*Journal of Elasticity* **52:** 289–292, 1999.



# On the Non-Uniqueness of Solution for Screw Dislocations in Multiply Connected Regions

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Received: 14 May 1997; in revised form: 18 December 1998

**Abstract.** The solution for a screw or edge dislocation in multiply connected regions is not unique in the sense that it depends on the cut along which displacement discontinuity is imposed. If the connectivity of the region is $n$, there are that many different solutions. This is demonstrated by considering screw dislocation in a hollow cylinder and near a cavity in an infinite medium.

**Mathematics Subject Classifications (1991):** 73C02 (73C99).

**Key words:** dislocations, cut dependence, multiply-connected regions.

## 1. Introduction

It is well-known that the solution for a straight dislocation in an infinite medium does not depend on the cut along which the displacement discontinuity is imposed. This is also true for a semi-infinite medium, or any simply-connected region. The stress and strain fields, energy and dislocation forces are the same regardless of the cut. Only displacements differ in the region between the two cuts by a rigid-body translation. For multiply connected regions, the solution depends on the cut used to create a dislocation, and if the connectivity of the region is $n$, there are that many possible solutions. This is demonstrated here by considering a screw dislocation in a hollow cylinder and near a cavity in an infinite medium. The consideration for a screw dislocation is simpler than for an edge dislocation, since it only requires appropriately located image dislocations to achieve the traction-free boundary conditions.

## 2. Screw Dislocation in a Hollow Cylinder

The solution for a screw dislocation eccentrically situated in a circular cylinder was given by Eshelby [1]. If the dislocation is at a distance $a$ from the center of a cylinder of radius $R$, an image dislocation of the opposite sign is located at the conjugate position, at the distance $R^2/a$ from the center. If a screw dislocation is in a hollow cylinder, an infinite set of image dislocations is needed to achieve

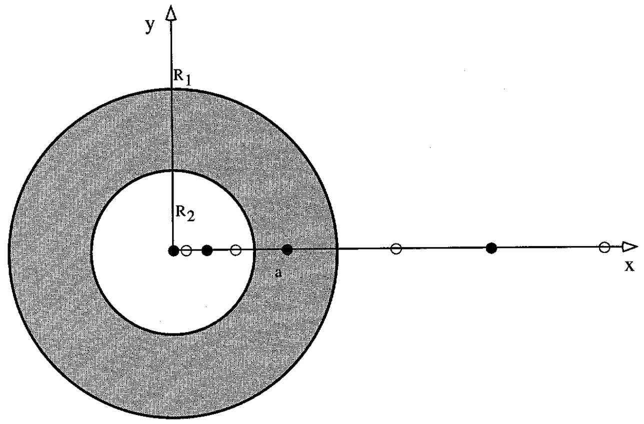


*Figure 1.* Screw dislocation at distance $a$ from the center of a hollow cylinder. Infinitely many image dislocations are needed at conjugate points inside and outside the cylinder.

the traction-free boundary condition at the inner and outer surface of the cylinder. These are located at distance $a_n = (R_1/R_2)^{2n}a$ from the center of cylinder for positive dislocations, and $\lambda a_n$ for negative dislocations, where $\lambda = R_1^2/a^2$ and $n$ runs from minus to plus infinity (Figure 1). The outer radius of the cylinder is $R_1$, and the inner radius is $R_2$. For example, the stress component $\sigma_{zy}$ is

$$\sigma_{zy} = \frac{\mu b_z}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \frac{x - a_n}{(x - a_n)^2 + y^2} - \frac{x - \lambda a_n}{(x - \lambda a_n)^2 + y^2} \right], \tag{1}$$

where $\mu$ is the shear modulus, and $b_z$ is the magnitude of the Burgers vector of the dislocation. In this representation, contributions from image dislocations are taken in pairs of positive and negative dislocations, so that the series clearly converges. This also implies that a dislocation at the position $a$ was introduced by imposing a displacement discontinuity along a cut from $a$ to the outer surface of the cylinder ($R_1$). Thus, the elastic strain energy per unit length of dislocation, excluding the energy of the dislocation core of radius $\rho$, is

$$\begin{aligned} E^{\mathrm{d}} &= \frac{1}{2} \int_{a+\rho}^{R_1} b_z \sigma_{zy}(x, 0)\, \mathrm{d}x \\ &= \frac{\mu b_z^2}{4\pi} \left[ \ln \frac{R_1^2 - a^2}{R_1 \rho} - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \ln \frac{(a - a_n)(R_1 - \lambda a_n)}{(a - \lambda a_n)(R_1 - a_n)} \right]. \end{aligned} \tag{2}$$

This has a maximum value, corresponding to an unstable equilibrium position of the dislocation, at $a = (R_1 R_2)^{1/2}$. However, dislocation at $a$ can also be created by imposing a displacement discontinuity along the cut from a point on the inner surface of the cylinder ($R_2$) to the point $a$. The difference in the two displacement

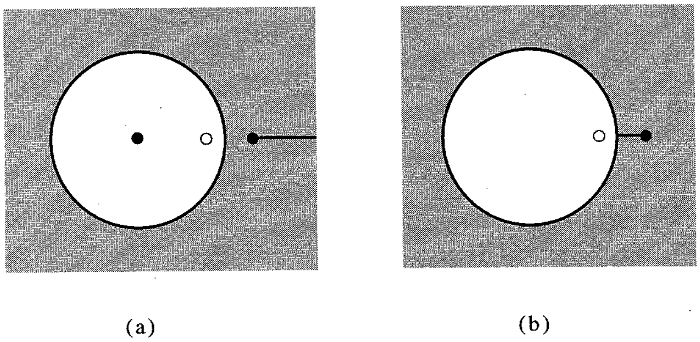


*Figure 2.* Screw dislocation near the cavity in an infinite body. Two solutions are possible depending on the cut along which displacement discontinuity is imposed. In case (a) there are two, and in case (b) there is one image dislocation within the cavity.

fields is a displacement field due to one image dislocation at the origin (outside the physical domain), thus one has different stresses and dislocation forces in the two cases. For example, numerical evaluations reveal that in the second case the dislocation is in an unstable equilibrium at the distance nearly equal to $[(R_1^2 + R_2^2)/2]^{1/2}$ from the origin. This is greater than $(R_1R_2)^{1/2}$, because the attraction of an added negative image dislocation at the origin has to be balanced by the attraction from the outer surface of the cylinder, which occurs if the dislocation is moved toward the outer surface more than in the first case.

## 3. Screw Dislocation Near Cavity in an Infinite Medium

If $R_1 \to \infty$ in the previous section, the screw dislocation is located near the cavity of radius $R_2 = R$ in an infinite medium. In this case, all $a_n$ and $\lambda a_n$ tend to zero for negative $n$, except $\lambda a_{-1} = R^2/a$. Thus, in the limiting process, when all other image dislocations collapse at the center, there is one more positive than negative dislocation there, and we are left with two image dislocations within the cavity (Figure 2(a)). Clearly, this implies a displacement discontinuity along the cut from $a$ to infinity (any cut from $a$ to infinity). The strain energy is infinite, since

$$E^d = \frac{\mu b_z^2}{4\pi} \ln \frac{R_1(a^2 - R^2)}{a^2\rho} \tag{3}$$

becomes infinite as $R_1 \to \infty$. This is also clear since the energy of a dipole of positive and negative dislocation is finite, but the energy of the remaining dislocation in an infinite body is infinite. The dislocation force exerted by the cavity is

$$F_x = -\frac{\partial E^d}{\partial a} = -\frac{\mu b_z^2}{2\pi} \frac{R^2}{a(a^2 - R^2)}, \tag{4}$$

in agreement with [2].

However, if a dislocation is created by imposing a displacement discontinuity along the cut from $R$ to $a$ (any cut from any point on the surface of the cavity to point $a$), rather than from point $a$ to any point at infinity, only one image dislocation is required, at the conjugate position $R^2/a$ (Figure 2(b)). In this case, the strain energy is finite and equal to

$$E^d = \frac{1}{2}\int_R^{a-\rho} b_z \sigma_{zy}(x, 0)\, \mathrm{d}x = \frac{\mu b_z^2}{4\pi} \ln \frac{a^2 - R^2}{R\rho}. \tag{5}$$

The corresponding dislocation force exerted by the cavity is

$$F_x = -\frac{\partial E^d}{\partial a} = -\frac{\mu b_z^2}{2\pi} \frac{a}{a^2 - R^2}. \tag{6}$$

The cavity now exerts a stronger attractive force on the dislocation, the difference being proportional to $1/a$, due to the absence of an alike image dislocation at the center of the cavity. Since the two solutions, one for a screw dislocation created by a cut from $a$ to infinity, and the other for a screw dislocation created by a cut from the surface of the cavity to $a$, differ by a right-hand screw at the origin, the difference in the two displacement fields is not simply a rigid-body translation. This gives different stress fields and different dislocation forces in the two cases. One can view this situation as the nonuniqueness of the solution for screw dislocation in a doubly connected region, in the sense that it depends on the cut along which the displacement discontinuity is imposed. Alternatively, one can consider the two solutions to correspond to two different physical problems. In a mathematical sense, to specify the problem one must either specify the far-field displacement discontinuity as an explicit boundary condition, or, alternatively, specify the value of a Burgers circuit around the cavity. In the case in Figure 2(a), the Burgers circuit around the dislocation at $a$ always gives $b_z$, regardless of whether the circuit encompasses the cavity or not. In the case in Figure 2(b), the Burgers circuit around the dislocation gives $b_z$ only if the circuit does not encompass the cavity. In the case of a screw dislocation between two cavities in an infinite medium, there are three different solutions, corresponding to three types of the cut used to create a dislocation. Details of the calculations are straightforward, and for brevity are omitted here.

## Acknowledgements

The author is grateful for research support provided by the Alcoa Center, and to the reviewer for his comments and suggestions.

## References

1. J.D. Eshelby, Screw dislocations in thin rods. *J. Appl. Phys.* **24** (1953) 176–179.
2. J. Friedel, *Dislocations*. Pergamon Press, Oxford (1964).