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Duality in constitutive formulation of finite-strain elastoplasticity based on $F = F_eF_p$ and $F = F^pF^e$ decompositions

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Abstract

A constitutive theory for large elastic–plastic deformations is presented by employing $\mathbf{F}=\mathbf{F}^p\mathbf{F}^e$ decomposition of the total deformation gradient. A duality in constitutive formulation based on this and the well-known Lee's decomposition $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$ is established for isotropic polycrystalline and single crystal plasticity.

Keywords: Elasto-plasticity; Constitutive equations; Finite strain; Deformation gradient; Multiplicative decomposition; Reversed decomposition

1. Introduction

The multiplicative decomposition of the deformation gradient into its elastic and plastic parts $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$, originally introduced by Lee (1969), has been frequently employed during the past three decades to study the constitutive behavior of elastoplastic polycrystalline materials and single crystals. The decomposition is introduced by defining at each stage of deformation process a stress-free intermediate configuration $\mathcal{B}_p$, obtained from elastoplastically deformed configuration $\mathcal{B}$ by conceptual elastic unloading to zero stress. Since elastic deformation is reversed during this unloading process, the intermediate configuration is deformed only plastically, i.e. it differs from the original (undeformed) configuration $\mathcal{B}_0$ by plastic part of the deformation gradient $\mathbf{F}_p$. Thus, the multiplicative decomposition $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$, where $\mathbf{F}_e$ represents elastic part of the deformation gradient from $\mathcal{B}_p$ to $\mathcal{B}$ (Fig. 1a). The elastoplastic deformation process is, therefore, imagined to take place in two stages. First, there is a plastic flow of material, at zero stress, from the initial configuration $\mathcal{B}_0$ to intermediate configuration $\mathcal{B}_p$, followed by elastic deformation from $\mathcal{B}_p$ to $\mathcal{B}$

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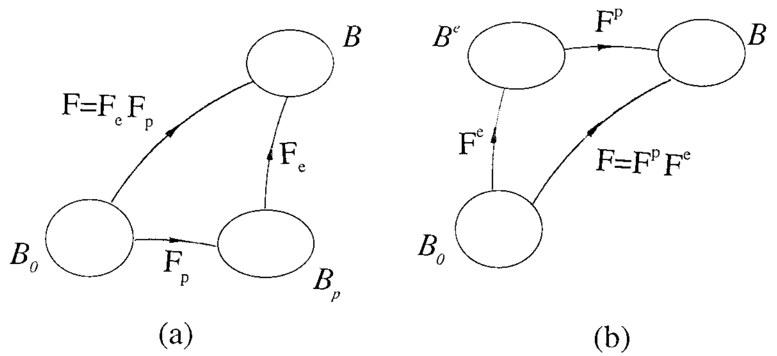

Fig. 1. (a) Intermediate configuration $\mathcal{B}_p$ is obtained from elastoplastically deformed configuration by elastic unloading to zero stress. Lee's decomposition $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$ results. (b) Intermediate configuration $\mathcal{B}^e$ is obtained from initial undeformed configuration by elastic stressing, keeping frozen mechanisms of plastic deformation. Reversed decomposition $\mathbf{F}=\mathbf{F}^p\mathbf{F}^e$ results.

due to total applied stress. The resulting decomposition of deformation gradient, $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$, provided a sound kinematic and kinetic basis for elastoplastic constitutive analysis of polycrystalline materials (e.g., Lubarda and Lee, 1981, Lubarda and Shih, 1994), and single crystals (e.g., Asaro, 1983, Havner, 1985,1992).

In the wake of $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$ decomposition, there have been several suggestions for an alternative, reversed decomposition $\mathbf{F}=\mathrm{F}^p\mathbf{F}^e$ (e.g. Clifton, 1972, Nemat-Nasser, 1979), but this decomposition remained virtually unemployed in subsequent constitutive analysis of elastoplastic behavior. The present paper is a result of my recent study of $\mathbf{F}=\mathrm{F}^p\mathbf{F}^e$ decomposition, which I never attempted to investigate before, being interested in $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$ decomposition only. Interestingly, but perhaps not surprisingly, it turned out that the constitutive analysis of elastoplastic behavior can be developed by using the reversed decomposition, quite analogously to using Lee's decomposition. The two formulations can, thus, be viewed as dual to each other, both leading to the same final structure of constitutive equations, although some of the derivation and interpretations are simpler in the case of Lee's decomposition.

The decomposition $\mathbf{F}=\mathrm{F}^p\mathbf{F}^e$ is introduced as follows. An arbitrary state of elastoplastic deformation, corresponding to deformation gradient $\mathbf{F}$, is again imagined to be reached in two stages. First, it is assumed that all internal mechanisms responsible for plastic deformation are frozen, so that, for example, the critical forces needed to drive dislocations, or critical resolved shear stresses on crystalline slip systems, are assigned infinitely large values. The application of total stress to such material, incapable of plastic deformation, results in pure elastic deformation $\mathbf{F}^e$ that carries the material from initial configuration $\mathcal{B}_0$ to intermediate configuration $\mathcal{B}^e$. Subsequently, the material is plastically unlocked, by unfreezing the mechanisms of plastic deformation, which enables material to flow at constant stress. The corresponding part of the deformation gradient, from intermediate configuration $\mathcal{B}^e$ to final configuration $\mathcal{B}$, is the plastic part of the deformation gradient $\mathbf{F}^p$. Thus, the reversed multiplicative decomposition $\mathbf{F}=\mathrm{F}^p\mathbf{F}^e$ (Fig. 1b).

2. Uniqueness and objectivity issues

The unstressed intermediate configuration $\mathcal{B}_p$ of Lee's decomposition is not unique, because a superimposed rotation $\mathbf{Q}$ leaves this configuration unstressed, i.e.

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p = \hat{\mathbf{F}}_e\hat{\mathbf{F}}_p, \quad \hat{\mathbf{F}}_e = \mathbf{F}_e\mathbf{Q}^T, \quad \hat{\mathbf{F}}_p = \mathbf{Q}\mathbf{F}_p. \tag{1}$$

Superimposed T denotes the transpose. If material is elastically isotropic and remains such in the course of plastic deformation, rotation of the intermediate configuration $\mathcal{B}_p$ has no effect on the stress response. The intermediate configuration can then be specified uniquely by choosing conveniently $\mathbf{F}_e = \mathbf{V}_e$ (the elastic left stretch tensor). Thus, $\mathbf{F} = \mathbf{V}_e\mathbf{F}_p$, which was a basis of Lee's (1969) and Lubarda and Lee's (1981) constitutive analysis.

In the case of reversed decomposition $\mathbf{F} = \mathbf{F}^p\mathbf{F}^e$, the intermediate elastically deformed configuration $\mathcal{B}^e$ is necessarily unique, because a rotation superimposed to it would rotate the stress state, as well, and the plastic flow from $\mathcal{B}^e$ to $\mathcal{B}$ would not be at constant stress any more. Furthermore, if material is elastically isotropic, an initial rotation at $\mathcal{B}_0$ does not affect the stress response, and the relevant part of the total deformation gradient is $\mathbf{F} = \mathbf{F}^p\mathbf{V}_e$. Therefore, for constitutive analysis we can write

$$\mathbf{F} = \mathbf{V}_e\mathbf{F}_p = \mathbf{F}^p\mathbf{V}_e. \tag{2}$$

If material is elastically anisotropic, then, relative to a given orientation of principal directions of elastic anisotropy, there is a unique $\mathbf{F}_e$ that gives rise to total stress in $\mathcal{B}^e$ or $\mathcal{B}$, which is

$$\boldsymbol{\tau} = \mathbf{F}_e\frac{\partial\psi}{\partial\mathbf{E}_e}\mathbf{F}_e^T \tag{3}$$

Here, ψ is the specific strain energy, $\mathbf{E}_e = (1/2)(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{I})$ is the Lagrangean elastic strain relative to its ground state ($\mathcal{B}_p$ in the case of Lee's decomposition and $\mathcal{B}_0$ in the case of reversed decomposition, both having the same orientation of the principal axes of anisotropy relative to fixed frame of reference). The Kirchhoff stress is $\boldsymbol{\tau} = |\mathbf{F}|\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ designates the Cauchy stress, and $||$ the determinant. It is also assumed that plastic flow is incompressible and that it does not affect the elastic properties. Therefore, we now have

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p = \mathbf{F}^p\mathbf{F}_e. \tag{4}$$

Note that the same elastic deformation gradient matrix $\mathbf{F}_e$ appears in both decompositions. The relationship between the two plastic parts of the deformation gradient is consequently

$$\mathbf{F}_p = \mathbf{F}_e^{-1}\mathbf{F}^p\mathbf{F}_e, \tag{5}$$

where -1 designates the inverse.

It is helpful for the subsequent development to briefly discuss objectivity requirements for the constituents of the two decompositions. If a rigid-body rotation $\mathbf{Q}$ is superimposed to elastoplastically deformed configuration $\mathcal{B}$, the deformation gradient $\mathbf{F}$ changes to $\hat{\mathbf{F}} = \mathbf{QF}$, while the Kirchhoff stress becomes $\hat{\boldsymbol{\tau}} = \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T$. Since plastic flow from $\mathcal{B}^e$ to $\mathcal{B}$ occurs at constant stress, $\mathcal{B}^e$ has to be rotated by $\mathbf{Q}$, as well. Consequently, $\hat{\mathbf{F}}^p = \mathbf{QF}^p\mathbf{Q}^T$ and $\hat{\mathbf{F}}_e = \mathbf{QF}_e$. On the other hand, $\boldsymbol{F}_p$ remains unchanged, $\hat{\boldsymbol{F}}_p = \boldsymbol{F}_p$. Moreover, since $\mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e$, the elastic stretch and rotation tensors change to $\hat{\mathbf{V}}_e = \mathbf{QV}_e\mathbf{Q}^T$ and $\hat{\mathbf{R}}_e = \mathbf{QR}_e$. For brevity, the objectivity requirements corresponding to an independent rotation of intermediate configuration $\mathcal{B}_p$ of Lee's decomposition are not discussed here, since they have been examined in detail by Lubarda (1991a) and Lubarda and Shih (1994).

3. Elastic unloading

During elastic loading from $\mathcal{B}_p$ to $\mathcal{B}$, or elastic unloading from $\mathcal{B}$ to $\mathcal{B}_p$, the plastic deformation gradient $\mathbf{F}_p$ of Lee's decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ remains constant. This greatly simplifies derivation of the corresponding constitutive equations. Indeed, the velocity gradient in $\mathcal{B}$ is

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1} + \mathbf{F}_e\left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right)\mathbf{F}_e^{-1}, \tag{6}$$

so that during elastic unloading $\dot{\mathbf{F}}_p = 0$ and $\mathbf{L} = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1}$. However, in the framework of reversed decomposition, $\mathbf{F} = \mathbf{F}^p\mathbf{F}_e$, the plastic part of the deformation gradient $\mathbf{F}^p$ does not remain constant during elastic unloading. In fact, upon complete unloading from an elastoplastic state of deformation to zero stress, the configuration $\mathcal{B}_p$ is reached, and $\mathbf{F}^p = \mathbf{F}_p$ at that instant (Fig. 2). Therefore, $\dot{\mathbf{F}}^p \neq 0$ during elastic unloading. This can also be recognized from the general relationship between $\mathbf{F}^p$ and $\mathbf{F}_p$. By differentiating Eq. (5), we obtain

$$\dot{\mathbf{F}}_p = \mathbf{F}_e^{-1}\overset{*}{\mathbf{F}}{}^p\boldsymbol{F}_e, \tag{7}$$

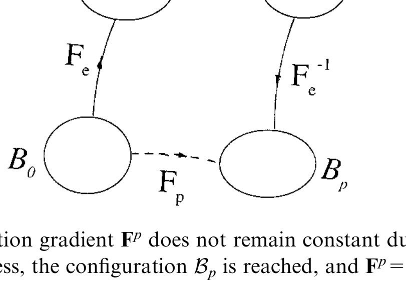

Fig. 2. Plastic part of deformation gradient $\mathbf{F}^p$ does not remain constant during elastic unloading. Upon complete unloading to zero stress, the configuration $\mathcal{B}_p$ is reached, and $\mathbf{F}^p = \mathbf{F}_p$ at that instant.

where

$$\overset{*}{\mathbf{F}}{}^{p} = \dot{\mathbf{F}}^{p} - \left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right)\mathbf{F}^{p} + \mathbf{F}^{p}\left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right) \tag{8}$$

is a convected-type derivative (Hill, 1978) of $\mathbf{F}^p$ relative to elastic deformation. Clearly, from Eq. (7), $\overset{*}{\mathbf{F}}{}^{p} = 0$ if $\dot{\mathbf{F}}_p = 0$, so that in that case

$$\dot{\mathbf{F}}^{p} = \left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right)\mathbf{F}^{p} - \mathbf{F}^{p}\left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right). \tag{9}$$

The last expression defines the change of $\mathbf{F}^p$ during elastic unloading. From Eq. (7), furthermore,

$$\dot{\mathbf{F}}_p\mathbf{F}_p^{-1} = \mathbf{F}_e^{-1}\left(\overset{*}{\mathbf{F}}{}^{p}\mathbf{F}^{p^{-1}}\right)\mathbf{F}_e, \tag{10}$$

and substitution into Eq. (6) gives

$$\mathbf{L} = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1} + \overset{*}{\mathbf{F}}{}^{p}\mathbf{F}^{p^{-1}} \tag{11}$$

This is a dual equation to Eq. (6) of Lee's theory, which will be frequently used in the subsequent analysis.

4. Polycrystalline plasticity

We now elaborate on the constitutive formulation of finite-strain elastoplasticity within the structure of reversed decomposition $\mathbf{F} = \mathbf{F}^p\mathbf{F}_e$. Phenomenological polycrystalline plasticity is first considered. For simplicity, elastic isotropy is assumed throughout the course of deformation. The elastic stress response from $\mathcal{B}_0$ to $\mathcal{B}^e$ is given by Eq. (3) with $\mathbf{F}_e = \mathbf{V}_e$. Thus, upon differentiation

$$\overset{*}{\boldsymbol{\tau}} = \mathbf{K}_e : \left(\dot{\mathbf{V}}_e\mathbf{V}_e^{-1}\right)_s, \quad \Lambda^e_{ijkl} = V^e_{im}V^e_{jn}\frac{\partial^2\psi}{\partial E^e_{mn}\partial E^e_{pq}}V^e_{pk}V^e_{ql}. \tag{12}$$

Here, $\overset{*}{\boldsymbol{\tau}}$ stands for the convected-type derivative of the Kirchhoff stress with respect to elastic deformation

$$\overset{*}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{V}}_e\mathbf{V}_e^{-1}\right)\boldsymbol{\tau} - \boldsymbol{\tau}\left(\dot{\mathbf{V}}_e\mathbf{V}_e^{-1}\right)^T. \tag{13}$$

The trace product in Eq. (12) is denoted by : and the subscript s stands for the symmetric part.

In terms of the Jaumann derivative

$$\dot{\boldsymbol{\tau}} = \overset{*}{\boldsymbol{\tau}} + \left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s \boldsymbol{\tau} + \boldsymbol{\tau}\left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s, \tag{14}$$

Eq. (12) can be rewritten as

$$\dot{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_e : \left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s, \quad \mathcal{L}^e_{ijkl} = \Lambda^e_{ijkl} + \frac{1}{2}\left(\delta_{ik}\tau_{jl} + \delta_{il}\tau_{jk} + \tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk}\right). \tag{15}$$

The components of the unit tensor are the Kronecker δ_{ij}.
By taking symmetric and antisymmetric parts of Eq. (11), we have:

$$\left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s = \mathbf{D} - \left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p-1}\right)_s, \tag{16}$$

$$\left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_a = \mathbf{W} - \left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a. \tag{17}$$

The strain rate and spin tensors in the configuration $\mathcal{B}$ are denoted by $\mathbf{D}$ and $\mathbf{W}$. Hence,

$$\dot{\boldsymbol{\tau}} = \overset{\circ}{\boldsymbol{\tau}} + \left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a \boldsymbol{\tau} - \boldsymbol{\tau}\left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a, \tag{18}$$

where $\overset{\circ}{\boldsymbol{\tau}} = \dot{\tau} - \mathbf{W}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{W}$ is the Jaumann derivative of the Kirchhoff stress with respect to $\mathbf{W}$. Consequently, substitution of Eq. (18) into Eq. (15) gives

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_e : \left\{\left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s - \boldsymbol{\mathcal{M}}_e : \left[\left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a \boldsymbol{\tau} - \boldsymbol{\tau}\left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a\right]\right\}. \tag{19}$$

The fourth-order tensor $\boldsymbol{\mathcal{M}}_e = \boldsymbol{\mathcal{L}}_e^{-1}$ is the instantaneous elastic compliance, the inverse of the elastic stiffness tensor $\boldsymbol{\mathcal{L}}_e$. The elastic strain rate $\mathbf{D}_e$, corresponding to stress rate $\overset{\circ}{\boldsymbol{\tau}}$, is evidently

$$\mathbf{D}_e = \left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s - \boldsymbol{\mathcal{M}}_e : \left[\left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a \boldsymbol{\tau} - \boldsymbol{\tau}\left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a\right]. \tag{20}$$

In view of Eq. (16) and additive decomposition of the strain rate into its elastic and plastic parts, the plastic part of the strain rate becomes

$$\mathbf{D}_p = \left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_s + \boldsymbol{\mathcal{M}}_e : \left[\left(\overset{*}{\mathbf{F}}_p \mathbf{F}^{p^{-1}}\right)_a \boldsymbol{\tau} - \boldsymbol{\tau}\left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p-1}\right)_a\right]. \tag{21}$$

Clearly, during elastic unloading $\mathbf{D}_p = 0$ and $\mathbf{D}_e = (\mathbf{V}_e \mathbf{V}_e^{-1})_s$, since $\overset{*}{\mathbf{F}}{}^p = 0$ in the unloading process.

Further constitutive analysis proceeds as in the case of Lee's decomposition. From Eq. (19), the elastic strain rate, expressed in terms of the stress rate, is

$$\mathbf{D}_e = \boldsymbol{\mathcal{M}}_e : \overset{\circ}{\boldsymbol{\tau}}. \tag{22}$$

This defines a reversible part of the strain increment, recovered upon unloading of the stress increment associated with the Jaumann stress rate $\overset{\circ}{\boldsymbol{\tau}}$. The remaining, plastic part of the strain rate gives a residual part of the strain increment. Under usual assumptions of classical plasticity, this is codirectional with the outward normal to a locally smooth yield surface f in the stress space. In the case of isotropic hardening

$$\mathbf{D}_p = \frac{1}{h}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}\right)\frac{\partial f}{\partial \boldsymbol{\tau}}. \tag{23}$$

where the scalar function h accounts for the plastic deformation history. Thus,

$$\mathbf{D} = \left[\boldsymbol{\mathcal{M}}_e + \frac{1}{h}\left(\frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}}\right)\right] : \overset{\circ}{\boldsymbol{\tau}}, \tag{24}$$

or, by inversion,

$$\overset{\circ}{\boldsymbol{\tau}} = \left[\boldsymbol{\mathcal{L}}_e - \frac{1}{h_0}\boldsymbol{\mathcal{L}}_e : \left(\frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}}\right) : \boldsymbol{\mathcal{L}}_e\right] : \mathbf{D}, \tag{25}$$

where $h_0 = h + (\partial f/\partial \boldsymbol{\tau}){:}\boldsymbol{\mathcal{L}}_e{:}(\partial f/\partial \boldsymbol{\tau})$.

5. Strain rate expressions in terms of $\mathbf{V}_e$, $\mathbf{F}^p$ and their rates

The elastic and plastic strain rates were expressed in terms of the constituents $\mathbf{V}_e$ and $\mathbf{F}_p$ of Lee's decomposition $\mathbf{F} = \mathbf{V}_e\mathbf{F}_p$ by Lubarda and Lee (1981), and in more general context by Lubarda (1991a,b) and Lubarda and Shih (1994). In this section the corresponding relations are derived in the framework of reversed decomposition $\mathbf{F} = \mathbf{F}^p\mathbf{V}_e$. Again, for simplicity, only isotropic response is considered. First, introduce the spin Ω as the solution of the following matrix equation

$$\mathbf{W} = \left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_a + \left(\mathbf{V}_e \Omega \mathbf{V}_e^{-1}\right)_a. \tag{26}$$

The corresponding Jaumann derivatives of the Kirchhoff stress $\boldsymbol{\tau}$ and the elastic stretch $\mathbf{V}_e$ are:

$$\overset{\nabla}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \Omega\boldsymbol{\tau} + \boldsymbol{\tau}\Omega, \quad \overset{\nabla}{\mathbf{V}}_e = \dot{\mathbf{V}}_e - \Omega\mathbf{V}_e + \mathbf{V}_e\Omega. \tag{27}$$

Applying these derivatives to Eq. (3), with $\mathbf{F}_e = \mathbf{V}_e$, there follows

$$\overset{\nabla}{\tau} = \left(\overset{\nabla}{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)\boldsymbol{\tau} + \boldsymbol{\tau}\left(\overset{\nabla}{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)^T + \Lambda_e : \left(\overset{\nabla}{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s. \tag{28}$$

In view of Eqs. (26) and $(27)_2$, the relationship holds

$$\overset{\nabla}{\mathbf{V}}_e \mathbf{V}_e^{-1} = \left(\overset{\nabla}{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s + \mathbf{W} - \Omega, \tag{29}$$

and substitution into Eq. (28) gives

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_e : \left(\overset{\nabla}{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s. \tag{30}$$

This identifies the elastic strain rate as

$$\mathbf{D}_e = \left(\overset{\nabla}{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s = \left(\dot{\mathbf{V}}_e \mathbf{V}_e^{-1}\right)_s + \left(\mathbf{V}_e \Omega \mathbf{V}_e^{-1}\right)_s. \tag{31}$$

On the other hand, comparing Eqs. (17) and (26) yields

$$\left(\mathbf{V}_e \Omega \mathbf{V}_e^{-1}\right)_a = \left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_a. \tag{32}$$

Therefore, from Eqs. (16) and (31) the plastic strain rate is given by

$$\mathbf{D}_p = \mathbf{D} - \mathbf{D}_e = \left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)_s - \left(\mathbf{V}_e \Omega \mathbf{V}_e^{-1}\right)_s. \tag{33}$$

This is identically equal to

$$\mathbf{D}_p = \overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}} - \mathbf{V}_e \Omega \mathbf{V}_e^{-1}, \tag{34}$$

since antisymmetric part of the difference on the right-hand side of Eq. (34) vanishes by Eq. (32).

If hardening is isotropic, the principal directions of the stress $\boldsymbol{\tau}$ and the elastic stretch $\mathbf{V}_e$ are parallel to those of the plastic strain rate $\mathbf{D}_p$. Thus by multiplying Eq. (34) by $\mathbf{V}_e^{-1}$ from the left, and by $\mathbf{V}_e$ from the right, it follows

$$\mathbf{D}_p = \mathbf{V}_e^{-1}\left(\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}}\right)\mathbf{V}_e - \Omega. \tag{35}$$

Taking symmetric and antisymmetric parts of Eq. (35) finally gives:

$$\mathbf{D}_p = \left[\mathbf{V}_e^{-1}\left(\overset{*}{\mathbf{F}}{}^p\mathbf{F}^{p^{-1}}\right)\mathbf{V}_e\right]_s, \tag{36}$$

$$\Omega = \left[\mathbf{V}_e^{-1}\left(\overset{*}{\mathbf{F}}{}^p\mathbf{F}^{p^{-1}}\right)\mathbf{V}_e\right]_a. \tag{37}$$

These expressions are in complete agreement with the original results obtained by Lubarda and Lee (1981) in the framework of Lee's decomposition $\mathbf{F}=\mathbf{V}_e\mathbf{F}_p$. Indeed, since $\mathbf{F}_p=\mathbf{V}_e^{-1}\mathbf{F}^p\mathbf{V}_e$, one has

$$\mathbf{V}_e^{-1}\left(\overset{*}{\mathbf{F}}{}^p\mathbf{F}^{p^{-1}}\right)\mathbf{V}_e = \dot{\mathbf{F}}_p\mathbf{F}_p^{-1}, \tag{38}$$

and Eqs. (36) and (37) reduce to Lubarda and Lee's results

$$\mathbf{D}_p = \left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right)_s, \quad \Omega = \left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right)_a. \tag{39}$$

This demonstrates a duality in constitutive formulations for isotropic elastoplasticity based on the two alternative decompositions $\mathbf{F}=\mathbf{V}_e\mathbf{F}_p$ and $\mathbf{F}=\mathbf{F}^p\mathbf{V}_e$.

6. Single crystal plasticity

We next consider the constitutive formulation for large elastoplastic deformations of single crystals, in which crystallographic slip is assumed to be the only mechanism of plastic deformation. The reverse multiplicative decomposition $\mathbf{F}=\mathbf{F}^p\mathbf{F}_e$ is employed to compare the formulation with the well-known formulation based on the decomposition $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$ (e.g. Asaro, 1983, where the notation $\mathbf{F}=\mathbf{F}^*\mathbf{F}^p$ is used). The total stress is applied to initial configuration $\mathcal{B}_0$, considering that crystallographic slip is prevented by assigning infinite value to critical resolved shear stress on each slip system. The resulting deformation $\mathbf{F}_e$ is purely elastic, and caused by lattice stretching and rotation that carries $\mathcal{B}_0$ into intermediate configuration $\mathcal{B}^e$. Subsequently, the critical resolved shear stresses are relaxed to their actual finite values, which enables material to flow through the crystalline lattice at constant stress. This part of deformation gradient is the plastic part $\mathbf{F}^p$, and the decomposition $\mathbf{F}=\mathbf{F}^p\mathbf{F}_e$ holds. Denote the unit slip direction and the normal to the corresponding slip plane in the undeformed configuration $\mathcal{B}_0$ by $\mathbf{s}_0^\alpha$ and $\mathbf{m}_0^\alpha$, where α designates the slip system. The vector $\mathbf{s}_0^\alpha$ is embedded in the lattice, so that it becomes $\mathbf{s}^\alpha=\mathbf{F}_e\cdot\mathbf{s}_0^\alpha$ in the intermediate configuration $\mathcal{B}^e$. Since the material flows through the lattice during the deformation $\mathbf{F}^p$, the slip direction $\mathbf{s}^\alpha$ remains the same in the final configuration $\mathcal{B}$ (Fig. 3). The normal to the slip plane in the configurations $\mathcal{B}^e$ and $\mathcal{B}$ is defined by the reciprocal vector $\mathbf{m}^\alpha=\mathbf{m}_0^\alpha\cdot\mathbf{F}_e^{-1}$. In general, $\mathbf{s}^\alpha$ and $\mathbf{m}^\alpha$ are not unit vectors, but are orthogonal to each other.

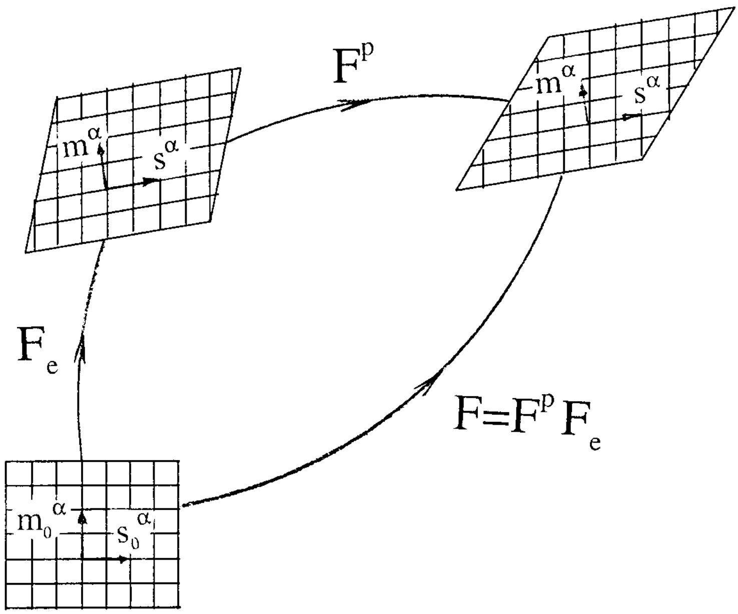

Fig. 3. Unit slip direction and unit normal to slip plane in undeformed configuration $\mathcal{B}_0$ are $\mathbf{s}_0^\alpha$ and $\mathbf{m}_0^\alpha$. The vector $\mathbf{s}_0^\alpha$ is embedded in the lattice, becoming $\mathbf{s}^\alpha = \mathbf{F}_e{\cdot}\mathbf{s}_0^\alpha$ in the intermediate configuration $\mathcal{B}^e$. As material flows through the lattice during deformation $\mathbf{F}^p$, the slip direction $\mathbf{s}^\alpha$ remains the same.

The velocity gradient in the deformed configuration $\mathcal{B}$ is a consequence of the slip rates $\dot{\gamma}^\alpha$ over n active slip systems, such that

$$\overset{*}{\mathbf{F}}{}^p \mathbf{F}^{p^{-1}} = \sum_{\alpha=1}^{n} (\mathbf{s}^\alpha \otimes \boldsymbol{m}^\alpha)\dot{\gamma}^\alpha, \tag{40}$$

where $\otimes$ designates a dyadic product, and $\overset{*}{\mathbf{F}}{}^p$ is a convected-type derivative defined by Eq. (8). This convected rate of $\mathbf{F}^p$ appears in Eq. (40) because, during elastoplastic increment of deformation, both the current and intermediate configurations, with respect to which plastic deformation gradient $\mathbf{F}^p$ is defined, deform. Furthermore, as discussed in Section 3, during elastic unloading from the current configuration, $\overset{*}{\mathbf{F}}{}^p = 0$, so that Eq. (40) is in accord with the requirement that during elastic unloading the slip rates $\dot{\gamma}^\alpha$ must also be equal to zero. In view of Eq. (10), Eq. (40) is equivalent to the corresponding equation from the formulation based on $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ decomposition,where the point of departure is an expression for $\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}$ in terms of the sum of slip contributions $(\mathbf{s}_0^\alpha \otimes \mathbf{m}_0^\alpha)\dot{\gamma}^\alpha$ in the intermediate configuration $\mathcal{B}_p$ (Asaro, 1983).

The elastic response of the lattice and the material embedded on it is described by Eq. (3). It is assumed that crystallographic slip is an isochoric deformation process, and that elastic properties of the crystal are unaffected by the slip. Thus,

$$\dot{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_e : \left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right)_s, \quad \dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right)_a\boldsymbol{\tau} + \boldsymbol{\tau}\left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right)_a. \tag{41}$$

The components of the instantaneous elastic stiffness tensor of the crystal are

$$\mathcal{L}_{ijkl} = F^e_{im}F^e_{jn}\frac{\partial^2\psi}{\partial E^e_{mn}\partial E^e_{pq}}F^e_{kp}F^e_{lq} + \frac{1}{2}\left(\delta_{ik}\tau_{jl} + \delta_{il}\tau_{jk} + \tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk}\right). \tag{42}$$

The strain rate and spin tensors can be expressed from Eqs. (11) and (40) as:

$$\mathbf{D} = \left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right)_s + \sum_{\alpha=1}^{n}\mathbf{P}^\alpha\dot{\gamma}^\alpha, \tag{43}$$

$$\mathbf{W} = \left(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}\right)_a + \sum_{\alpha=1}^{n}\mathbf{W}^\alpha\dot{\gamma}^\alpha, \tag{44}$$

where the second-order tensors are introduced:

$$\mathbf{P}^\alpha = \frac{1}{2}(\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha + \mathbf{m}^\alpha \otimes \mathbf{s}^\alpha), \quad \mathbf{W}^\alpha = \frac{1}{2}(\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha - \mathbf{m}^\alpha \otimes \mathbf{s}^\alpha), \tag{45}$$

as commonly done in crystal plasticity.

Further analysis proceeds identically as in the case of decomposition $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$. Since

$$\dot{\boldsymbol{\tau}} = \overset{\circ}{\boldsymbol{\tau}} + \sum_{\alpha=1}^{n}(\mathbf{W}^\alpha\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W}^\alpha)\dot{\gamma}^\alpha, \tag{46}$$

Eq. (41) becomes

$$\dot{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_e : \left\{\mathbf{D} - \sum_{\alpha=1}^{n}[\mathbf{P}^\alpha + \boldsymbol{\mathcal{M}}_e : (\mathbf{W}^\alpha\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W}^\alpha)]\dot{\gamma}^\alpha\right\}, \tag{47}$$

which defines the plastic strain rate corresponding to $\overset{\circ}{\boldsymbol{\tau}}$ as

$$\mathbf{D}_p = \sum_{\alpha=1}^{n}[\mathbf{P}^\alpha + \boldsymbol{\mathcal{M}}_e : (\mathbf{W}^\alpha\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W}^\alpha)]\dot{\gamma}^\alpha. \tag{48}$$

Physically, $\mathbf{D}_p$ yields the residual strain increment left in the crystal upon an infinitesimal loading/unloading cycle associated with the stress rate $\overset{\circ}{\boldsymbol{\tau}}$. The reversible part is the elastic part of the strain rates, $\mathbf{D}_e=\boldsymbol{\mathcal{M}}_e:\overset{\circ}{\boldsymbol{\tau}}$. The structure of the strain rate expressions corresponding to other choices of the stress rates has been discussed by Lubarda (1994a,1999).

Eq. (47) can be rewritten as

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_e : \mathbf{D} - \sum_{\alpha=1}^{n} \mathbf{K}^{\alpha} \dot{\gamma}^{\alpha}, \quad \mathbf{K}^{\alpha} = \boldsymbol{\mathcal{L}}_e : \mathbf{P}^{\alpha} + \mathbf{W}^{\alpha} \boldsymbol{\tau} - \boldsymbol{\tau} \mathbf{W}^{\alpha}. \tag{49}$$

To complete the constitutive formulation, the slip rates $\dot{\gamma}^{\alpha}$ have to be expressed in terms of the rate of deformation or the rate of stress. This is done in a standard manner. For rate-independent materials it is assumed that plastic flow occurs on a slip system when the resolved shear stress on that system reaches the critical value ($\tau^{\alpha} = \tau^{\alpha}_{cr}$). The generalized Schmid stress τ^{α} can be defined as the work conjugate to the slip rate $\dot{\gamma}^{\alpha}$, such that (Hill and Rice, 1972; Hill and Havner, 1982; Asaro, 1983)

$$\sum_{\alpha=1}^{n} \tau^{\alpha} \dot{\gamma}^{\alpha} = \boldsymbol{\tau} : \sum_{\alpha=1}^{n} \mathbf{P}^{\alpha} \dot{\gamma}^{\alpha}. \tag{50}$$

Therefore, $\tau^{\alpha} = \mathbf{P}^{\alpha} : \boldsymbol{\tau}$, its rate being

$$\dot{\tau}^{\alpha} = \mathbf{K}^{\alpha} : \left(\dot{\mathbf{F}}_e \mathbf{F}_e^{-1} \right)_s. \tag{51}$$

If the rate of change of the critical resolved shear stress on a given slip system is defined as a linear combination of the slip rates $\dot{\gamma}^{\alpha}$, the coefficients being the slip-plane hardening rates $h_{\alpha\beta}$, the consistency condition for the slip on the system α is

$$\dot{\tau}^{\alpha} = \sum_{\beta=1}^{n} h_{\alpha\beta} \dot{\gamma}^{\beta}. \tag{52}$$

In view of Eqs. (51) and (43), this becomes

$$\mathbf{K}^{\alpha} : \mathbf{D} = \sum_{\beta=1}^{n} g_{\alpha\beta} \dot{\gamma}^{\beta}, \quad g_{\alpha\beta} = h_{\alpha\beta} + \mathbf{K}^{\alpha} : \mathbf{P}^{\beta}. \tag{53}$$

Thus, the slip rates can be determined from

$$\dot{\gamma}^{\alpha} = \sum_{\beta=1}^{n} g^{-1}_{\alpha\beta} \mathbf{K}^{\beta} : \mathbf{D}, \tag{54}$$

provided that the matrix with components $g_{\alpha\beta}$ is positive-definite (Hill and Rice, 1972). Substitution of Eq. (54) into Eq. (49) finally gives

$$\overset{\circ}{\boldsymbol{\tau}} = \left(\boldsymbol{\mathcal{L}}_e - \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \mathbf{K}^{\alpha} g^{-1}_{\alpha\beta} \mathbf{K}^{\beta} \right) : \mathbf{D}, \tag{55}$$

which is the same constitutive structure of single crystal plasticity as presented by Asaro (1983) in the framework of the decomposition $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$. For prescribed components of velocity gradient, the slip rates are determined from Eq. (54), and $\mathbf{F}_e$ by integration from Eq. (11) with the incorporated Eq. (40). The plastic deformation gradient is then $\mathbf{F}^p = \mathbf{F}\mathbf{F}_e^{-1}$.

7. Conclusion

We have demonstrated in this paper a duality in constitutive formulation of large-deformation elastoplasticity based on Lee's decomposition $\mathbf{F}=\mathbf{F}_e\mathbf{F}_p$ and reversed decomposition $\mathbf{F}=\mathbf{F}^p\mathbf{F}_e$, at least for the constitutive models considered in this paper. It is shown that for materials that preserve elastic properties during plastic deformation, the same elastic deformation gradient $\mathbf{F}_e$ appears in both decompositions, while plastic deformation gradients $\mathbf{F}_p$ and $\mathbf{F}^p$ naturally differ. Expressions for elastic and plastic strain rates, derived in the framework of reversed decomposition, are given by Eqs. (20) and (21), and Eqs. (31) and (36). A kinematic representation of the velocity gradient in Eq. (11) was used in this derivation. The structure of the obtained expressions is found to be more involved than of those based on Lee's decomposition, cf. Eqs. (36) and (39). This is partly because during elastic unloading the plastic gradient $\mathbf{F}_p$ of Lee's decomposition remains constant, while $\mathbf{F}^p$ of reversed decomposition changes, albeit in a definite manner specified by Eq. (9). Nonetheless, within the same physical ingredients, both formulations lead to the same final structure of elastoplastic constitutive equations for isotropic polycrystalline materials and single crystals. These are given by Eqs. (25) and (55).

In addition to conceptual importance of demonstrating a duality, the analysis presented in this paper may be of interest because it is possible that in some applications the reversed decomposition has certain advantages. For example, Clifton (1972) found it to be slightly more convenient for the analysis of one-dimensional wave propagation in elastic-viscoplastic solids. It should, however, be pointed out that Lee's decomposition has definite advantages in modeling plasticity with evolving elastic properties. In this case, a set of structural tensors can be attached to the intermediate configuration $\mathcal{B}_p$ to represent its current state of elastic anisotropy. The structural tensors evolve during plastic deformation, depending on the nature of microscopic inelastic processes, their change being represented by appropriate evolution equations. The stress response at each instant of deformation is given in terms of the gradient of elastic strain energy with respect to elastic strain, at the current values of the structural tensors. This type of analysis was used by Lubarda (1994b) and Lubarda and Krajcinovic (1995) in their study of damage-elastoplasticity. In the case of reversed decomposition, however, the elastic response is defined relative to the initial configuration $\mathcal{B}_0$, which does not contain any information about evolving elastic properties or subsequently developed elastic anisotropy. Additional remedy has to be introduced to deal with these features of material response, which is likely to make the reversed decomposition less attractive than the original decomposition.

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References

Asaro, R.J., 1983. Crystal plasticity. J. Appl. Mech. 50, 921.

Clifton, R.J., 1972. On the equivalence of $\mathbf{F}^e\mathbf{F}^p$ and $\bar{\mathbf{F}}^p\bar{\mathbf{F}}^e$. J. Appl. Mech. 39, 288.

Havner, K.S., 1985. Comparisons of crystal hardening laws in multiple slip. Int. J. Plasticity 2, 111.

Havner, K.S., 1985. Finite plastic deformation of crystalline solids. Int. J. Plasticity 2, 111.

Hill, R., 1978. Aspects of invariance in solid mechanics. In: Yih, C.-S. (Ed.), Advances in Applied Mechanics. Academic Press, New York, pp. 1–75.

Hill, R., Havner, K.S., 1982. Perspectives in the mechanics of elastoplastic crystals. J. Mech. Phys. Solids 30, 5.

Hill, R., Rice, J.R., 1972. Constitutive analysis of elastic-plastic crystals at arbitrary strain. J. Mech. Phys. Solids 20, 401.

Lee, E.H., 1969. Elastic-plastic deformation at finite strains. J. Appl. Mech. 36, 1.

Lubarda, V.A., 1991a. Constitutive analysis of large elasto-plastic deformation based on the multiplicative decomposition of deformation gradient. Int. J. Solids Struct. 27, 885.

Lubarda, V.A., 1991b. Some aspects of elasto-plastic constitutive analysis of elastically anisotropic materials. Int. J. Plasticity 7, 625.

Lubarda, V.A., 1994a. Elastoplastic constitutive analysis with the yield surface in strain space. J. Mech. Phys. Solids 42, 931.

Lubarda, V.A., 1994b. An analysis of large-strain damage elastoplasticity. Int. J. Solids Struct. 31, 2951.

Lubarda, V.A., 1999. On the partition of rate of deformation in crystal plasticity. Int. J. Plasticity 15, 721.

Lubarda, V.A., Krajcinovic, D., 1995. Some fundamental issues in rate theory of damage-elastoplasticity. Int. J. Plasticity 11, 763.

Lubarda, V.A., Lee, E.H., 1981. A correct definition of elastic and plastic deformation and its computational significance. J. Appl. Mech. 48, 35.

Lubarda, V.A., Shih, C.F., 1994. Plastic spin and related issues in phenomenological plasticity. J. Appl. Mech. 61, 524.

Nemat-Nasser, S., 1979. Decomposition of strain measures and their rates in finite deformation elastoplasticity. Int. J. Solids Struct. 15, 155.