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Second-order elastic analysis of dilute distribution of spherical inclusions

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Abstract

The volume change and the strain energy stored in an isotropic elastic matrix by a disperse substitution of spherical inclusions are determined by a non-linear analysis which accounts for the second-order elastic effects. When reduced to linear theory, the results are compared with those of Eshelby.

Keywords: Inclusions; Dilute distribution; Elasticity; Second-order effects; Strain energy

1. Introduction

A second-order elastic analysis, in which the strain energy is assumed to be a cubic polynomial in strain invariants, has been successfully applied in the past to study material response under high pressure, the Poynting effect, material behavior in the localized regions of severe deformation, and other problems (Murnaghan, 1951; Seeger and Mann, 1959; Seeger and Buck, 1960; Toupin and Bernstein, 1961; Gschneidner and Vineyard, 1962 etc.). There has also been a significant amount of research devoted to the evaluation of effective elastic properties of non-linear composites and pollycrystalline materials (e.g., Ogden, 1974; Hashin, 1985; Chen and Jiang, 1993; Imam et al., 1995; Lubarda, 1997, 1998). In the present paper we develop a second-order elastic analysis of a dilute distribution of spherical inclusions, by extending the corresponding linear analysis of Eshelby (1956). We first derive the volume change associated with a disperse substitution of spherical inclusions in an isotropic matrix, and then derive an expression for the total elastic energy stored in the composite. When second-order effects are ommitted, the results are compared with those derived by Eshelby in his study of substitutional atoms and binary alloys. A mistake in his final expression for the energy of an alloy is indicated.

2. Spherical inclusion in second-order elasticity

A solid spherical inclusion inserted into a spherical hole of a matrix material is a classic elasticity problem of importance for various applications. For example, it has been often considered as a simplified elastic model of a

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substitutional or interstitial atom (e.g., Friedel, 1954; Eshelby, 1956). If the size of an inclusion is only slightly different from a size of the hole, linear elasticity can be employed to determine the stress and deformation fields in the inclusion and the matrix. For larger differences in size, non-linear effects become important. In the present analysis, it is assumed that the material response is elastic, and that the non-linear effects are accounted for by taking the strain energy (per unit initial volume) to be a cubic polynomial in the strain invariants (second-order elasticity). For an isotropic solid, this is (Murnaghan, 1951)

$$\psi = \frac{3\kappa + 4\mu}{6} I^2 - 2\mu II + \frac{l + 2m}{3} I^3 - 2mI \cdot II + nIII, \tag{1}$$

where κ and μ are the second-order bulk and shear moduli, while l, m and n are the third-order elastic moduli. The three invariants of the Lagrangian strain, $\mathbf{E} = (\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})/2$, $\mathbf{F}$ being the deformation gradient and $\mathbf{I}$ the second-order unit tensor, are:

$$I = \mathbf{E}\colon\mathbf{I}, \qquad II = \frac{1}{2}(I^2 - \mathbf{E}\colon\mathbf{E}),$$
$$III = \frac{1}{3}(-I^3 + 3I \cdot II + \mathbf{E}^2\colon\mathbf{E}). \tag{2}$$

The symmetric Piola–Kirchhoff stress $\mathbf{S}$ is the gradient of ψ with respect to $\mathbf{E}$, which yields a quadratic relationship between $\mathbf{S}$ and $\mathbf{E}$ given by

$$\mathbf{S} = [(\kappa - 2\mu/3)I + lI^2 - (2m - n)II]\mathbf{I} + [2\mu + (2m - n)I]\mathbf{E} + n\mathbf{E}^2. \tag{3}$$

In the absence of body forces, equilibrium equations are the conditions for a divergence-free non-symmetric Piola–Kirchhoff stress tensor $\mathbf{P} = \mathbf{S}\mathbf{F}^{\mathrm{T}}$. In a problem with spherical symmetry, the equilibrium equations reduce to a single equation

$$\frac{\mathrm{d}P_r}{\mathrm{d}r} + \frac{2}{r}(P_r - P_\theta) = 0, \tag{4}$$

written with respect to spherical coordinates (r, θ, ϕ) in the undeformed configuration. The only non-vanishing displacement component is the radial displacement $u = u(r)$, so that the deformation gradient becomes $\mathbf{F} = \mathbf{I} + \boldsymbol{\epsilon}$, and the Lagrangian strain $\mathbf{E} = \boldsymbol{\epsilon} + \boldsymbol{\epsilon}^2/2$. The components of the strain matrix $\boldsymbol{\epsilon}$ are $\epsilon_r = \mathrm{d}u/\mathrm{d}r$, and $\epsilon_\theta = \epsilon_\phi = u/r$. Substitution into Eq. (4) yields a differential equation for u, which is highly non-linear and difficult to solve analytically. Murnaghan (1951) accordingly used a systematic approximation procedure to simplify the analysis, and for a pressurized spherical hole in an infinite matrix, the following expression for the displacement is obtained,

$$u_1(r) = \left[\frac{p}{4\mu_1} + (a_1 - \alpha_1)\left(\frac{p}{4\mu_1}\right)^2\right]\frac{R_1^3}{r^2} + \alpha_1\left(\frac{p}{4\mu_1}\right)^2\frac{R_1^6}{r^5}. \tag{5}$$

The applied pressure is p, and the initial radius of the hole is R_1. The introduced parameters are:

$$a_1 = \frac{11\mu_1 + n_1}{4\mu_1}, \qquad \alpha_1 = 1 + \frac{\mu_1 + 2m_1}{\gamma_1\kappa_1}, \tag{6}$$

$$\gamma_1 = 1 + \frac{4\mu_1}{3\kappa_1}. \tag{7}$$

The subscript 1 indicates the matrix material. Note that the final expression for a radial displacement, given by Murnaghan (1951) at the bottom of p. 124, has a printing error, since the term proportional to R_1^6/r^5 is there missing. The points on the surface of the hole are displaced by

$$u_1(R_1) = \left[\frac{p}{4\mu_1} + a_1\left(\frac{p}{4\mu_1}\right)^2\right]R_1. \tag{8}$$

On the other hand, the stress and deformation in a solid sphere of material 2, under pressure p, are uniform and associated with the displacement field (Murnaghan, 1951)

$$u_2(r) = -\left[\frac{p}{3\kappa_2} + a_2\left(\frac{p}{3\kappa_2}\right)^2\right]r,$$
$$a_2 = \frac{9l_2 + n_2}{3\kappa_2} - \frac{1}{2}. \tag{9}$$

If a spherical inclusion of radius R_2 is inserted into a spherical hole of radius R_1, within an infinite matrix material, the required pressure for insertion of the inclusion is obtained from the misfit condition $u_1(R_1) - u_2(R_2) = \Delta$, where $\Delta = R_2 - R_1$ is the misfit between the inclusion and the hole. In view of Eqs. (8) and (9), this gives the misfit-pressure relationship

$$\frac{\Delta}{\gamma R_1} = \frac{p}{4\mu_1} + b\left(\frac{p}{4\mu_1}\right)^2, \tag{10}$$

where

$$b = 1 + \frac{1}{\gamma}[a_1 - 1 + (a_2 + 1)(\gamma - 1)^2]. \tag{11}$$

The parameter γ is

$$\gamma = 1 + \frac{4\mu_1}{3\kappa_2}. \tag{12}$$

By inverting Eq. (10), the pressure can be expressed in terms of the misfit as

$$\frac{p}{4\mu_1} = \frac{\Delta}{\gamma R_1} - b\left(\frac{\Delta}{\gamma R_1}\right)^2, \tag{13}$$

to second-order terms in Δ/R_1. If the misfit is sufficiently small, this reduces to the well-known linear elasticity result, $p = 4\mu_1\Delta/R_1\gamma$.

The displacement field outside the inclusion can be expressed in terms of Δ/R_1 by substituting Eq. (13) into Eq. (5). The result is

$$u_1(r) = \left[\frac{\Delta}{\gamma R_1} + (a_1 - \alpha_1 - b)\left(\frac{\Delta}{\gamma R_1}\right)^2\right]\frac{R_1^3}{r^2} + \alpha_1\left(\frac{\Delta}{\gamma R_1}\right)^2\frac{R_1^6}{r^5}, \tag{14}$$

to second-order terms in Δ/R_1. The displacement in the inclusion is similarly

$$u_2(r) = -(\gamma - 1)\left\{\frac{\Delta}{\gamma R_1} + [a_2(\gamma - 1) - b]\left(\frac{\Delta}{\gamma R_1}\right)^2\right\} r. \tag{15}$$

Sufficiently distant from the inclusion, the term proportional to r^{-5} in Eq. (14) can be neglected, and the displacement there becomes

$$u_1^\infty(r) = \frac{C}{r^2},\quad C = R_1^3\left[\frac{\Delta}{\gamma R_1} + (a_1 - \alpha_1 - b)\left(\frac{\Delta}{\gamma R_1}\right)^2\right]. \tag{16}$$

In the case of linear elasticity, $C = R_1^2\Delta/\gamma$, and everywhere in the matrix material the displacement is $\mathbf{u}_1 = C\mathbf{r}/r^3$. This is a divergence free field, giving no volume strain anywhere in the matrix (Eshelby, 1956).

The radius of the inclusion after its insertion in the matrix is $R = R_2 + u_2(R_2)$. This is also equal to the radius of the hole after insertion of the inclusion, i.e. $R = R_1 + u_1(R_1)$, which gives

$$R = R_1\left\{1 + \left[\frac{\Delta}{\gamma R_1} + (a_1 - b)\left(\frac{\Delta}{\gamma R_1}\right)^2\right]\right\}. \tag{17}$$

If the inclusion is incompressible, so that the second-order bulk modulus κ_2 is infinitely large, the parameter $\gamma = 1$ from Eq. (12), and $a_1 = b$ from Eq. (11).

3. Volume change due to inclusions

Consider a finite matrix of volume V enclosed by the surface S, free of any external load or surface constraint. Remove a small sphere of radius R_1 deep inside the volume V, and replace it by an inclusion of radius $R_2 > R_1$. (An analogous derivation proceeds in the case $R_2 < R_1$.) The external surface S expands, causing an increase ΔV of the volume within S. Eshelby's (1956) analysis for the calculation of the volume change can be applied as follows. The inclusion is first considered to be inserted in an infinite matrix. The points on an imagined internal surface, having the size and shape of S, are far from the inclusion and, thus, have displacements given by Eq. (16). The volume increase within the surface S is accordingly

$$\Delta V^\infty = \int_S \mathbf{u}_1^\infty \cdot \mathbf{n}\, \mathrm{d}S = 4\pi C, \tag{18}$$

where $\mathbf{n}$ is the outward normal to S. This follows by using Eq. (16) and from the Gauss divergence theorem, recalling that for any n and the three-dimensional vector $\mathbf{r}$, $\operatorname{div}(r^n\mathbf{r}) = (n + 3)r^n$. The effects of the second-order elasticity are included in the quadratic term of the expression for the constant C in Eq. (16).

An auxiliary problem is next considered, in which the Cauchy image traction, $\mathbf{t}^{\mathrm{im}} = -\mathbf{n}\cdot\boldsymbol{\sigma}_1^\infty = -2\mu_1\,\mathbf{n}\cdot\boldsymbol{\epsilon}_1^\infty$, is applied over the surface S of the body V. The superposition of two problems makes the total traction over S equal to zero, and thus represents the solution of the original problem. Since the volume of inserted inclusion is small comparing to the matrix volume V, in the image problem we can neglect the fact that the material of the inclusion is different from the surrounding matrix material. Consequently, the volume change due to image traction can be calculated from a linear elasticity formula (Eshelby, 1956)

$$\Delta V^{\mathrm{im}} = \frac{1}{3\kappa_1}\int_S \mathbf{r}\cdot\mathbf{t}^{\mathrm{im}}\,\mathrm{d}S = \frac{4\mu_1}{3\kappa_1}\,\Delta V^{\infty}. \tag{19}$$

The last step follows because the strain $\boldsymbol{\epsilon}^{\infty}$ is traceless, and because the displacement $u_1^{\infty}(r)$ is a homogeneous function of degree −2.

Upon summation of Eqs. (18) and (19), the total volume increase produced by inserted inclusion is

$$\Delta V = 4\pi\gamma_1 C. \tag{20}$$

In the case of small misfit, the term proportional to $(\Delta/R_1)^2$ in Eq. (16) for the constant C can be omitted, and Eq. (20) reduces to $\Delta V = V_{\mathrm{mis}}\gamma_1/\gamma$, where $V_{\mathrm{mis}} = 4\pi R_1^2\Delta$ is the misfit volume. In this case $\Delta V^{\infty} = V_{\mathrm{mis}}/\gamma$ and $\Delta V^{\mathrm{im}} = (\gamma_1 - 1)V_{\mathrm{mis}}/\gamma$, both in accord with Eshelby's linear elasticity results.

In order to calculate the volume change associated with a dilute distribution of N inserted inclusions within a finite volume V, imagine that all inclusions are first inserted into an infinite matrix. Since non-linear elasticity effects are localized to regions around each small inclusion, and assuming these to be distant enough so that regions of non-linearity do not overlap, we can calculate the volume change ΔV^{∞} within the surface S by superposition from Eq. (18), as $\Delta V^{\infty} = 4\pi NC$. The image traction is then applied over the external surface S of the body V, associated with the displacement field of each inserted inclusion. The volume change in the image problem is, therefore, $\Delta V^{\mathrm{im}} = (4\mu_1/3\kappa_1)\Delta V^{\infty}$. Upon the summation, the total volume change produced by all inclusions is

$$\Delta V = 4\pi N\gamma_1 R_1^3\left[\frac{\Delta}{\gamma R_1} + (a_1 - \alpha_1 - b)\left(\frac{\Delta}{\gamma R_1}\right)^2\right]. \tag{21}$$

If the quadratic term in Δ/R_1 is neglected, Eq. (21) reduces to the linear elasticity result of Eshelby (1956), $\Delta V = N(\gamma_1/\gamma)V_{\mathrm{mis}}$.

4. Strain energy of a single inclusion

The strain energy stored in the matrix by insertion of inclusion is equal to the work done by the pressure p over the surface of the hole, i.e.,

$$E_1 = \int_0^{u_1} 4\pi(R_1 + u_1)^2 p\,\mathrm{d}u_1, \tag{22}$$

where $u_1 = u_1(R_1)$ is the displacement of the points on the surface of the hole. Since, by inverting Eq. (8),

$$\frac{p}{4\mu_1} = \frac{u_1}{R_1} - a_1\left(\frac{u_1}{R_1}\right)^2, \tag{23}$$

to second degree in (u_1/R_1), the substitution into Eq. (22) gives, upon integration,

$$E_1 = 6\mu_1 V_1\left[\left(\frac{u_1}{R_1}\right)^2 - \frac{2}{3}(a_1 - 2)\left(\frac{u_1}{R_1}\right)^3\right]. \tag{24}$$

The volume $V_1 = 4\pi R_1^3/3$ is the initial volume of the hole. Expressed in terms of the relative misfit Δ/R_1, the energy in the matrix is

$$E_1 = 6\mu_1 V_1\left\{\left(\frac{\Delta}{\gamma R_1}\right)^2 + 2\left[\frac{2}{3}(a_1 + 1) - b\right]\left(\frac{\Delta}{\gamma R_1}\right)^3\right\}. \tag{25}$$

The strain energy in the inclusion is $E_2 = V_2\psi$, since the inclusion is deformed uniformly. The specific strain energy ψ is calculated from Eq. (1) with the invariants: $I = 3[(u_2/R_2) + (u_2/R_2)^2/2]$, $II = I^2/3$, and $III = I^3/27$, and with the displacement $u_2 = u_2(R_2)$ defined by Eq. (9). This gives

$$E_2 = \frac{9}{2}\kappa_2 V_2\left[\left(\frac{u_2}{R_2}\right)^2 + \frac{2}{3}(a_2 + 2)\left(\frac{u_2}{R_2}\right)^3\right]. \tag{26}$$

Expressed in terms of Δ/R_1, the energy in the inclusion is

$$E_2 = 6\mu_1 V_2(\gamma - 1)\left\{\left(\frac{\Delta}{\gamma R_1}\right)^2 + 2\left[\frac{2}{3}(\gamma - 1)(a_2 - 1) - b\right]\left(\frac{\Delta}{\gamma R_1}\right)^3\right\}. \quad (27)$$

The total strain energy stored in the inclusion and the matrix is $E = E_1 + E_2$. Thus, upon substitution of Eqs. (25) and (27),

$$E = 6\mu_1 V_1\left[\gamma\left(\frac{\Delta}{\gamma R_1}\right)^2 + \varphi\left(\frac{\Delta}{\gamma R_1}\right)^3\right], \quad (28)$$

where

$$\varphi = \gamma - \frac{1}{3}[2a_1 - 1 + (2a_2 + 1)(\gamma - 1)^2]. \quad (29)$$

If the inclusion is incompressible ($\gamma = 1$), the parameter $\varphi = 2(2 - a_1)/3$.

The total energy can be rewritten in terms of the misfit volume $V_{\text{mis}} = 4\pi(R_2^3 - R_1^3)/3$ by using the approximation

$$\frac{\Delta}{R_1} = \frac{1}{3}\left[\frac{V_{\text{mis}}}{V_1} - \frac{1}{3}\left(\frac{V_{\text{mis}}}{V_1}\right)^2\right]. \quad (30)$$

For example, in the case of small misfit and linear elasticity, the total energy is

$$E = \frac{2}{3}\frac{\mu_1}{\gamma}\frac{V_{\text{mis}}^2}{V_1}, \qquad V_{\text{mis}} = 4\pi R_1^2\Delta. \quad (31)$$

The derived energy expressions also apply when a small inclusion is inserted deep inside a large, but finite body. This is because the strain energy due to image tractions, required in the transition from an infinite to a finite body, can be neglected comparing to E, since it is of the order of $(V_1/V)E$, V being a large volume of the body ($V \gg V_1$).

The gradients of the two energies with respect to the corresponding boundary displacements are:

$$\frac{\partial E_1}{\partial u_1} = 4R_1^2\pi p_1^0, \qquad \frac{\partial E_2}{\partial u_2} = -4R_2^2\pi p_2^0, \quad (32)$$

where p_1^0 and p_2^0 are the nominal or first Piola–Kirchhoff pressures on the surface of the hole and the inclusion, i.e.,

$$p_1^0 = 4\mu_1\left[\frac{u_1}{R_1} - (a_1 - 2)\left(\frac{u_1}{R_1}\right)^2\right], \quad (33)$$

$$p_2^0 = -3\kappa_2\left[\frac{u_2}{R_2} + \frac{1}{3}(a_2 + 5)\left(\frac{u_2}{R_2}\right)^2\right]. \quad (34)$$

The derivative of the total strain energy with respect to the radius R of the inclusion, after its insertion into the matrix, is likewise

$$\frac{\partial E}{\partial R} = \frac{\partial E_1}{\partial u_1} + \frac{\partial E_2}{\partial u_2} = 4\pi(R_1^2 p_1^0 - R_2^2 p_2^0), \quad (35)$$

which is equal to zero, since $R_1^2 p_1^0 = R_2^2 p_2^0 = R^2 p$. Therefore, the inclusion will adopt the size in its deformed state which minimizes the total strain energy of the inclusion and the matrix. In the case of linear elasticity, the distinction between the pressures p_1^0 and p_2^0 disappears, and $4\mu_1 u_1 = -3\kappa_2 u_2$. Since $u_1 - u_2 = \Delta$, it follows that $u_1 = \Delta/\gamma$, in agreement with a linearized form of the result following from Eq. (14).

5. Strain energy of two distant inclusions

The strain energy due to one inclusion (say, inclusion a), expressed in terms of the pressure p^a required for insertion of the inclusion into the matrix, is

$$E^a = 6\mu_1 V_1\left[\left(\frac{p^a}{4\mu_1}\right)^2 + \frac{4}{3}(a_1 + 1)\left(\frac{p^a}{4\mu_1}\right)^3\right] + \frac{9}{2}\kappa_2 V_2\left[\left(\frac{p^a}{3\kappa_2}\right)^2 + \frac{4}{3}(a_2 - 1)\left(\frac{p^a}{3\kappa_2}\right)^3\right]. \quad (36)$$

This is obtained by summing Eqs. (24) and (26), and by incorporating the pressure-displacement relationships given by Eqs. (8) and (9). When the second inclusion is inserted into a matrix, far from its boundary and from the inclusion a, the additional strain energy stored in the system can be calculated as the work required for the second insertion. This is

$$E^b = 6\mu_1 V_1\left[\left(\frac{p^b}{4\mu_1}\right)^2 + \frac{4}{3}(a_1 + 1)\left(\frac{p^b}{4\mu_1}\right)^3\right] + \int_{S_b} \mathbf{t}_n^a \cdot \mathbf{u}_1^b \, \mathrm{d}S_b + \frac{9}{2}\kappa_2 V_2\left[\left(\frac{p^b + p_{\text{im}}^a}{3\kappa_2}\right)^2 + \frac{4}{3}(a_2 - 1)\left(\frac{p^b + p_{\text{im}}^a}{3\kappa_2}\right)^3\right]. \quad (37)$$

The contribution given by the integral over S_b in Eq. (37) follows because, when a hole is cut in the matrix to insert the inclusion b, its surface S_b is already under the traction $\mathbf{t}_n^a$ due to the stress field of the previously inserted inclusion a. The work of this traction on the displacement required for the insertion of inclusion b is

$$\int_{S_b} \mathbf{t}_n^a \cdot \mathbf{u}_1^b \, \mathrm{d}S_b = -\sigma_{ij}^a u_1^b \int_{S_b} n_i n_j \, \mathrm{d}S_b = 4\pi R_1^2 p_{\mathrm{im}}^a u_1^b. \quad (38)$$

The last step follows because the integral of $n_i n_j$ is equal to $4\pi R_1^2 \delta_{ij}/3$, and because σ_{ij}^a has a non-deviatoric contribution only from the image stress, i.e., $\sigma_{kk}^a = -p_{\mathrm{im}}^a$, where p_{im}^a is the pressure over S_b due to the image field of the inclusion a. To insert the inclusion, the pressure p^b is added to the surface of the hole S_b, which gives rise to the energy contribution represented by the first term in Eq. (37). In order to insure the continuity of traction between the inclusion b and the matrix, the total pressure applied to the inclusion b is $p^b + p_{\mathrm{im}}^{\mathrm{a}}$. This pressure appears in the last part of the energy in Eq. (37).

The misfit condition for the inclusion b is

$$u_1^b - u_2^b = \Delta + \frac{p_{\mathrm{im}}^a}{3\kappa_1} R_1. \quad (39)$$

The term $R_1 p_{\mathrm{im}}^a / 3\kappa_1$ is included because it represents how much the spherical portion of the matrix material, to be replaced by the inclusion b, has changed its size by the image field of the inclusion a, before the insertion of the inclusion b took place. By inversion of Eqs. (8) and (9), we have:

$$u_1^b = \left[\frac{p^b}{4\mu_1} + a_1 \left(\frac{p^b}{4\mu_1} \right)^2 \right] R_1, \quad (40)$$

$$u_2^b = -\left[\frac{p^b + p_{\mathrm{im}}^a}{3\kappa_2} + a_2 \left(\frac{p^b + p_{\mathrm{im}}^a}{3\kappa_2} \right)^2 \right] R_2, \quad (41)$$

and the substitution into the misfit condition of Eq. (39) gives

$$\frac{\Delta}{\gamma R_1} = \frac{p^b}{4\mu_1} + b \left(\frac{p^b}{4\mu_1} \right)^2 + \left[1 - \frac{\gamma_1}{\gamma} + d_1 \frac{p^b}{4\mu_1} + d_2 \left(\frac{p^b}{4\mu_1} \right)^2 \right] \frac{p_{\mathrm{im}}^a}{4\mu_1}, \quad (42)$$

to within linear terms in the image pressure. The introduced parameters are:

$$d_1 = \frac{\gamma - 1}{\gamma} [2\gamma - \gamma_1 + 2a_2(\gamma - 1)], \quad (43)$$

$$d_2 = \frac{\gamma - 1}{\gamma} [a_1 + a_2(\gamma - 1)(6\gamma - \gamma_1 - 3)]. \quad (44)$$

Solving Eq. (42) for the pressure p^b, we obtain

$$p^b = p^a - \left\{ 1 - \frac{\gamma_1}{\gamma} - \left[2b \left(1 - \frac{\gamma_1}{\gamma} \right) - d_1 \right] \frac{\Delta}{\gamma R_1} + (d_2 - 3bd_1) \left(\frac{\Delta}{\gamma R_1} \right)^2 \right\} p_{\mathrm{im}}^a. \quad (45)$$

Substituting Eq. (45) into Eq. (37) gives, after a lengthy but straightforward derivation, the following expression for the energy stored in the system by insertion of the inclusion b,

$$E^b = E^a + E^{\mathrm{int}}. \quad (46)$$

The energy E^{int} is the interaction energy between the stress and deformation fields of two inclusions, i.e.,

$$E^{\mathrm{int}} = 3V_1 \left[\gamma_1 \left(\frac{\Delta}{\gamma R_1} \right) + \phi \left(\frac{\Delta}{\gamma R_1} \right)^2 \right] p_{\mathrm{im}}^a, \quad (47)$$

where

$$\phi = (a_1 - 1)\left(1 - \frac{\gamma_1}{\gamma} \right) - (a_2 + 1) \frac{\gamma_1}{\gamma} (\gamma - 1)^2 + \gamma_1 - 1. \quad (48)$$

The interaction energy in Eq. (47) is much smaller than the self-energy due to one inclusion, given by Eq. (28), because $p_{\mathrm{im}}^a \ll \mu_1 \Delta / \gamma R_1$, the latter being of the order of the pressure between the matrix and inclusion, which is much greater than p_{im}^a. The expression for the interaction energy is, however, needed because the interaction energy becomes of the same order as the self-energy when a large number of inclusions is substituted in the matrix (see next section). Note also that for an incompressible inclusion, the parameter $\phi = (1-\gamma_1)(a_1-2)$.

Eq. (47) can be rewritten in terms of the misfit volume by using Eq. (30). This yields

$$E^{\rm int} = V_1\left[\frac{\gamma_1}{\gamma}\left(\frac{V_{\rm mis}}{V_1}\right) + \frac{1}{3\gamma^2}(\phi-\gamma\gamma_1)\left(\frac{V_{\rm mis}}{V_1}\right)^2\right] p^a_{\rm im}. \tag{49}$$

The interaction energy can also be expressed in terms of the volume change ΔV given by Eq. (20). The result is

$$E^{\rm int} = V_1\left[\frac{\Delta V}{V_1} + \frac{1}{3\gamma_1^2}\left(\gamma_1 + \frac{6m_1-n_1}{3\kappa_1}\right)\left(\frac{\Delta V}{V_1}\right)^2\right] p^a_{\rm im}. \tag{50}$$

In the case of linear elasticity, Eq. (50) reduces to $E^{\rm int} = p^a_{\rm im}\Delta V$, in agreement with Eq. (8.9) of Eshelby (1956), and confirming that the two distant inclusions interact only through their image fields. A reciprocity property also holds in the linear case: the image pressure at b due to inclusion at a, is equal to the image pressure at a due to inclusion at b.

6. Strain energy of dilute distribution of inclusions

When a large number (N) of inclusions is inserted into the matrix in a dilute manner, the interaction energy among inclusions is of the same order as the self-energy of all inclusions, because a substitution of N inclusions introduces $N(N-1)/2$ interactions among them. Thus, from Eq. (47), the total interaction energy is

$$E^{\rm int} = \frac{3}{2}NV_1\left[\gamma_1\left(\frac{\Delta}{\gamma R_1}\right) + \phi\left(\frac{\Delta}{\gamma R_1}\right)^2\right] p^{\rm im}, \tag{51}$$

where $p^{\rm im}$ is the image pressure at the location of one inclusion due to image fields of the remaining $(N-1)$ inclusions. For a dilute distribution of inclusions throughout the volume V, $p^{\rm im}$ can be calculated from the average dilatation produced by image fields. This gives $p^{\rm im} = -\kappa_1\Delta V^{\rm im}/V$, where $\Delta V^{\rm im} = 16\pi N\mu_1 C/3\kappa_1$, from the analysis in Section 3. Therefore,

$$E^{\rm int} = -8\pi\mu_1 C\frac{V}{V_1}\, x^2\left[\gamma_1\left(\frac{\Delta}{\gamma R_1}\right) + \phi\left(\frac{\Delta}{\gamma R_1}\right)^2\right], \tag{52}$$

where $x = NV_1/V$ is a measure of the volume concentration of inclusions. It is noted that image tractions do not change the volume only, but also the shape of a matrix, and the total interaction energy cannot be calculated simply as $p^{\rm im}\Delta V^{\rm im}/2$.

In view of Eq. (16) for the constant C, Eq. (52) becomes

$$E^{\rm int} = -6\mu_1 Vx^2\left[\gamma_1\left(\frac{\Delta}{\gamma R_1}\right)^2 + \psi\left(\frac{\Delta}{\gamma R_1}\right)^3\right], \tag{53}$$

within cubic terms in (Δ/R_1), where

$$\psi = (a_1-1)\left(1+\gamma_1-\frac{2\gamma_1}{\gamma}\right) - 2(a_2+1)\frac{\gamma_1}{\gamma}(\gamma-1)^2 - \frac{3\gamma_1+1}{4} - \frac{2m_1}{\kappa_1}. \tag{54}$$

Since a dilute distribution of inclusions is considered, $E^{\rm int}$ does not depend on a distance between inclusions, and consequently there are no long-range interaction forces among the inclusions.

The self-energy contribution of all inclusions is N times the self-energy of a single inclusion, given by Eq. (28), i.e.,

$$E^{\rm self} = 6\mu_1 Vx\left[\gamma\left(\frac{\Delta}{\gamma R_1}\right)^2 + \varphi\left(\frac{\Delta}{\gamma R_1}\right)^3\right]. \tag{55}$$

The total strain energy is the sum of the right-hand sides of Eqs. (53) and (55). Thus,

$$E = 6\mu_1 Vx\left[(\gamma-\gamma_1 x)\left(\frac{\Delta}{\gamma R_1}\right)^2 + (\varphi-\psi x)\left(\frac{\Delta}{\gamma R_1}\right)^3\right]. \tag{56}$$

In the case of small misfit, the cubic term in Eq. (56) can be neglected, and the total energy becomes

$$E = \frac{V}{V_1}E_0\, x\left(1-\frac{\gamma_1}{\gamma}x\right), \tag{57}$$

where $E_0 = 6\mu_1 V_1\gamma(\Delta/\gamma R_1)^2$ denotes the strain energy due to only one inserted inclusion. Eq. (57) differs from Eshelby's (1956) result, which contains the coefficient $\gamma_1(\gamma_1-1)/\gamma(\gamma-1)$, rather than γ_1/γ, in his (8.26). That seems to be a mistake, because for incompressible inclusions $\gamma = 1$, and Eshelby's expression would give a negative infinite energy.

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