bounding a regular subregion of V.

Proof: By differentiating Eq. (15) with respect to η and then setting $\eta = 0$, there follows

$$\left(\frac{\partial E_\eta}{\partial \eta}\right)_0 = \int_V \left[W f_{k,k}'(\mathbf{x}, 0) + \frac{\partial W}{\partial \gamma_{ij}}\left(\frac{\partial \hat{\gamma}_{ij}}{\partial \eta}\right)_0 + \frac{\partial W}{\partial \kappa_{ij}}\left(\frac{\partial \hat{\kappa}_{ij}}{\partial \eta}\right)_0\right] \mathrm{d}V\,. \tag{32}$$

The partial derivatives with respect to η appearing in Eq. (32) can be evaluated by using Eqs. (20) and (21). This gives

$$\left(\frac{\partial \hat{\gamma}_{ij}}{\partial \eta}\right)_0 = \frac{\partial g_j'(\mathbf{u}, 0)}{\partial u_k}\, u_{k,i} - f_{k,i}'(\mathbf{x}, 0)\, u_{j,k} - e_{ijk}\, h_k'(\boldsymbol{\varphi}, 0)\,, \tag{33}$$

$$\left(\frac{\partial \hat{\kappa}_{ij}}{\partial \eta}\right)_0 = \frac{\partial h'_j(\boldsymbol{\varphi}, 0)}{\partial \varphi_k}\,\kappa_{ik} - f'_{k,i}(\mathbf{x}, 0)\,\kappa_{kj}\,, \tag{34}$$

$$\left[\frac{\partial(\mathrm{d}\hat{V})}{\partial \eta}\right]_0 = f'_{k,k}(\mathbf{x}, 0)\,\mathrm{d}V\,. \tag{35}$$

The integrand in Eq. (32) is continuous on V, so that the integral vanishes if and only if its integrand vanishes at each $\mathbf{x}$. The leading term of the integrand can be eliminated by using the identity

$$\frac{\partial}{\partial x_k}\left[W f'_k(\mathbf{x}, 0)\right] = W f'_{k,k}(\mathbf{x}, 0) + f'_k(\mathbf{x}, 0)\left(\frac{\partial W}{\partial \gamma_{ij}}\,\gamma_{ij,k} + \frac{\partial W}{\partial \kappa_{ij}}\,\kappa_{ij,k}\right). \tag{36}$$

Accordingly, the integrand in Eq. (32) becomes

$$\frac{\partial}{\partial x_k}\left[W f'_k(\mathbf{x}, 0)\right] + D_{ij}\,\frac{\partial W}{\partial \gamma_{ij}} + d_{ij}\,\frac{\partial W}{\partial \kappa_{ij}} = 0\,, \tag{37}$$

where

$$D_{ij} = \left(\frac{\partial \hat{\gamma}_{ij}}{\partial \eta}\right)_0 - f'_k(\mathbf{x}, 0)\,\gamma_{ij,k}\,, \tag{38}$$

$$d_{ij} = \left(\frac{\partial \hat{\kappa}_{ij}}{\partial \eta}\right)_0 - f'_k(\mathbf{x}, 0)\,\kappa_{ij,k}\,. \tag{39}$$

Introducing the vectors b_i and c_i, defined by Eqs. (28) and (29), it can be readily verified that

$$b_{j,i} = D_{ij} + e_{ijk}\,c_k\,, \quad c_{j,i} = d_{ij}\,. \tag{40}$$

Thus, there is an identity

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(\frac{\partial W}{\partial \gamma_{ij}}\,b_j + \frac{\partial W}{\partial \kappa_{ij}}\,c_j\right) &= \frac{\partial W}{\partial \gamma_{ij}}\,D_{ij} + \frac{\partial W}{\partial \kappa_{ij}}\,d_{ij} \\ &+ \frac{\partial W}{\partial \gamma_{ij}}\,e_{ijk}\,c_k + \frac{\partial}{\partial x_i}\left(\frac{\partial W}{\partial \gamma_{ij}}\right)b_j + \frac{\partial}{\partial x_i}\left(\frac{\partial W}{\partial \kappa_{ij}}\right)c_j\,.\end{aligned} \tag{41}$$

In view of the equilibrium equations (25), the last two terms on the right-hand side of Eq. (41) are together equal to $-c_j\,e_{jkl}\,t_{kl}$, so that

$$\frac{\partial W}{\partial \gamma_{ij}}\,e_{ijk}\,c_k - c_j\,e_{jkl}\,t_{kl} = 0\,. \tag{42}$$

Consequently, Eq. (41) reduces to

$$\frac{\partial}{\partial x_i}\,(t_{ij}\,b_j + m_{ij}\,c_j) = \frac{\partial W}{\partial \gamma_{ij}}\,D_{ij} + \frac{\partial W}{\partial \kappa_{ij}}\,d_{ij}\,. \tag{43}$$

Substituting Eq. (43) into Eq. (37) gives the desired result of Eq. (26). The conservation law (31) follows by applying the Gauss divergence theorem. The Knowles and Sternberg (1972) proof for infinitesimal non-polar elasticity follows by taking $W = W(\epsilon_{ij})$, and by setting $m_{ij} = 0$ and $t_{<ij>} = 0$.

The formulation and the proof of Noether's type theorem for micropolar elasticity presented here can be compared with the results of Lubarda and Markenscoff (1999,2000) for the couple stress elasticity, in which there is a rotation constraint given by Eq. (16). Since

$$\hat{\varphi}_i = \frac{1}{2}\,e_{ijk}\,\frac{\partial \hat{u}_k}{\partial \hat{x}_j} = \frac{1}{2}\,e_{ijk}\,\frac{\partial \hat{u}_k}{\partial u_m}\,\frac{\partial u_m}{\partial x_n}\,\frac{\partial x_n}{\partial \hat{x}_j}\,, \tag{44}$$

there follows

$$h_i'(\boldsymbol{\varphi},0) = \frac{1}{2}\,e_{ijk}\left(\frac{\partial g_k'(\mathbf{u},0)}{\partial u_l}\,u_{l,j} - f_{l,j}'(\mathbf{x},0)\,u_{k,l}\right). \tag{45}$$

Using this, in conjunction with Eqs. (18) and (19) of Lubarda and Markenscoff (2000), it can be readily shown that their condition (16) ($d_{ij}\tau_{ji} = e_{ijk}\,c_i\tau_{jk}$) is identically satisfied. This was not explicitly demonstrated in their paper with this generality, although it was proven in each particular case there considered.

§ 4. CONSERVATION INTEGRALS IN MICROPOLAR ELASTICITY

The strain energy E in micropolar elasticity is invariant under the mappings

$$\hat{x}_i = x_i^0\,\eta + Q_{ij}(\eta)\,x_j\,, \quad \hat{u}_i = Q_{ij}(\eta)\,u_j\,, \quad \hat{\varphi}_i = Q_{ij}(\eta)\,\varphi_j\,, \tag{46}$$

where x_i^0 is a constant vector, and $Q_{ij}(\eta)$ is an orthogonal tensor in the case of an isotropic material, and $Q_{ij} = \delta_{ij}$ (Kronecker delta) in the case of a fully anisotropic material. Thus, since the invariance necessarily implies an infinitesimal invariance, the corresponding conservation laws follow from Eq. (31). Indeed, we have

$$f_i'(\mathbf{x},0) = x_i^0 + q_{ij}\,x_j\,, \quad g_i'(\mathbf{u},0) = q_{ij}\,u_j\,, \quad h_i'(\boldsymbol{\varphi},0) = q_{ij}\,\varphi_j\,, \tag{47}$$

and

$$a_i = x_i^0 + q_{ij}\,x_j\,, \tag{48}$$

$$b_i = q_{ij}\,u_j - (x_m^0 + q_{mn}\,x_n)\,u_{i,m}\,, \tag{49}$$

$$c_i = q_{ij}\,\varphi_j - (x_m^0 + q_{mn}\,x_n)\,\varphi_{i,m}\,, \tag{50}$$

where $q_{ij} = Q_{ij}(0)$. When this is substituted into Eq. (31), we obtain

$$\begin{aligned} & x_i^0 \int_S (W\,n_i - T_i\,u_{i,j} - M_i\,\varphi_{i,j})\,\mathrm{d}S \\ & \quad + q_{ij} \int_S (W\,n_i\,x_j + T_i\,u_j + M_i\,\varphi_j - T_l\,u_{l,i}\,x_j - M_l\,\varphi_{l,i}\,x_j)\,\mathrm{d}S = 0\,. \end{aligned} \tag{51}$$

For a fully anisotropic material $q_{ij} = 0$, and by choosing the vector x_i^0 to be a unit vector in the direction k ($x_i^0 = \delta_{ik}$, for each value of $k = 1,2,3$), Eq. (51) gives

$$J_k = \int_S (W\,n_k - T_k\,u_{k,j} - M_k\,\varphi_{k,j})\,\mathrm{d}S = 0\,. \tag{52}$$

For an isotropic material q_{ij} are the components of an orthogonal tensor, and we can take $q_{ij} = e_{ijk}$ for $k = 1,2,3$, so that, in addition to (52), there is a conservation law

$$L_k = e_{ijk} \int_S (W\,n_i\,x_j + T_i\,u_j + M_i\,\varphi_j - T_l\,u_{l,i}\,x_j - M_l\,\varphi_{l,i}\,x_j)\,\mathrm{d}S = 0\,. \tag{53}$$

If the micropolar terms are omitted, the above conservation laws reduce to those of the classical non-polar elasticity (Knowles and Sternberg 1972, Budiansky and Rice 1973). The micropolar conservation law (52) was earlier derived without referral to Noether's theorem by Dai (1986) and Jarić (1986). Pucci and Saccomandi (1990) deduced (52) and (53) from Noether's theorem, but unnecessarily restricted their considerations to linear isotropic materials in the case of (52), and linear materials in the case of (53). See also a Corollary 4.30 of Olver (1986). Conservation laws for the Cosserat continuum under finite deformations were derived by Nikitin and Zubov (1998).

Introducing the Eshelby's (1970,1975) energy-momentum tensor of micropolar elastic field,

$$P_{ij} = W\,\delta_{ij} - t_{ik}\,u_{k,j} - m_{ik}\,\varphi_{k,j}\,, \tag{54}$$

the derived conservation integrals can be recast as

$$J_k = \int_S P_{jk}\,n_j\,\mathrm{d}S\,, \tag{55}$$

$$L_k = e_{ijk}\int_S (P_{li}\,x_j + t_{li}\,u_j + m_{li}\,\varphi_j)\,n_l\,\mathrm{d}S\,. \tag{56}$$

§ 5. Conservation Laws of Plane-Strain Micropolar Elasticity

In two-dimensional plane-strain problems within (x_1, x_2) plane, the components φ_3, M_3, m_{13}, and m_{23} are generally different from zero, while other rotation, moment, and couple stress components are equal to zero. By taking S to be a cylindrical surface with its generatrix parallel to x_3 axis and with its two flat bases bounded by a curve C, integration in Eqs. (55) and (56) gives (per unit length in x_3 direction)

$$J_\alpha = \int_C P_{\beta\alpha}\,n_\beta\,\mathrm{d}C\,, \tag{57}$$

$$L = e_{\alpha\beta 3}\int_S (P_{\gamma\alpha}\,x_\beta + t_{\gamma\alpha}\,u_\beta)\,n_\gamma\,\mathrm{d}C\,. \tag{58}$$

The energy-momentum tensor of the plane-strain micropolar elasticity is

$$P_{\alpha\beta} = W\,\delta_{\alpha\beta} - t_{\alpha\gamma}\,u_{\gamma,\beta} - m_{\alpha 3}\,\varphi_{3,\beta}\,. \tag{59}$$

The summation in repeated Greek indices is over 1 and 2. The J_1 integral from Eq. (57) was used by Atkinson and Leppington (1974) to calculate the energy release rate for a semi-infinite crack within a strip of thickness h. Xia and Hutchinson (1996) also used the J_1 integral to study the elastoplastic crack tip field in a strain-gradient dependent material described by the deformation-type theory of plasticity.

§ 6. M INTEGRAL OF MICROPOLAR ELASTICITY

In contrast to classical elasticity, there is no M conservation law of micropolar elasticity. In two-dimensional case this was originally observed by Atkinson and Leppington (1977), and later discussed in the three-dimensional context by Pucci and Saccomandi (1990), and for the couple stress elasticity by Lubarda and Markenscoff (2000). To elaborate, consider a family of scale-changes

$$\hat{x}_i = (1+\eta)x_i\,, \quad \hat{u}_i = \left(1 - \frac{\eta}{2}\right)u_i\,. \tag{60}$$

It is easily verified that the total strain energy of a non-polar elastic material with a quadratic strain energy representation is infinitesimally invariant under (60). This is, however, not so in the case of micropolar elasticity, because the material length parameter, whose square is the ratio of the representative micropolar elastic moduli $l^2 = [K]/[C]$, remains unaltered by the transformation (60). Indeed, since the angle change corresponding to (60) is

$$\hat{\varphi}_i = \frac{1-\eta/2}{1+\eta}\,\varphi_i\,, \tag{61}$$

which follows from a simple dimensional argument ($u \sim x\varphi$, $\hat{u} \sim \hat{x}\hat{\varphi}$), we have

$$\hat{\gamma}_{ij} = \frac{1-\eta/2}{1+\eta}\,\gamma_{ij}\,, \quad \hat{\kappa}_{ij} = \frac{1-\eta/2}{(1+\eta)^2}\,\kappa_{ij}\,. \tag{62}$$

We assume here that $\eta_* < 1$. Thus, for a linear micropolar material with a quadratic strain energy representation,

$$E_\eta = \frac{1}{2}\,\left(1 - \frac{\eta}{2}\right)^2 \int_V \left[(1+\eta)\,t_{ij}\,\gamma_{ij} + \frac{1}{1+\eta}\,m_{ij}\,\kappa_{ij}\right]\mathrm{d}V\,, \tag{63}$$

so that $E_\eta \neq E_0 = E$, and

$$\left(\frac{\partial E_\eta}{\partial \eta}\right)_0 = -\int_V m_{ij}\,\kappa_{ij}\,\mathrm{d}V \neq 0\,. \tag{64}$$

This shows that the total strain energy E is not infinitesimally invariant with respect to the considered family of scale-changes. Consequently, there is no M conservation law in micropolar elasticity. Actually, the value of the M integral is equal to the expression in Eq. (64). This follows from

$$f_i'(\mathbf{x},0) = x_i\,, \quad g_i'(\mathbf{u},0) = -\frac{1}{2}\,u_i\,, \quad h_i'(\boldsymbol{\varphi},0) = -\frac{3}{2}\,\varphi_i\,, \tag{65}$$

and

$$M = \int_S \left(W\,x_i\,n_i - \frac{1}{2}\,T_i\,u_i - \frac{3}{2}\,M_i\,\varphi_i - T_i\,x_j\,u_{i,j} - M_i\,x_j\,\varphi_{i,j}\right)\mathrm{d}S\,. \tag{66}$$

Upon using the Gauss divergence theorem, the evaluation of the last integral gives

$$M = -\int_V m_{ij}\,\kappa_{ij}\,\mathrm{d}V\,. \tag{67}$$

In the derivation it should be observed that

$$W_{,k} = t_{ij}\,\gamma_{ij,k} + m_{ij}\,\kappa_{ij,k}\,, \tag{68}$$

and that, for the quadratic strain energy representation, the identities hold

$$t_{ij}\,\gamma_{ij,k} = t_{ij,k}\,\gamma_{ij}\,, \quad m_{ij}\,\kappa_{ij,k} = m_{ij,k}\,\kappa_{ij}\,. \tag{69}$$

In terms of the energy-momentum tensor (54), the M integral of Eq. (66) can be rewritten as

$$M = \int_S \left(P_{ij}\,x_j - \frac{1}{2}\,t_{ij}\,u_j - \frac{3}{2}\,m_{ij}\,\varphi_j\right)n_i\,\mathrm{d}S\,. \tag{70}$$

The plane-strain counterpart is

$$M = \int_C (P_{\alpha\beta}\,x_\beta - m_{\alpha 3}\,\varphi_3)\,n_\alpha\,\mathrm{d}C\,. \tag{71}$$

If the polar effects are neglected, the couple stress vanishes and the conservation law $M = 0$ of the classical linear isotropic elasticity is recovered. Its applications in two-dimensional fracture mechanics were explored by Freund (1978), Kubo (1982), Lubarda (1993), and others.

§ 7. CONCLUSIONS

We have derived in this paper two conservation laws of micropolar elasticity ($J_k = 0$ and $L_k = 0,\ k = 1, 2, 3$) by using the framework of Noether's theorem on invariant variational principles. These laws can also be proven independently of Noether's theorem by direct evaluation of the considered integrals. For example, in evaluating the L_k integral one first applies the divergence theorem, and incorporates the equilibrium conditions and the expression

$$W_{,k} = t_{ij}\, u_{j,ik} + m_{ij}\, \varphi_{j,ik} - e_{ijl}\, t_{ij}\, \varphi_{l,k}\ . \tag{72}$$

Since,

$$e_{uij}\,(t_{vi}\,\gamma_{vj} - t_{jv}\,\gamma_{iv}) = 0\,, \quad e_{uij}\,(m_{vi}\,\kappa_{vj} - m_{jv}\,\kappa_{iv}) = 0\,, \tag{73}$$

because the tensors within the brackets are symmetric in (i, j) by Eq. (11) (in the case of material linearity), after a straightforward but lengthy derivation it follows that $L_k = 0$. In the case of material nonlinearity, there is an extra term in the stress expression (14) proportional to $e_{ikl}\, e_{jmn}\, \gamma_{km}\, \gamma_{ln}$, but its contribution to either $e_{uij}\, t_{vi}\, \gamma_{vj}$ or $e_{uij}\, t_{jv}\, \gamma_{iv}$ vanishes (similarly for the couple stress and curvature terms). For instance,

$$e_{uij}\, t_{vi}\, \gamma_{vj} = 2\, e_{vkl}\, \gamma_{vj}\, \gamma_{kj}\, \gamma_{lu}\ . \tag{74}$$

Since the alternating tensor is antisymmetric, the right-hand side vanishes because $\gamma_{vj}\, \gamma_{kj}$ is symmetric in (v, k). An analogous derivation proceeds in evaluating the integrals appearing in the definitions of J_k and M. Noether's theorem was, however, of fundamental importance in arriving at the proper representation of the considered integrals in terms of the kinematic and kinetic quantities that appear in the structure of micropolar elasticity (u_i, φ_i, T_i, M_i, and W). In the less general context of the couple stress elasticity this was already discussed by Lubarda and

Markenscoff (*op. cit.*). For example, another non-conserved integral of micropolar elasticity can be introduced as

$$N = M + \int_S M_i\, \varphi_i \, \mathrm{d}S\,. \tag{75}$$

The evaluation of this integral gives

$$N = -\int_V t_{ij}\, e_{ijk}\, \varphi_k \, \mathrm{d}V = -\int_V t_{ij}\,(\gamma_{<ij>} - \omega_{ij})\, \mathrm{d}V\,. \tag{76}$$

In the couple stress elasticity this simplifies because $\gamma_{ij} = \epsilon_{ij}$, and $\gamma_{<ij>} = 0$. The plane-strain counterpart of (76) is

$$N = \int_C P_{\alpha\beta}\, x_\beta\, n_\alpha \, \mathrm{d}C\,, \tag{77}$$

with the energy-momentum tensor $P_{\alpha\beta}$ defined by Eq. (59). The N integral was used by Atkinson and Leppington (1977) to show that the energy release rate for the finite crack in an infinite medium under remote tension reduces from its polar to non-polar value when the micropolar parameter tends to zero. The energy release rate decreases as the material length scale increases relative to the crack length. For an analysis of the fractal cracks in micropolar elastic solids, a recent paper by Yavari, Sarkani and Moyer (2002) can be consulted.

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