Pergamon

J. Mech. Phys. Solids, Vol. 42, No. 6, pp. 931–952, 1994

0022-5096/94 $7.00 + 0.00

0022-5096(94)E0008-R

ELASTOPLASTIC CONSTITUTIVE ANALYSIS WITH THE YIELD SURFACE IN STRAIN SPACE

V. A. LUBARDA

Department of Mechanical and Aerospace Engineering, Arizona State University,
Tempe, AZ 85287, U.S.A.

(*Received* 19 *June* 1993; *in revised form* 29 *January* 1994)

ABSTRACT

A COMMON APPROACH used in formulating elastoplastic constitutive equations is to partition the total strain rate into its elastic and plastic parts, and then develop the constitutive expression for the plastic strain rate using the concept of the yield surface in stress space. An alternative approach is to partition the stress rate into its elastic and plastic parts, and then develop the constitutive expression for the plastic stress rate using the concept of the yield surface in strain space. Both of these approaches are used in this paper to derive and compare the final structures of the corresponding constitutive equations. It is shown that the preferable choice of the yield surface may be in either stress or strain space depending on the selected strain and conjugate stress measures utilized to construct the constitutive formulation.

1. INTRODUCTION

THE PROCEDURE that is normally used in developing the elastoplastic constitutive equations involves decomposing the total strain rate into its elastic and plastic parts, and then deriving the constitutive expressions for each part. The elastic part of the strain rate is defined as the reversible part of the total strain rate, in the sense that it gives a strain increment that is recovered after the loading–unloading cycle of the appropriate stress increment. The remaining part of the total strain rate is the plastic (residual) contribution which under certain assumptions is codirectional with the outward normal to the current locally smooth yield surface in the appropriate stress space.

An alternative but rarely utilized approach is to partition the stress rate into its elastic and plastic parts, and then derive the constitutive expressions for each of these parts. The elastic part of the stress rate is defined as the stress rate that would correspond to a prescribed strain rate if the instantaneous material response was purely elastic. The plastic part of the stress rate then gives a residual stress decrement in an infinitesimal loading–unloading strain cycle. As discussed in Section 3, this part of the stress rate under certain assumptions is codirectional with the inward normal to the current locally smooth yield surface in the strain space.

Although the general constitutive framework which employs both of the described approaches has been outlined by HILL (1959, 1967, 1978), IL'YUSHIN (1961) and HILL and RICE (1972, 1973), the second approach has received little attention. It is the purpose of this paper to elaborate on this issue, i.e. to derive explicit representations of the elastoplastic constitutive equations using the stress rate decomposition and the yield surface in strain space. These results are then compared with those of the traditional approach using the strain rate decomposition and the yield surface in stress space. It is shown that the preferable choice of the yield surface may be in either stress or strain space depending on the selected strain and conjugate stress measures utilized to construct the constitutive formulation.

The formulation of elastoplasticity with the yield surface in strain space has been studied during the past two decades by Naghdi and his coworkers, i.e. NAGHDI and TRAPP (1975), CASEY and NAGHDI (1981, 1983, 1984), NAGHDI (1990). The differences in the structure of the loading conditions that arise in the formulations using the yield surface in stress and strain space are examined in the context of hardening, softening and perfectly plastic behavior. The plastic strain is regarded as a primitive variable defined by its rate through an appropriate constitutive equation. The constitutive equation for the plastic strain rate is then constructed in both formulations, i.e. with reference to the yield surface in stress and strain space. A different approach is taken by YODER and IWAN (1981). By restricting their attention to small strains, they obtain the stress response by subtracting the so-called stress relaxation from the stress that would arise elastically from the current strain. Here, the increment of the relaxation stress is assumed to be normal to the introduced relaxation surface in strain space. The relationship with the traditional formulation using the yield surface in stress space is then obtained. Constitutive inequalities and normality properties in elastoplastic analysis with the yield surface in strain space have been studied by LUBLINER (1986). Duality of stress- and strain-based plasticity formulations, as delineated by HILL (1967), is also discussed by NEMAT-NASSER (1992).

The contents of the present paper are as follows. Section 2 contains kinematic and kinetic preliminaries. The finite strain kinematics and the multiplicative decomposition of deformation gradient into its elastic and plastic parts (LEE, 1969) are conveniently utilized. In Section 3 a kinetic basis for the partition of the stress and strain rates is introduced according to the procedure presented by HILL and RICE (1973). These results are then applied to some convenient choices of reference state and corresponding conjugate measures of stress and strain, and their rates. Section 4 gives explicit relationships between introduced elastic and plastic parts of the stress and strain rates, and constituents of the multiplicative decomposition of the deformation gradient. The constitutive equation for the plastic stress rate is derived in Section 5, using the concept of the yield surface in strain space. The overall elastoplastic constitutive equations are derived with an explicit representation of the current elastoplastic stiffness and compliance tensors. For the sake of comparison, the elastoplastic constitutive analysis with reference to the yield surface in stress space is presented in Section 6. Lastly, the discussion and conclusions are given in Section 7.

2. KINEMATIC AND KINETIC PRELIMINARIES

Consider the current elastoplastically deformed configuration of the material sample $\mathscr{B}_t$, whose initial undeformed configuration is $\mathscr{B}_0$. Let $\mathbf{F}$ be the deformation gradient that maps an infinitesimal material element $d\mathbf{X}$ in $\mathscr{B}_0$ to $d\mathbf{x}$ in $\mathscr{B}_t$, i.e. $d\mathbf{x} = \mathbf{F}\,d\mathbf{X}$. Both the initial $\mathbf{X}$ and current $\mathbf{x}$ locations of the material particle are referred to the same fixed set of the rectangular coordinate axis. Let $\mathscr{P}_t$ be the intermediate configuration obtained from $\mathscr{B}_t$ by elastic distressing to zero stress. Such a configuration differs from the initial configuration by a residual (plastic) deformation, and from the current configuration by a reversible (elastic) deformation. By introducing $\mathbf{F}_e$ and $\mathbf{F}_p$ as deformation gradients associated with transformations $\mathscr{P}_t \to \mathscr{B}_t$ and $\mathscr{B}_0 \to \mathscr{P}_t$, respectively, the multiplicative decomposition of deformation gradient follows (LEE, 1969)

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p. \tag{2.1}$$

$\mathbf{F}_e$ and $\mathbf{F}_p$ are customarily called the elastic and plastic parts of the total deformation gradient $\mathbf{F}$. For inhomogeneous deformations only $\mathbf{F}$ is a true deformation gradient, whose components are the partial derivatives $\partial\mathbf{x}/\partial\mathbf{X}$. In contrast, the mappings $\mathscr{P}_t \to \mathscr{B}_t$ and $\mathscr{B}_0 \to \mathscr{P}_t$ are not, in general, continuous one-to-one mappings, so that $\mathbf{F}_e$ and $\mathbf{F}_p$ are not defined as the gradients of the respective mappings (which may not exist), but as the point functions (local deformation gradients). In the case when elastic distressing to zero stress ($\mathscr{B}_t \to \mathscr{P}_t$) is not physically achievable due to the onset of reverse inelastic deformation before the zero stress is reached (which often occurs at advanced stages of deformation due to anisotropic hardening and strong Bauschinger effect), the intermediate configuration can be conceptually introduced by virtual distressing to zero stress, locking all inelastic structural changes that would occur during the actual distressing.

The Lagrangian strains corresponding to deformation gradients $\mathbf{F}_e$ and $\mathbf{F}_p$ are

$$\mathbf{E}_e = \tfrac{1}{2}(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{I}), \quad \mathbf{E}_p = \tfrac{1}{2}(\mathbf{F}_p^T\mathbf{F}_p - \mathbf{I}) \tag{2.2}$$

where $\mathbf{I}$ denotes the second-order identity tensor and $(\ )^T$ the transpose. The total Lagrangian strain can consequently be expressed as

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \mathbf{E}_p + \mathbf{F}_p^T\mathbf{E}_e\mathbf{F}_p. \tag{2.3}$$

The elastic and plastic strain measures $\mathbf{E}_e$ and $\mathbf{E}_p$ do not sum to give the total strain $\mathbf{E}$, because $\mathbf{E}$ and $\mathbf{E}_p$ are defined relative to the initial configuration $\mathscr{B}_0$ as a reference configuration, while $\mathbf{E}_e$ is defined relative to the intermediate configuration $\mathscr{P}_t$ as a reference configuration. Consequently, it is the strain $\mathbf{F}_p^T\mathbf{E}_e\mathbf{F}_p$, induced from elastic strain $\mathbf{E}_e$ by plastic deformation $\mathbf{F}_p$, that sums up with plastic strain $\mathbf{E}_p$ to give the total strain $\mathbf{E}$.

The work conjugate stress to the Lagrangian strain $\mathbf{E}$ is the Piola–Kirchhoff stress

$$\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}. \tag{2.4}$$

In (2.4), $(\ )^{-1}$ designates the inverse, and $\boldsymbol{\tau} = |\mathbf{F}|\boldsymbol{\sigma}$ is the Kirchhoff stress, i.e. the Cauchy stress $\boldsymbol{\sigma}$ multiplied by the determinant of deformation gradient $\mathbf{F}$. We also introduce the stress tensor

$$\mathbf{S}_e = \mathbf{F}_e^{-1}\boldsymbol{\tau}_e\mathbf{F}_e^{-T}, \tag{2.5}$$

where $\boldsymbol{\tau}_e = |\mathbf{F}_e|\boldsymbol{\sigma}$. In what follows the plastic deformation will be assumed to be incompressible, so that $|\mathbf{F}_p| = 1$, and $\boldsymbol{\tau}_e = \boldsymbol{\tau}$.

The deformation gradients $\mathbf{F}_e$ and $\mathbf{F}_p$ are not uniquely defined because arbitrary local material element rotations superposed to the unstressed state give alternate intermediate configurations. However, if material is elastically isotropic and remains such during inelastic deformation (preserving its elastic properties), then the elastic strain energy ψ per unit unstressed volume is an isotropic function of the Lagrangian strain $\mathbf{E}_e$, i.e. $\psi(\mathbf{Q}\mathbf{E}_e\mathbf{Q}^T) = \psi(\mathbf{E}_e)$, where $\mathbf{Q}$ is an orthogonal tensor corresponding to arbitrary rigid-body rotation superposed to the unstressed state. The elastic stress response from $\mathscr{P}_t \to \mathscr{B}_t$ is, therefore, not influenced by the non-uniqueness of intermediate configuration and is given by the well-known finite elasticity expression (TRUESDELL and NOLL, 1965)

$$\mathbf{S}_e = \frac{\partial\psi(\mathbf{E}_e)}{\partial\mathbf{E}_e}. \tag{2.6}$$

For the sake of clarity, the main issues discussed in this paper and the subsequent analysis will be restricted to isotropic elastic behavior. This can also be extended to include anisotropy along the lines presented in other papers on the related subject (LUBARDA, 1991, 1993).

2.1. *Strain and stress rate measures*

Consider the velocity gradient in the current configuration at time t, as defined by $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, where the dot designates the material time derivative. By introducing the multiplicative decomposition (2.1) of the deformation gradient $\mathbf{F}$, the velocity gradient becomes

$$\mathbf{L} = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1} + \mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}. \tag{2.7}$$

The strain rate $\mathbf{D}$ and spin $\mathbf{W}$ are given by the symmetric and antisymmetric parts of $\mathbf{L}$ as

$$\mathbf{D} = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s + [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s \tag{2.8}$$

$$\mathbf{W} = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_a + [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a. \tag{2.9}$$

The following relationships can be easily derived for the rates of the Lagrangian strains

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F} \tag{2.10}$$

$$\dot{\mathbf{E}}_p = \mathbf{F}_p^T(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})_s\mathbf{F}_p \tag{2.11}$$

$$\dot{\mathbf{E}}_e = \mathbf{F}_e^T(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s\mathbf{F}_e = \mathbf{F}_e^T\{\mathbf{D} - [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s\}\mathbf{F}_e. \tag{2.12}$$

The rate of the Piola–Kirchhoff stress (2.4) is

$$\dot{\mathbf{S}} = \mathbf{F}^{-1}\overset{*}{\boldsymbol{\tau}}\mathbf{F}^{-T}, \tag{2.13}$$

where

$$\overset{*}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{L}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{L}^T \tag{2.14}$$

is a convected type stress rate that is observed in the frame that deforms with the material in the current configuration. Furthermore, the rate of stress (2.5) is

$$\dot{\mathbf{S}}_e = \mathbf{F}_e^{-1}\overset{*c}{\boldsymbol{\tau}}\mathbf{F}_e^{-T}. \tag{2.15}$$

The convected derivative $\overset{*c}{\boldsymbol{\tau}}$, in the case of plastic incompressibility, is given by

$$\overset{*c}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})\boldsymbol{\tau} - \boldsymbol{\tau}(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})^T. \tag{2.16}$$

In view of (2.7), (2.16) can be written in terms of the convected stress rate (2.14) as

$$\overset{*c}{\boldsymbol{\tau}} = \overset{*}{\boldsymbol{\tau}} + [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]\boldsymbol{\tau} + \boldsymbol{\tau}[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]^T. \tag{2.17}$$

2.2. *Rate-type elasticity equation*

If the current increment of deformation is purely elastic (as during elastic unloading or loading within the elastic range), one has

$$\dot{\mathbf{E}}_e = \mathbf{F}_e^T\mathbf{D}\mathbf{F}_e, \quad \dot{\mathbf{S}}_e = \mathbf{F}_e^{-1}\overset{*}{\boldsymbol{\tau}}\mathbf{F}_e^{-T}. \tag{2.18}$$

By differentiating (2.6)

$$\dot{\mathbf{S}}_e = \frac{\partial^2\psi}{\partial\mathbf{E}_e \otimes \partial\mathbf{E}_e} : \dot{\mathbf{E}}_e, \tag{2.19}$$

where $\otimes$ denotes the outside tensor product, and (:) denotes the trace product. Using (2.18) with (2.19) gives

$$\mathbf{F}_e^{-1}\overset{*}{\boldsymbol{\tau}}\mathbf{F}_e^{-T} = \boldsymbol{\Lambda}_e : (\mathbf{F}_e^T\mathbf{D}\mathbf{F}_e). \tag{2.20}$$

In (2.20), the fourth-order tensor

$$\boldsymbol{\Lambda}_e = \frac{\partial^2\psi}{\partial\mathbf{E}_e \otimes \partial\mathbf{E}_e} \tag{2.21}$$

is the corresponding instantaneous elastic stiffness tensor. Expression (2.20) can be rewritten as

$$\overset{*}{\boldsymbol{\tau}} = \mathscr{L}_e : \mathbf{D}, \tag{2.22}$$

where

$$\mathscr{L}_e = \mathbf{F}_e\mathbf{F}_e\boldsymbol{\Lambda}_e\mathbf{F}_e^T\mathbf{F}_e^T. \tag{2.23}$$

In the component form (2.23) reads

$$\mathscr{L}^e_{ijkl} = F^e_{im}F^e_{jn}\Lambda^e_{mnpq}F^e_{kp}F^e_{lq}. \tag{2.23'}$$

Note that the instantaneous elastic moduli tensor $\mathscr{L}_e$ is independent of a superposed rotation to the reference state, which follows by inspection of (2.23′), or directly from (2.22), because neither $\mathbf{D}$ nor $\overset{*}{\boldsymbol{\tau}}$ depends on such a rotation. Of course, in general, the tensor $\boldsymbol{\Lambda}_e$ in (2.21) does depend on the rotation of the reference state. Recall that under superposed rotation $\mathbf{Q}$ of the reference state, $\mathbf{F}_e$ changes to $\mathbf{F}_e\mathbf{Q}^T$, while $\mathbf{E}_e$ changes to $\mathbf{Q}\mathbf{E}_e\mathbf{Q}^T$.

By introducing the complementary elastic strain energy by a Legendre transformation $\phi = \mathbf{S}_e : \mathbf{E}_e - \psi$, one has

$$\mathbf{E}_e = \frac{\partial \phi}{\partial \mathbf{S}_e}, \tag{2.24}$$

so that by differentiation

$$\mathbf{F}_e^T\mathbf{D}\mathbf{F}_e = \mathbf{M}_e : (\mathbf{F}_e^{-1}\overset{*}{\boldsymbol{\tau}}\mathbf{F}_e^{-T}). \tag{2.25}$$

In (2.25), the fourth-order tensor

$$\mathbf{M}_e = \frac{\partial^2 \phi}{\partial \mathbf{S}_e \otimes \partial \mathbf{S}_e} \tag{2.26}$$

is the corresponding instantaneous elastic compliance tensor, which is the inverse of the instantaneous elastic stiffness tensor $\boldsymbol{\Lambda}_e$. Expression (2.25) can be rewritten as

$$\mathbf{D} = \mathscr{M}_e : \overset{*}{\boldsymbol{\tau}}, \tag{2.27}$$

where

$$\mathscr{M}_e = \mathbf{F}_e^{-T}\mathbf{F}_e^{-T}\mathbf{M}_e\mathbf{F}_e^{-1}\mathbf{F}_e^{-1} \tag{2.28}$$

is the inverse of the stiffness tensor $\mathscr{L}_e$. In the component form (2.28) reads

$$\mathscr{M}^e_{ijkl} = F^{e-1}_{mi}F^{e-1}_{nj}M^e_{mnpq}F^{e-1}_{pk}F^{e-1}_{ql}. \tag{2.28'}$$

By using the Jaumann corotational derivative of the Kirchhoff stress

$$\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{W} = \overset{*}{\boldsymbol{\tau}} + \mathbf{D}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{D}, \tag{2.29}$$

(2.22) can be alternatively written as

$$\overset{\circ}{\boldsymbol{\tau}} = \hat{\mathscr{L}}_e : \mathbf{D}. \tag{2.30}$$

In (2.30),

$$\hat{\mathscr{L}}_e = \mathscr{L}_e + \boldsymbol{\Sigma}, \tag{2.31}$$

where $\boldsymbol{\Sigma}$ is the fourth-order tensor with the rectangular components

$$\Sigma_{ijkl} = \delta_{ik}\tau_{jl} + \delta_{il}\tau_{jk} + \tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk}. \tag{2.32}$$

The inverse of (2.30) is

$$\mathbf{D} = \hat{\mathscr{M}}_e : \overset{\circ}{\boldsymbol{\tau}}, \quad \hat{\mathscr{M}}_e = \hat{\mathscr{L}}_e^{-1}. \tag{2.33}$$

For small elastic deformations of metals, when the elastic moduli are far greater than the applied stresses, it follows that $\hat{\mathscr{L}}_e \approx \mathscr{L}_e \approx \boldsymbol{\Lambda}_e$, with the components

$$\Lambda^e_{ijkl} \approx \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}, \tag{2.34}$$

where λ and μ are the Lamé elastic constants. The computational advantages of using the rate-type elasticity equation (2.30) or (2.33), associated with a truly hyperelastic response (2.6) or (2.24), are discussed by SIMO and ORTIZ (1985).

3. PARTITIONING OF STRAIN AND STRESS RATES

Following HILL and RICE (1973), let $\mathbf{e}$ be any objective symmetric strain tensor that measures deformation from an arbitrary reference state. Let $\mathbf{t}$ be the symmetric conjugate stress such that the Pfaffian $\mathbf{t}:\mathrm{d}\mathbf{e}$ represents the work per unit volume of the adopted reference state in any deformation increment $\mathrm{d}\mathbf{e}$. Further, let $\boldsymbol{\Lambda}$ and $\mathbf{M}$ be the corresponding instantaneous elastic moduli and compliance tensors, such that for purely elastic increment of strain $\delta\mathbf{e}$ and corresponding stress increment $\delta\mathbf{t}$

$$\delta\mathbf{t} = \boldsymbol{\Lambda}:\delta\mathbf{e}, \quad \delta\mathbf{e} = \mathbf{M}:\delta\mathbf{t}. \tag{3.1}$$

If the increment of strain $\mathrm{d}\mathbf{e}$ is elastoplastic, its plastic part is defined by

$$\mathrm{d}^{\mathrm{p}}\mathbf{e} = \mathrm{d}\mathbf{e} - \mathbf{M}:\mathrm{d}\mathbf{t}. \tag{3.2}$$

In a hardening material (i.e. when the yield surface in stress space moves locally outwards), $\mathrm{d}^{\mathrm{p}}\mathbf{e}$ corresponds to the residual strain increment after an infinitesimal loading–unloading stress cycle $\mathrm{d}\mathbf{t}$. Applying the Il'yushin postulate (IL'YUSHIN, 1961) to certain finite and infinitesimal strain cycles (HILL, 1968; HILL and RICE, 1973), it follows that the plastic strain increment $\mathrm{d}^{\mathrm{p}}\mathbf{e}$ is codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{t}$ space. Indeed, for any increment of stress $\delta\mathbf{t}$ emanating from the same stress state $\mathbf{t}$ on the yield surface f and directed inside the yield surface [Fig. 1(a)],

$$\delta\mathbf{t}:\mathrm{d}^{\mathrm{p}}\mathbf{e} < 0, \tag{3.3}$$

hence

$$\mathrm{d}^{\mathrm{p}}\mathbf{e} \sim \frac{\partial f}{\partial \mathbf{t}}. \tag{3.4}$$

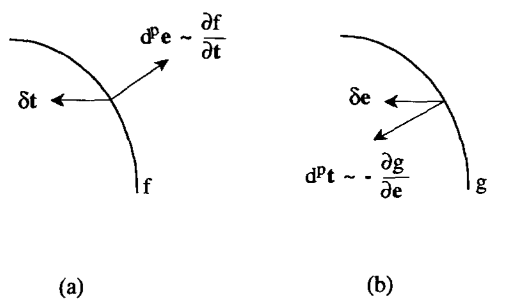

FIG. 1. (a) Plastic strain increment $\mathrm{d}^{\mathrm{p}}\mathbf{e}$ is codirectional with the outward normal to the yield surface f in stress space; $\delta\mathbf{t}$ is the stress increment directed inside the yield surface. (b) Plastic strain increment $\mathrm{d}^{\mathrm{p}}\mathbf{t}$ is codirectional with the inward normal to the yield surface g in strain space; $\delta\mathbf{e}$ is the strain increment directed inside the yield surface.

Likewise, the plastic stress change $d^p\mathbf{t}$ is defined as a residual stress decrement in an infinitesimal strain cycle, i.e.

$$d^p\mathbf{t} = d\mathbf{t} - \mathbf{\Lambda} : d\mathbf{e}. \tag{3.5}$$

Comparing (3.2) and (3.5), clearly:

$$d^p\mathbf{t} = -\mathbf{\Lambda} : d^p\mathbf{e}, \quad d^p\mathbf{e} = -\mathbf{M} : d^p\mathbf{t}. \tag{3.6}$$

Consider next the yield surface g in strain space $\mathbf{e}$. For any increment of strain $\delta\mathbf{e}$ emanating from the same strain $\mathbf{e}$ on the yield surface g and directed inside the yield surface, it follows that

$$\delta\mathbf{e} : d^p\mathbf{t} = \delta\mathbf{e} : (-\mathbf{\Lambda} : d^p\mathbf{e}) = -\delta\mathbf{t} : d^p\mathbf{e} > 0. \tag{3.7}$$

Hence, $d^p\mathbf{t}$ is codirectional with the inward normal to a locally smooth yield surface in strain space $\mathbf{e}$ [Fig. 1(b)], i.e.

$$d^p\mathbf{t} \sim -\frac{\partial g}{\partial \mathbf{e}}. \tag{3.8}$$

In retrospect, if the material is in a hardening range, i.e. if the yield surface in stress space locally expands during plastic deformation $d^p\mathbf{e}$, the corresponding stress increment $d\mathbf{t}$ is directed outside the current yield surface, and $d\mathbf{t} : d^p\mathbf{e} > 0$. If the material is in a softening range, i.e. if the yield surface locally shrinks during plastic deformation $d^p\mathbf{e}$, the corresponding stress increment $d\mathbf{t}$ is directed inside the current yield surface, and $d\mathbf{t} : d^p\mathbf{e} < 0$. However, as discussed by PALMER *et al.* (1967), in both cases $d^p\mathbf{e}$ is codirectional with the outward normal to a locally smooth yield surface in stress space $\mathbf{t}$. In the formulation based on the yield surface in strain space $\mathbf{e}$, when the yield surface moves locally outwards during plastic loading regardless of the hardening or softening features in the corresponding stress space, the actual strain increment $d\mathbf{e}$ is always directed outside the current yield surface, and $d\mathbf{e} : d^p\mathbf{t} < 0$. It should be remembered, however, that hardening and softening are relative terms, and depend on the strain and conjugate stress measures that are employed. A material that is judged to be strain hardening with one choice of the measures may be strain softening relative to another choice (HILL, 1967, 1978; PALGEN and DRUCKER, 1983).

3.1. *Partitions corresponding to Lagrangian strain and its conjugate stress*

Let $\mathbf{e}$ be the Lagrangian strain $\mathbf{E}$ and $\mathbf{t}$ its conjugate Piola–Kirchhoff stress $\mathbf{S}$. Adopting the reference state for these measures to always coincide with intermediate-unstressed state $\mathscr{P}_i$, it follows that

$$\mathbf{e} = \mathbf{E}_e, \quad \mathbf{t} = \mathbf{S}_e \tag{3.9}$$

$$\dot{\mathbf{e}} = \mathbf{F}_e^T\mathbf{D}\mathbf{F}_e, \quad \dot{\mathbf{t}} = \mathbf{F}_e^{-1}\overset{*}{\boldsymbol{\tau}}\mathbf{F}_e^{-T}. \tag{3.10}$$

During purely elastic response (3.1) holds. In the case of the measures defined by (3.9) and (3.10), this is

$$\mathbf{F}_e^T\mathbf{D}\mathbf{F}_e = \mathbf{M}_e : (\mathbf{F}_e^{-1}\overset{*}{\boldsymbol{\tau}}\mathbf{F}_e^{-T}), \quad \mathbf{F}_e^{-1}\overset{*}{\boldsymbol{\tau}}\mathbf{F}_e^{-T} = \mathbf{\Lambda}_e : (\mathbf{F}_e^T\mathbf{D}\mathbf{F}_e) \tag{3.11}$$

where Λ_e and $\mathbf{M}_e$ are defined by (2.21) and (2.26). Denoting by $\mathbf{D}^e$ and $\mathbf{D}^p$ the elastic and plastic parts of the total strain rate $\mathbf{D}$, corresponding to the above choice of the stress and strain measures, (3.2) gives

$$(\mathbf{F}_e^T\mathbf{D}\mathbf{F}_e)^p = \mathbf{F}_e^T\mathbf{D}\mathbf{F}_e - \mathbf{M}_e : (\mathbf{F}_e^{-1}\,\overset{*}{\boldsymbol{\tau}}\,\mathbf{F}_e^{-T}). \tag{3.12}$$

From here,

$$\mathbf{D}^p = \mathbf{D} - \mathscr{M}_e : \overset{*}{\boldsymbol{\tau}}, \tag{3.13}$$

with the elastic compliance $\mathscr{M}_e$ defined by (2.28). The plastic strain rate (3.13) corresponds to the selection of the Lagrangian strain and its conjugate stress, and the corresponding rate measures, defined relative to the current state as a reference.

Likewise, from (3.5) it follows that

$$(\mathbf{F}_e^{-1}\,\overset{*}{\boldsymbol{\tau}}\,\mathbf{F}_e^{-T})^p = \mathbf{F}_e^{-1}\,\overset{*}{\boldsymbol{\tau}}\,\mathbf{F}_e^{-T} - \Lambda_e : (\mathbf{F}_e^T\mathbf{D}\mathbf{F}_e), \tag{3.14}$$

i.e.

$$(\overset{*}{\boldsymbol{\tau}})^p = \overset{*}{\boldsymbol{\tau}} - \mathscr{L}_e : \mathbf{D}, \tag{3.15}$$

with the elastic moduli tensor $\mathscr{L}_e$ defined by (2.23). Observe that

$$(\overset{*}{\boldsymbol{\tau}})^p = -\mathscr{L}_e : \mathbf{D}^p, \quad \mathbf{D}^p = -\mathscr{M}_e : (\overset{*}{\boldsymbol{\tau}})^p. \tag{3.16}$$

As will be seen in Section 5, the selection of the Lagrangian strain and its conjugate Piola–Kirchhoff stress is most suitable in the development of the elastoplastic constitutive analysis using the stress rate decomposition into its elastic and plastic parts, and the concept of the yield surface in strain space.

3.2. *Partitions corresponding to logarithmic strain and its conjugate stress*

Hill's general constitutive formulation employing arbitrary conjugate stress and strain measures, has been frequently utilized by adopting the reference configuration to always coincide with the current configuration. With this choice of reference state, a particularly convenient selection of the strain and stress measures is the logarithmic strain and its conjugate stress (HILL, 1978). In this case

$$\mathbf{e} = \mathbf{0}, \quad \mathbf{t} = \boldsymbol{\sigma} \tag{3.17}$$

$$\dot{\mathbf{e}} = \mathbf{D}, \quad \dot{\mathbf{t}} = |\mathbf{F}|^{-1}\overset{\circ}{\boldsymbol{\tau}}, \tag{3.18}$$

where $\overset{\circ}{\boldsymbol{\tau}}$ is the Jaumann corotational derivative of the Kirchhoff stress. Denoting by $\hat{\mathbf{D}}^e$ and $\hat{\mathbf{D}}^p$ the elastic and plastic parts of the total strain rate $\mathbf{D}$, corresponding to the above choice of stress and strain measures, (3.2) gives

$$\hat{\mathbf{D}}^p = \mathbf{D} - \hat{\mathscr{M}}_e : \overset{\circ}{\boldsymbol{\tau}}. \tag{3.19}$$

The elastic compliance tensor $\hat{\mathscr{M}}_e$ is the inverse of the elastic moduli tensor $\hat{\mathscr{L}}_e$, explicitly given by (2.31).

Likewise, from (3.5) it follows that

$$(\overset{\circ}{\boldsymbol{\tau}})^{\mathrm{p}} = \overset{\circ}{\boldsymbol{\tau}} - \hat{\mathscr{L}}_{\mathrm{e}} : \mathbf{D}, \tag{3.20}$$

so that

$$(\overset{\circ}{\boldsymbol{\tau}})^{\mathrm{p}} = -\hat{\mathscr{L}}_{\mathrm{e}} : \hat{\mathbf{D}}^{\mathrm{p}}, \quad \hat{\mathbf{D}}^{\mathrm{p}} = -\hat{\mathscr{M}}_{\mathrm{e}} : (\overset{\circ}{\boldsymbol{\tau}})^{\mathrm{p}}. \tag{3.21}$$

As described in Section 6, the selection of the logarithmic strain and its conjugate stress, with the reference configuration always being the current configuration, is most suitable to derive the elastoplastic constitutive equations using the strain rate decomposition into its elastic and plastic parts, and the yield surface in stress space.

Observe from (3.20) and (3.15) that the plastic parts of the Jaumann and convected stress rates are equal to each other, since

$$(\overset{\circ}{\boldsymbol{\tau}})^{\mathrm{p}} = \overset{\circ}{\boldsymbol{\tau}} - \hat{\mathscr{L}}_{\mathrm{e}} : \mathbf{D} = \overset{*}{\boldsymbol{\tau}} - \mathscr{L}_{\mathrm{e}} : \mathbf{D} \equiv (\overset{*}{\boldsymbol{\tau}})^{\mathrm{p}}. \tag{3.22}$$

4. Relationships Between Elastic and Plastic Strain Rates and Constituents of $\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{p}}$ Decomposition

If the current increment of deformation is elastoplastic, substitution of (2.12) and (2.15) into

$$\dot{\mathbf{S}}_{\mathrm{e}} = \boldsymbol{\Lambda}_{\mathrm{e}} : \dot{\mathbf{E}}_{\mathrm{e}} \tag{4.1}$$

gives

$$\overset{*\mathrm{e}}{\boldsymbol{\tau}} = \mathscr{L}_{\mathrm{e}} : \mathbf{D} - \mathscr{L}_{\mathrm{e}} : [\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]_{\mathrm{s}}. \tag{4.2}$$

In (4.1) and (4.2), $\boldsymbol{\Lambda}_{\mathrm{e}}$ and $\mathscr{L}_{\mathrm{e}}$ are elastic stiffness tensors defined by (2.21) and (2.23). Substituting (2.17) to express $\overset{*\mathrm{e}}{\boldsymbol{\tau}}$ in terms of $\overset{*}{\boldsymbol{\tau}}$, (4.2) becomes

$$\overset{*}{\boldsymbol{\tau}} = \mathscr{L}_{\mathrm{e}} : \mathbf{D} - \mathscr{L}_{\mathrm{e}} : [\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]_{\mathrm{s}} - [\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]\boldsymbol{\tau} - \boldsymbol{\tau}[\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]^{\mathrm{T}}. \tag{4.3}$$

Since, according to (3.15),

$$(\overset{*}{\boldsymbol{\tau}})^{\mathrm{e}} = \mathscr{L}_{\mathrm{e}} : \mathbf{D} \tag{4.4}$$

is the elastic part of the stress rate $\overset{*}{\boldsymbol{\tau}}$, from (4.3) it follows that the plastic part is

$$(\overset{*}{\boldsymbol{\tau}})^{\mathrm{p}} = -\{\mathscr{L}_{\mathrm{e}} : [\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]_{\mathrm{s}} + [\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]\boldsymbol{\tau} + \boldsymbol{\tau}[\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]^{\mathrm{T}}\}. \tag{4.5}$$

Comparing (4.5) with the first of the expressions in (3.16), the plastic strain rate expression is identified as

$$\mathbf{D}^{\mathrm{p}} = [\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]_{\mathrm{s}} + \mathscr{M}_{\mathrm{e}} : \{[\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]\boldsymbol{\tau} + \boldsymbol{\tau}[\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]^{\mathrm{T}}\}. \tag{4.6}$$

Using the strain rate decomposition $\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}}$, the expression for the elastic strain rate follows from (4.6) and (2.8) as

$$\mathbf{D}^{\mathrm{e}} = (\dot{\mathbf{F}}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}} - \mathscr{M}_{\mathrm{e}} : \{[\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]\boldsymbol{\tau} + \boldsymbol{\tau}[\mathbf{F}_{\mathrm{e}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\mathbf{F}_{\mathrm{e}}^{-1}]^{\mathrm{T}}\}. \tag{4.7}$$

If, instead of the convected derivative $\overset{*}{\tau}$, the Jaumann derivative $\overset{\circ}{\tau}$ is used, it follows from (4.3) and (2.9) that

$$\overset{\circ}{\tau} = \hat{\mathscr{L}}_e : \mathbf{D} - \hat{\mathscr{L}}_e : [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s - [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a\boldsymbol{\tau} + \boldsymbol{\tau}[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a. \quad (4.8)$$

Therefore, since according to (3.20),

$$(\overset{\circ}{\tau})^e = \hat{\mathscr{L}}_e : \mathbf{D} \quad (4.9)$$

is the elastic part of the stress rate $\overset{\circ}{\tau}$, from (4.8) it follows that the plastic part is

$$(\overset{\circ}{\tau})^p = -\{\hat{\mathscr{L}}_e : [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s + [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a\boldsymbol{\tau} - \boldsymbol{\tau}[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a\}. \quad (4.10)$$

Comparing (4.10) with the first of the expressions in (3.21), the corresponding plastic strain rate expression is

$$\hat{\mathbf{D}}^p = [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s + \hat{\mathscr{M}}_e : \{[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a\boldsymbol{\tau} - \boldsymbol{\tau}[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a\}. \quad (4.11)$$

By using the strain rate decomposition $\mathbf{D} = \hat{\mathbf{D}}_e + \hat{\mathbf{D}}_p$, the expression for the elastic strain rate follows from (4.11) and (2.8) as

$$\hat{\mathbf{D}}^e = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s - \hat{\mathscr{M}}_e : \{[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a\boldsymbol{\tau} - \boldsymbol{\tau}[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a\}. \quad (4.12)$$

Note that $\hat{\mathbf{D}}^e$ does not depend on the superposed rotation of the intermediate configuration, since in

$$\hat{\mathbf{D}}^e = \hat{\mathscr{M}}_e : \overset{\circ}{\tau}, \quad (4.13)$$

neither $\hat{\mathscr{M}}_e$ nor $\overset{\circ}{\tau}$ depend on such rotation. In contrast, the constituents of $\hat{\mathbf{D}}^e$, namely the strain rate $(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s$ and spin

$$\boldsymbol{\omega} = [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a, \quad (4.14)$$

do depend on the choice of an intermediate configuration. Similar remarks apply to $\hat{\mathbf{D}}^p$ and its representation (4.11) (Lubarda and Shih, 1994). As discussed in the context of crystal plasticity by Hill and Rice (1972), Hill and Havner (1982) and Asaro (1983), the plastic strain rate $\hat{\mathbf{D}}^p$ does not come from the slip deformation alone, which is the first term on the right-hand side of (4.11). There is a further net elastic contribution from the lattice, which is caused by the slip-induced rotation of the lattice relative to the material, given by the $\hat{\mathscr{M}}_e : (\boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega})$ term in (4.11).

Observe from (4.6) and (4.11) that

$$\mathscr{L}_e : \mathbf{D}^p = \hat{\mathscr{L}}_e : [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s + \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega} \equiv \hat{\mathscr{L}}_e : \hat{\mathbf{D}}^p, \quad (4.15)$$

as it should be, because $(\overset{*}{\tau})^p \equiv (\overset{\circ}{\tau})^p$. Since $(\overset{*}{\tau})^p = -\mathscr{L}_e : \mathbf{D}^p$, from (4.15) one has

$$[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s = -\hat{\mathscr{M}}_e : (\overset{*}{\tau})^p - \hat{\mathscr{M}}_e : (\boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega}), \quad (4.16)$$

which is a useful relationship used in the next section of this paper.

5. ELASTOPLASTIC CONSTITUTIVE EQUATIONS WITH THE YIELD SURFACE IN STRAIN SPACE

5.1. *Constitutive equation for the plastic stress rate*

In deriving a constitutive equation for the plastic stress rate, a convenient choice of the strain and stress measures is the Lagrangian strain and its conjugate Piola–Kirchhoff stress, defined relative to the reference state that always coincides with the intermediate–unstressed state, as defined by (3.9) and (3.10). The plastic part of the stress rate, given by (3.14), is then codirectional with the inward normal to the yield surface in strain $\mathbf{E}_e$ space, i.e.

$$\mathbf{F}_e^{-1}(\overset{*}{\boldsymbol{\tau}})^p\mathbf{F}_e^{-T} = -\dot{\gamma}\frac{\partial g}{\partial \mathbf{E}_e}. \tag{5.1}$$

In (5.1), g is the yield function and $\dot{\gamma} > 0$ the loading index. From the expression (5.1) the plastic part of the convected stress derivative is

$$(\overset{*}{\boldsymbol{\tau}})^p = -\dot{\gamma}\mathbf{A}_e, \quad \mathbf{A}_e = \mathbf{F}_e\frac{\partial g}{\partial \mathbf{E}_e}\mathbf{F}_e^T \equiv \frac{\partial g}{\partial \mathscr{E}_e} \tag{5.2}$$

where $\mathscr{E}_e = \mathbf{F}_e^{-T}\mathbf{E}_e\mathbf{F}_e^{-1}$ is the elastic Eulerian strain.

Consider the simplest case of isotropic hardening for which

$$g(\mathbf{E}_e) = \kappa(\varphi). \tag{5.3}$$

The yield function g is an isotropic function of the strain $\mathbf{E}_e$, and κ is a scalar function of the parameter φ. This parameter can be defined as a dual of the plastic work per unit initial volume

$$\varphi = -\int_0^t \frac{\rho_0}{\rho_r}\mathbf{e} : \dot{\mathbf{t}}^p \, dt, \tag{5.4}$$

where ρ_0 and ρ_r are the mass densities in the initial and reference states. Alternatively, the equivalent (generalized) plastic stress can be used, i.e.

$$\varphi = -\int_0^t (\dot{\mathbf{t}}^p : \dot{\mathbf{t}}^p)^{1/2} \, dt. \tag{5.4'}$$

These are the analogous definitions to corresponding definitions of plastic work and equivalent plastic strain, used in a familiar plasticity formulation with the yield surface in stress space.

Using, for example, the definition (5.4), and substituting $\mathbf{e} = \mathbf{E}_e$ and expression (5.1) for the stress rate $\dot{\mathbf{t}}^p$, the rate of φ becomes in the case of plastic incompressibility ($\rho_r = \rho_0$)

$$\dot{\varphi} = \dot{\gamma}\left(\mathbf{E}_e : \frac{\partial g}{\partial \mathbf{E}_e}\right). \tag{5.5}$$

If g is a homogeneous function of $\mathbf{E}_e$ of degree n, the above is also equal to $\dot{\gamma}(n\kappa)$. From (5.3) the consistency condition for the continuing plastic deformation is

$$\frac{\partial g}{\partial \mathbf{E}_e} : \dot{\mathbf{E}}_e = \frac{d\kappa}{d\varphi}\dot{\varphi}. \tag{5.6}$$

In view of the expression (2.12) for the rate of strain $\dot{\mathbf{E}}_e$, (5.6) can be written as

$$\mathbf{A}_e : \{\mathbf{D} - [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s\} = \frac{d\kappa}{d\varphi}\dot{\varphi}. \tag{5.7}$$

Using (4.16), the second part on the left-hand side of (5.7) is

$$\mathbf{A}_e : [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s = -\mathbf{A}_e : \hat{\mathscr{M}}_e : (\overset{*}{\boldsymbol{\tau}})^p. \tag{5.8}$$

Observe that due to elastic isotropy, the strain rate $\hat{\mathscr{M}}_e : (\boldsymbol{\omega\tau} - \boldsymbol{\tau\omega})$ has parallel principal directions to those of the associated stress rate $(\boldsymbol{\omega\tau} - \boldsymbol{\tau\omega})$. Because the direction of the above stress rate is normal to $\boldsymbol{\tau}$, and since $\mathbf{A}_e$ is codirectional with $\boldsymbol{\tau}$ for isotropic function g, it follows that

$$\mathbf{A}_e : \hat{\mathscr{M}}_e : (\boldsymbol{\omega\tau} - \boldsymbol{\tau\omega}) = 0, \tag{5.9}$$

which was used in arriving at (5.8).

Substituting (5.2) into (5.8) and this into (5.7), the loading index is found to be

$$\dot{\gamma} = \frac{1}{h}(\mathbf{A}_e : \mathbf{D}). \tag{5.10}$$

The hardening parameter h is given by

$$h = \frac{d\kappa}{d\varphi}\left(\mathbf{E}_e : \frac{\partial g}{\partial \mathbf{E}_e}\right) + \mathbf{A}_e : \hat{\mathscr{M}}_e : \mathbf{A}_e. \tag{5.11}$$

Since during continuing plastic deformation $\dot{\gamma} > 0$, the plastic loading condition is given by the positiveness of the right-hand side of (5.10).

For example, if the yield function is defined by

$$g = \tfrac{1}{2}\mathbf{E}'_e : \mathbf{E}'_e, \tag{5.12}$$

where $\mathbf{E}'_e$ is the deviatoric part of the strain $\mathbf{E}_e$, it follows that

$$\frac{\partial g}{\partial \mathbf{E}_e} = \mathbf{E}'_e, \quad h = \frac{d}{d\varphi}(\kappa^2) + \mathbf{A}_e : \hat{\mathscr{M}}_e : \mathbf{A}_e. \tag{5.13}$$

If the elastic component of strain is small ($\mathbf{E}_e = \boldsymbol{\varepsilon}_e$), the strain-space yield function (5.12) corresponds to von Mises stress-space yield function $f = \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'$, where $\boldsymbol{\sigma}' = 2\mu\boldsymbol{\varepsilon}'_e$ is the deviatoric part of the Cauchy stress, and μ the elastic shear modulus.

Substituting (5.10) into (5.2), the constitutive structure for the plastic stress rate is

$$(\overset{*}{\boldsymbol{\tau}})^p = -\frac{1}{h}(\mathbf{A}_e \otimes \mathbf{A}_e) : \mathbf{D}. \tag{5.14}$$

Note from (5.14) that

$$\mathbf{D}:(\overset{*}{\boldsymbol{\tau}})^{\mathrm{p}} = -\frac{1}{h}(\mathbf{A}_{\mathrm{e}}:\mathbf{D})^2 < 0, \tag{5.15}$$

for $h > 0$.

5.2. *Elastoplastic constitutive equations*

Summing the elastic and plastic stress rates according to

$$\overset{*}{\boldsymbol{\tau}} = (\overset{*}{\boldsymbol{\tau}})^{\mathrm{e}} + (\overset{*}{\boldsymbol{\tau}})^{\mathrm{p}}, \tag{5.16}$$

where the elastic stress rate $(\overset{*}{\boldsymbol{\tau}})^{\mathrm{e}}$ is given by (4.4) and plastic stress rate $(\overset{*}{\boldsymbol{\tau}})^{\mathrm{p}}$ by (5.14), it follows that

$$\overset{*}{\boldsymbol{\tau}} = \left[\mathscr{L}_{\mathrm{e}} - \frac{1}{h}(\mathbf{A}_{\mathrm{e}} \otimes \mathbf{A}_{\mathrm{e}})\right]:\mathbf{D}. \tag{5.17}$$

This can be rewritten in terms of the Jaumann derivative as

$$\overset{\circ}{\boldsymbol{\tau}} = \left[\hat{\mathscr{L}}_{\mathrm{e}} - \frac{1}{h}(\mathbf{A}_{\mathrm{e}} \otimes \mathbf{A}_{\mathrm{e}})\right]:\mathbf{D}. \tag{5.18}$$

This is the final elastoplastic constitutive structure obtained by using the stress rate decomposition in its elastic and plastic parts, and the concept of the yield surface in strain space.

To obtain the inverse relationship, we first form a trace product of (5.18) with the elastic compliance tensor $\hat{\mathscr{M}}_{\mathrm{e}}$ to obtain

$$\hat{\mathscr{M}}_{\mathrm{e}}:\overset{\circ}{\boldsymbol{\tau}} = \mathbf{D} - \frac{1}{h}(\hat{\mathscr{M}}_{\mathrm{e}}:\mathbf{A}_{\mathrm{e}})(\mathbf{A}_{\mathrm{e}}:\mathbf{D}). \tag{5.19}$$

Applying to (5.19) the trace product with $\mathbf{A}_{\mathrm{e}}$, it then follows that

$$\frac{1}{h}(\mathbf{A}_{\mathrm{e}}:\mathbf{D}) = \frac{1}{H}(\mathbf{A}_{\mathrm{e}}:\hat{\mathscr{M}}_{\mathrm{e}}:\overset{\circ}{\boldsymbol{\tau}}), \tag{5.20}$$

where

$$H = h - \mathbf{A}_{\mathrm{e}}:\hat{\mathscr{M}}_{\mathrm{e}}:\mathbf{A}_{\mathrm{e}} = \frac{\mathrm{d}\kappa}{\mathrm{d}\varphi}\left(\mathbf{E}_{\mathrm{e}}:\frac{\partial g}{\partial \mathbf{E}_{\mathrm{e}}}\right). \tag{5.21}$$

Substituting (5.20) back into (5.19) and solving for the strain rate $\mathbf{D}$, the inverse constitutive relationship to (5.18) is obtained as

$$\mathbf{D} = \left[\hat{\mathscr{M}}_{\mathrm{e}} + \frac{1}{H}\hat{\mathscr{M}}_{\mathrm{e}}:(\mathbf{A}_{\mathrm{e}} \otimes \mathbf{A}_{\mathrm{e}}):\hat{\mathscr{M}}_{\mathrm{e}}\right]:\overset{\circ}{\boldsymbol{\tau}}. \tag{5.22}$$

The self-adjoint and other symmetries of the elastoplastic stiffness and compliance tensors appearing in (5.18) and (5.22) are evident. Note also that the first part on the right-hand side of (5.22) is the elastic strain rate corresponding to $\mathring{\tau}$, while the remaining part is the plastic strain rate.

5.3. *Additional analysis with the yield surface in strain space*

If the Lagrangian strain $\mathbf{E}$ relative to initial (fixed) reference configuration is used, with its conjugate Piola–Kirchhoff stress $\mathbf{S}$, the yield function g is defined in the strain space $\mathbf{E}$. According to (3.5) and the normality condition (3.8), the plastic part of the stress rate is given by

$$(\dot{\mathbf{S}})^{\mathrm{p}} = \dot{\mathbf{S}} - \mathbf{\Lambda}_0 : \dot{\mathbf{E}} = -\dot{\gamma}\frac{\partial g}{\partial \mathbf{E}}, \tag{5.23}$$

where $\mathbf{\Lambda}_0$ is the corresponding instantaneous elastic stiffness tensor. The plastic part of the strain rate is by (3.6)

$$(\dot{\mathbf{E}})^{\mathrm{p}} = -\mathbf{M}_0 : (\dot{\mathbf{S}})^{\mathrm{p}} = \dot{\gamma}\mathbf{M}_0 : \frac{\partial g}{\partial \mathbf{E}}, \tag{5.24}$$

where $\mathbf{M}_0$ is the instantaneous elastic compliance tensor, the inverse of $\mathbf{\Lambda}_0$.

By an analogous derivation to that of Subsection 5.1 it then follows that the loading index can be written as

$$\dot{\gamma} = \frac{1}{h_0}\left(\frac{\partial g}{\partial \mathbf{E}} : \dot{\mathbf{E}}\right) > 0. \tag{5.25}$$

The explicit representation of the expression for the hardening parameter h_0 depends on the specific structure of the introduced yield condition in strain space $\mathbf{E}$. Combining (5.23) and (5.25) gives the overall rate-type elastoplastic constitutive structure as

$$\dot{\mathbf{S}} = \left[\mathbf{\Lambda}_0 - \frac{1}{h_0}\left(\frac{\partial g}{\partial \mathbf{E}} \otimes \frac{\partial g}{\partial \mathbf{E}}\right)\right] : \dot{\mathbf{E}}. \tag{5.26}$$

From (5.26) it can be shown that

$$\frac{1}{h_0}\left(\frac{\partial g}{\partial \mathbf{E}} : \dot{\mathbf{E}}\right) = \frac{1}{H_0}\left(\frac{\partial q}{\partial \mathbf{E}} : \mathbf{M}_0 : \dot{\mathbf{S}}\right), \tag{5.27}$$

where

$$H_0 = h_0 - \frac{\partial q}{\partial \mathbf{E}} : \mathbf{M}_0 : \frac{\partial g}{\partial \mathbf{E}}. \tag{5.28}$$

Hence, the inverse of (5.26) is found to be

$$\dot{\mathbf{E}} = \left[\mathbf{M}_0 + \frac{1}{H_0}\mathbf{M}_0 : \left(\frac{\partial g}{\partial \mathbf{E}} \otimes \frac{\partial g}{\partial \mathbf{E}}\right) : \mathbf{M}_0\right] : \dot{\mathbf{S}}. \tag{5.29}$$

The elastoplastic constitutive equations (5.26) and (5.29) are identical to those given by expressions (2.45) of HILL (1978). Observe that the instantaneous elastic stiffness tensor Λ_0 can be expressed in terms of the tensor Λ_e defined in (2.21) as

$$\Lambda_0 = \mathbf{F}_p^{-1}\mathbf{F}_p^{-1}\Lambda_e\mathbf{F}_p^{-T}\mathbf{F}_p^{-T}. \tag{5.30}$$

This follows by partial differentiation from the definition $\Lambda_0 = \partial^2\psi/(\partial\mathbf{E}\otimes\partial\mathbf{E})$, and the expression for the work potential ψ (per unit initial volume) in terms of the total strain $\mathbf{E}$, i.e. $\psi = \psi(\mathbf{E}_e) = \psi[\mathbf{F}_p^{-T}(\mathbf{E}-\mathbf{E}_p)\mathbf{F}_p^{-1}]$. The stress response is given by $\mathbf{S} = \partial\psi/\partial\mathbf{E}$, which is equivalent to (2.6).

Assuming that during plastic deformation the yield surface in strain space moves locally outwards, from (5.25) it follows that h_0 is necessarily positive, while from (5.28) it follows that H_0 is not necessarily one-signed. Three types of stress response may, therefore, occur, i.e. strain hardening, strain softening and perfect plasticity, depending on whether H_0 is positive, negative or equal to zero. In the case of strain hardening, the stress rate $\dot{\mathbf{S}}$ is directed inside the yield surface in stress space. The inverse of (5.26) is then not unique, since either (5.29) or the elastic unloading expression $\dot{\mathbf{E}} = \mathbf{M}_0 : \dot{\mathbf{S}}$ applies. In the case of perfect plasticity, the plastic part of the strain rate becomes indeterminate to the extent of an arbitrary positive multiple.

The yield condition in the strain space $\mathbf{E}$, corresponding to isotropic yield condition in elastic strain space $\mathbf{E}_e$ is suggested to be

$$g(\mathbf{E}-\boldsymbol{\alpha}) = \kappa(\varphi), \tag{5.31}$$

where the second-order tensor $\boldsymbol{\alpha}$ represents the current center of the yield surface in the strain space $\mathbf{E}$. Within the context of small strain theory, an analogous structure of the yield condition was used by YODER and IWAN (1981). Similarly to the model of kinematic hardening in stress space, it can be assumed that the yield surface in strain space instantaneously translates in the direction of the plastic part of the stress rate, i.e.

$$\dot{\boldsymbol{\alpha}} = -k(\dot{\mathbf{S}})^p = k\dot{\gamma}\frac{\partial g}{\partial\mathbf{E}}, \quad (k>0). \tag{5.32}$$

A more general evolution expression for $\boldsymbol{\alpha}$ can be used, which is similar to the generalizations of the evolution expression for the back stress in the plasticity theory with the yield surface defined in the stress space. For example, the right-hand side of (5.24) can be used to represent the evolution of $\boldsymbol{\alpha}$. The scalar κ in (5.31) defines the size of the yield surface. The hardening parameter h_0, appearing in (5.25) and corresponding to the yield surface (5.31), is found to be

$$h_0 = k\left(\frac{\partial g}{\partial\mathbf{E}}:\frac{\partial g}{\partial\mathbf{E}}\right) + \frac{d\kappa}{d\varphi}\left(\mathbf{E}:\frac{\partial g}{\partial\mathbf{E}}\right). \tag{5.33}$$

If h_0 is positive, the yield surface in strain space $\mathbf{E}$ moves locally outwards during plastic deformation, regardless of whether the yield surface in the corresponding stress space (and the elastic strain space) moves locally outwards (hardening), inwards (softening) or is stationary (perfect plasticity). It is because of this that some authors prefer elastoplasticity formulation with the yield surface in strain space (CASEY and NAGHDI, 1981, 1983, 1984).

6. Elastoplastic Constitutive Equations with the Yield Surface in Stress Space

6.1. *Constitutive equations with the yield surface in Cauchy stress space*

To make a comparison with results from the previous section, the well-known procedure of deriving the constitutive expression for the plastic strain rate is presented. To that goal it is convenient to introduce the logarithmic strain and its conjugate stress measure defined relatively to the reference state that always coincides with the current state, which gives (3.17) and (3.18). The plastic part of the strain rate, defined by (3.19), is then codirectional with the outward normal to the yield surface in Cauchy (or Kirchhoff) stress space, i.e.

$$\hat{\mathbf{D}}^{\mathrm{p}} = \dot{\gamma}\frac{\partial f}{\partial \boldsymbol{\tau}}, \tag{6.1}$$

where f is the yield function and $\dot{\gamma}$ the loading index. Consider again the simplest case of isotropic hardening, for which

$$f(\boldsymbol{\tau}) = k(\vartheta). \tag{6.2}$$

The yield function f is an isotropic function of the stress $\boldsymbol{\tau}$, and k is a scalar function of the parameter ϑ, such as the plastic work per unit initial volume

$$\vartheta = \int_0^t \boldsymbol{\tau} : \hat{\mathbf{D}}^{\mathrm{p}}\,\mathrm{d}t. \tag{6.3}$$

In view of (6.1), the rate of ϑ is

$$\dot{\vartheta} = \dot{\gamma}\left(\boldsymbol{\tau} : \frac{\partial f}{\partial \boldsymbol{\tau}}\right). \tag{6.4}$$

Substitution of (6.4) into the consistency conduction

$$\frac{\partial f}{\partial \boldsymbol{\tau}} : \mathring{\boldsymbol{\tau}} = \frac{\mathrm{d}k}{\mathrm{d}\vartheta}\dot{\vartheta}, \tag{6.5}$$

gives the expression for the loading index

$$\dot{\gamma} = \frac{1}{\hat{H}}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \mathring{\boldsymbol{\tau}}\right). \tag{6.6}$$

In (6.6), the hardening parameter $\hat{H}$ is equal to

$$\hat{H} = \frac{\mathrm{d}k}{\mathrm{d}\vartheta}\left(\boldsymbol{\tau} : \frac{\partial f}{\partial \boldsymbol{\tau}}\right). \tag{6.7}$$

Because f is an isotropic function of $\boldsymbol{\tau}$, the Jaumann derivative $\mathring{\boldsymbol{\tau}}$ was conveniently used in (6.5), in place of the material derivative $\dot{\boldsymbol{\tau}}$.

Substituting (6.6) into (6.1) gives the well-known constitutive structure for the plastic strain rate as

$$\hat{\mathbf{D}}^{\mathrm{p}} = \frac{1}{\hat{H}}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}}\right) : \overset{\circ}{\boldsymbol{\tau}}. \tag{6.8}$$

Clearly,

$$\overset{\circ}{\boldsymbol{\tau}} : \hat{\mathbf{D}}^{\mathrm{p}} = \frac{1}{\hat{H}}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}\right)^2 > 0, \tag{6.9}$$

for $\hat{H} > 0$, which is the counterpart of the condition (5.15) from the previous section.

Summing the elastic and plastic strain rates according to

$$\mathbf{D} = \hat{\mathbf{D}}^{\mathrm{e}} + \hat{\mathbf{D}}^{\mathrm{p}}, \tag{6.10}$$

where elastic strain rate $\hat{\mathbf{D}}^{\mathrm{e}}$ is given by (4.13) and plastic strain rate $\hat{\mathbf{D}}^{\mathrm{p}}$ by (6.8), it follows that

$$\mathbf{D} = \left[\hat{\mathscr{M}}_{\mathrm{e}} + \frac{1}{\hat{H}}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}}\right)\right] : \overset{\circ}{\boldsymbol{\tau}}. \tag{6.11}$$

This is the final elastoplastic constitutive structure obtained by using the strain rate decomposition in its elastic and plastic parts, and the notion of yield surface in stress space.

Since

$$\frac{1}{\hat{H}}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}\right) = \frac{1}{\hat{h}}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \hat{\mathscr{L}}_{\mathrm{e}} : \mathbf{D}\right), \tag{6.12}$$

where

$$\hat{h} = \hat{H} + \frac{\partial f}{\partial \boldsymbol{\tau}} : \hat{\mathscr{L}}_{\mathrm{e}} : \frac{\partial f}{\partial \boldsymbol{\tau}}, \tag{6.13}$$

the inverse relationship to (6.11) is found to be

$$\overset{\circ}{\boldsymbol{\tau}} = \left[\hat{\mathscr{L}}_{\mathrm{e}} - \frac{1}{\hat{h}}\hat{\mathscr{L}}_{\mathrm{e}} : \left(\frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}}\right) : \hat{\mathscr{L}}_{\mathrm{e}}\right] : \mathbf{D}. \tag{6.14}$$

The corresponding plastic loading condition is given by the positiveness of the right-hand side of (6.12). When specialized to small elastic components of strain, the constitutive equations (6.11) and (6.14) reduce to the well-known elastoplastic constitutive equations, given by McMEEKING and RICE (1975), LUBARDA and LEE (1981), and others.

6.2. *Constitutive equations with the yield surface in Piola–Kirchhoff stress space*

If the yield surface f is introduced in the Piola–Kirchhoff stress space $\mathbf{S}$, the plastic part of the Lagrangian strain rate is, according to (3.2) and (3.4),

$$(\dot{\mathbf{E}})^{\mathrm{p}} = \dot{\mathbf{E}} - \mathbf{M}_0 : \dot{\mathbf{S}} = \dot{\gamma}\frac{\partial f}{\partial \mathbf{S}}, \tag{6.15}$$

where $\mathbf{M}_0$ is the instantaneous elastic compliance tensor. The loading index can be expressed from the consistency condition as

$$\dot{\gamma} = \frac{1}{\hat{H}_0}\left(\frac{\partial f}{\partial \mathbf{S}} : \dot{\mathbf{S}}\right) > 0, \tag{6.16}$$

so that the overall rate-type elastoplastic constitutive structure is

$$\dot{\mathbf{E}} = \left[\mathbf{M}_0 + \frac{1}{\hat{H}_0}\left(\frac{\partial f}{\partial \mathbf{S}} \otimes \frac{\partial f}{\partial \mathbf{S}}\right)\right] : \dot{\mathbf{S}}. \tag{6.17}$$

In (6.17), $\hat{H}_0$ is the hardening parameter which depends on the specific structure of the introduced yield condition in stress space $\mathbf{S}$. The same comments as those given in the paragraph following (5.30) apply to (6.17) regarding hardening, softening or ideally plastic behavior.

The inverted form of (6.17) is

$$\dot{\mathbf{S}} = \left[\mathbf{\Lambda}_0 - \frac{1}{\hat{h}_0}\mathbf{\Lambda}_0 : \left(\frac{\partial f}{\partial \mathbf{S}} \otimes \frac{\partial f}{\partial \mathbf{S}}\right) : \mathbf{\Lambda}_0\right] : \dot{\mathbf{E}}, \tag{6.18}$$

where $\mathbf{\Lambda}_0$ is the instantaneous elastic stiffness tensor, and

$$\hat{h}_0 = \hat{H}_0 + \frac{\partial f}{\partial \mathbf{S}} : \mathbf{\Lambda}_0 : \frac{\partial f}{\partial \mathbf{S}}. \tag{6.19}$$

As seen, the formulation with the yield surface in the Piola–Kirchhoff stress space $\mathbf{S}$ and the constitutive structures (6.17) and (6.18) are in complete duality with the formulation using the yield surface in the Lagrangian strain space $\mathbf{E}$ and the constitutive structures (5.26) and (5.29). The formulation with the yield surface in strain space may be preferred because an explicit representation of the yield condition in the Piola–Kirchhoff stress space appears to be more involved than the yield condition in the Lagrangian strain space. The elastoplastic constitutive theory with the yield surface in the Piola–Kirchhoff stress space has been studied by Green and Naghdi (1965), Mandel (1973), Naghdi and Trapp (1974), Lubliner (1986) and others.

7. Discussion and Conclusions

To compare the elastoplastic constitutive equations derived using the yield functions in stress and strain spaces, we consider the common case of metal plasticity for which the elastic component of strain is infinitesimally small. The components of elastic compliance tensor $\hat{\mathscr{M}}_e$, appearing in (6.11), are then approximately given by

$$\hat{\mathscr{M}}^e_{ijkl} \approx M^e_{ijkl} \approx \frac{1}{2\mu}\left[\tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\lambda}{2\mu + 3\lambda}\delta_{ij}\delta_{kl}\right], \tag{7.1}$$

which is the inverse of the elastic stiffness tensor (2.34). With the von Mises type yield condition in stress space

$$\tfrac{1}{2}\boldsymbol{\tau}':\boldsymbol{\tau}' = k(\vartheta), \quad \vartheta = \int_0^t \boldsymbol{\tau}:\hat{\mathbf{D}}^{\mathrm{p}}\,\mathrm{d}t \tag{7.2}$$

the constitutive structure (6.11) becomes

$$\mathbf{D} = \left[\mathbf{M}_{\mathrm{e}} + \frac{1}{\hat{H}}(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}')\right]:\mathring{\boldsymbol{\tau}}, \tag{7.3}$$

where $\hat{H} = \mathrm{d}(k^2)/\mathrm{d}\vartheta$.

On the other hand, when the yield surface is introduced in strain space,

$$\tfrac{1}{2}\boldsymbol{\varepsilon}_{\mathrm{e}}':\boldsymbol{\varepsilon}_{\mathrm{e}}' = \kappa(\varphi), \quad \varphi = -\int_0^t \boldsymbol{\varepsilon}_{\mathrm{e}}:(\overset{*}{\boldsymbol{\tau}})^{\mathrm{p}}\,\mathrm{d}t, \tag{7.4}$$

the constitutive structure (5.22) becomes, to within the same order of approximation,

$$\mathbf{D} = \left[\mathbf{M}_{\mathrm{e}} + \frac{1}{H}\mathbf{M}_{\mathrm{e}}:(\boldsymbol{\varepsilon}_{\mathrm{e}}'\otimes\boldsymbol{\varepsilon}_{\mathrm{e}}'):\mathbf{M}_{\mathrm{e}}\right]:\mathring{\boldsymbol{\tau}}. \tag{7.5}$$

Using $\boldsymbol{\varepsilon}_{\mathrm{e}}' = \mathbf{M}_{\mathrm{e}}:\boldsymbol{\tau}'$, (7.5) can be rewritten as

$$\mathbf{D} = \left(\mathbf{M}_{\mathrm{e}} + \frac{1}{16\mu^4}\frac{1}{H}\boldsymbol{\tau}'\otimes\boldsymbol{\tau}'\right):\mathring{\boldsymbol{\tau}}. \tag{7.6}$$

Since $k = 4\mu^2\kappa$, and

$$\varphi = -\int_0^t \boldsymbol{\varepsilon}_{\mathrm{e}}:(\overset{*}{\boldsymbol{\tau}})^{\mathrm{p}}\,\mathrm{d}t = -\int_0^t \boldsymbol{\varepsilon}_{\mathrm{e}}:(\mathring{\boldsymbol{\tau}})^{\mathrm{p}}\,\mathrm{d}t = \int_0^t \boldsymbol{\varepsilon}_{\mathrm{e}}:(\hat{\mathscr{L}}_{\mathrm{e}}:\hat{\mathbf{D}}^{\mathrm{p}})\,\mathrm{d}t = \int_0^t \boldsymbol{\tau}:\hat{\mathbf{D}}^{\mathrm{p}}\,\mathrm{d}t \equiv \vartheta, \tag{7.7}$$

from (5.21) it follows that the hardening parameter H is equal to $\hat{H}/16\mu^4$. Substituting this into (7.6) makes the constitutive expressions (7.3) and (7.6) identically equal. This was expected to be the case, because the utilized stress and strain yield conditions (7.2) and (7.4) physically represent the same yield condition.

In the case of the yield surface in strain space $\mathbf{E}$, such as (5.31), the corresponding elastoplastic constitutive structure (5.26) is most easily compared with those of (5.17) and (6.14) by substituting into (5.26) the expression (2.10) for the strain rate $\dot{\mathbf{E}}$, and (2.13) for the stress rate $\dot{\mathbf{S}}$. In view of the relationships (2.23) and (5.30), it follows that

$$\overset{*}{\boldsymbol{\tau}} = \left[\mathscr{L}_{\mathrm{e}} - \frac{1}{h_0}\left(\frac{\partial g}{\partial \mathscr{E}}\otimes\frac{\partial g}{\partial \mathscr{E}}\right)\right]:\mathbf{D}, \tag{7.8}$$

where $\mathscr{E} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}\mathbf{F}^{-1}$ is the Eulerian strain. Clearly, (7.8) is an analogous structure to (5.17), obtained by using the yield surface in the elastic strain space. With an appropriately specified evolution equation for $\boldsymbol{\alpha}$, (5.17) and (7.8) predict the same material response.

In conclusion, we have demonstrated in this paper that the elastoplastic constitutive analysis can be performed by using either the yield surface in stress or strain space. If the yield surface is introduced in stress space, the strain rate is partitioned into its elastic and plastic parts. The plastic part of the strain rate is under certain kinetic assumptions codirectional with the outward normal to the locally smooth yield surface in stress space. If the yield surface is introduced in strain space, the stress rate is partitioned into its elastic and plastic parts. The plastic part of the stress rate is then codirectional with the inward normal to the locally smooth yield surface in strain space. Which of the two approaches to use, i.e. whether to introduce the yield surface in stress or strain space, depends on the analytical and computational conveniences that result from the particular choice of the conjugate stress and strain measures that are used to cast the constitutive formulation. For example, if the Lagrangian strain and its conjugate Piola–Kirchhoff stress are used, it is easier to construct and use the yield surface representation, such as (5.31), in the strain space. On the other hand, if the logarithmic strain and its conjugate stress are used, the structure of the yield surface in stress space, such as (6.2), is a preferable choice. Furthermore, by comparing the formulations corresponding to the two different choices of the conjugate measures of stress and strain, one using the yield surface in stress space and another using the yield surface in strain space, it is found that the structure of the yield condition in stress space will be most likely simpler in form, as exemplified by the structures (5.31) and (6.2). The results are also more readily compared with available experimental data, customarily reported in terms of stress components. However, an advantage of the formulation with the yield surface in strain space is that it directly leads to a single loading condition and corresponding constitutive equations, regardless of the hardening or softening features observed in the description with the yield surface in stress space.

Acknowledgements

The author is grateful for the support provided by a research grant from the Waterways Experiment Station, U.S. Army Corps of Engineers, and U.S. Department of Energy, Office of Basic Energy Sciences, Division of Engineering and Geosciences, to the Arizona State University, and to the reviewers for their comments and suggestions.

References

Asaro, R. J. (1983) Crystal plasticity. *J. Appl. Mech.* **50,** 921–934.

Casey, J. and Naghdi, P. M. (1981) On the characterization of strain-hardening in plasticity. *J. Appl. Mech.* **48,** 285–296.

Casey, J. and Naghdi, P. M. (1983) On the nonequivalence of the stress and strain space formulations of plasticity theory. *J. Appl. Mech.* **50,** 350–354.

Casey, J. and Naghdi, P. M. (1984) Further constitutive results in finite plasticity. *Q. J. Mech. Appl. Math.* **37,** 231–249.

Green, A. E. and Naghdi, P. M. (1965) A general theory of an elastic–plastic continuum. *Arch. Rat. Mech. Anal.* **18,** 251–281.

Hill, R. (1959) Some basic principles in the mechanics of solids without a natural time. *J. Mech. Phys. Solids* **7,** 209–225.

Hill, R. (1967) On the classical constitutive relations for elastic–plastic solids. *Recent Progress*

in Applied Mechanics, Folke Odqvist Volume (ed. B. BROBERG, J. HULT and F. NIORDSON), pp. 241–249. Almqvist and Wiksell, Stockholm.
HILL, R. (1968) On constitutive inequalities for simple materials—II. *J. Mech. Phys. Solids* **16,** 315–322.
HILL, R. (1978) Aspects of invariance in solid mechanics. *Advances in Applied Mechanics* Vol. 18 (ed. C.-S. YIH), pp. 1–75. Academic Press, New York.
HILL, R. and HAVNER, K. S. (1982) Perspectives in the mechanics of elastoplastic crystals. *J. Mech. Phys. Solids* **30,** 5–22.
HILL, R. and RICE, J. R. (1972) Constitutive analysis of elastic–plastic crystals at arbitrary strain. *J. Mech. Phys. Solids* **20,** 401–413.
HILL, R. and RICE, J. R. (1973) Elastic potentials and the structure of inelastic constitutive laws. *SIAM J. Appl. Math.* **25,** 448–461.
IL'YUSHIN, A. A. (1961) On the postulate of plasticity. *Prikl. Mat. Mekh.* **25,** 503–507.
LEE, E. H. (1969) Elastic–plastic deformation at finite strains. *J. Appl. Mech.* **36,** 1–6.
LUBARDA, V. A. (1991) Constitutive analysis of large elasto-plastic deformation based on the multiplicative decomposition of deformation gradient. *Int. J. Solids Struct.* **27,** 885–895.
LUBARDA, V. A. (1993) An analysis of large-strain damage elastoplasticity. Submitted.
LUBARDA, V. A. and LEE, E. H. (1981) A correct definition of elastic and plastic deformation and its computational significance. *J. Appl. Mech.* **48,** 35–40.
LUBARDA, V. A. and SHIH, C. F. (1994) Plastic spin and related issues in phenomenological plasticity. *J. Appl. Mech.*, in print.
LUBLINER, J. (1986) Normality rules in large-deformation plasticity. *Mech. Mater.* **5,** 29–34.
MANDEL, J. (1973) Equations constitutives et directeurs dans les milieux plastiques et visco-plastiques. *Int. J. Solids Struct.* **9,** 725–740.
MCMEEKING, R. M. and RICE, J. R. (1975) Finite-element formulations for problems of large elastic–plastic deformation. *Int. J. Solids Struct.* **11,** 601–616.
NAGHDI, P. M. (1990) A critical review of the state of finite plasticity. *Z. angew. Math. Phys.* **41,** 315–394.
NAGHDI, P. M. and TRAPP, J. A. (1974) On finite elastic–plastic deformation of metals. *J. Appl. Mech.* **41,** 254–260.
NAGHDI, P. M. and TRAPP, J. A. (1975) The significance of formulating plasticity theory with reference to loading surfaces in strain space. *Int. J. Engng Sci.* **13,** 785–797.
NEMAT-NASSER, S. (1992) Phenomenological theories of elastoplasticity and strain localization at high strain rates. *Appl. Mech. Rev.* **45,** S19–S45.
PALGEN, L. and DRUCKER, D. C. (1983) The structure of stress–strain relations in finite elasto-plasticity. *Int. J. Solids Struct.* **19,** 519–531.
PALMER, A. C., MAIER, G. and DRUCKER, D. C. (1967) Normality relations and convexity of yield surfaces for unstable materials and structural elements. *J. Appl. Mech.* **34,** 464–470.
SIMO, J. C. and ORTIZ, M. (1985) A unified approach to finite deformation elastoplasticity based on the use of hyperelastic relations. *Comput. Meth. Appl. Mech. Engng* **49,** 221–245.
TRUESDELL, C. and NOLL, W. (1965) The nonlinear field theories of mechanics. *Handbuch der Physik*, Band III/3. Springer, Berlin.
YODER, P. J. and IWAN, W. D. (1981) On the formulation of strain space plasticity with multiple loading surfaces. *J. Appl. Mech.* **48,** 773–778.