PERGAMON

International Journal of Plasticity 15 (1999) 721–736

INTERNATIONAL JOURNAL OF Plasticity

On the partition of rate of deformation in crystal plasticity

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Received in revised form 18 January 1999

Abstract

Two different partitions of the rate of deformation tensor into its elastic and plastic parts are derived for elastic–plastic crystals in which crystallographic slip is the only cause of plastic deformation. One partition is associated with the Jaumann, and the other with convected rate of the Kirchhoff stress. Different expressions for the plastic part of the rate of deformation are obtained, and corresponding constitutive inequalities discussed. Relationship with the plastic part of the rate of the Lagrangian strain is also given.

Keywords: B. Crystal plasticity; Elastic-plastic material

1. Introduction

What is left as a residual increment of plastic deformation upon an infinitesimal cycle of stress depends on the employed measures of stress and strain. Thus, there are infinitely many ways to partition the rate of deformation tensor into its elastic and plastic parts. The general framework for the partition, corresponding to any measure of the strain and its work-conjugate stress, was developed for elastic–plastic crystals at finite strain by Hill and Rice (1972). By using this theory, we have derived the plastic parts of the rate of deformation tensor corresponding to the Jaumann and convected rates of the Kirchhoff stress. The first one is well-known, but the second one less so. The differences between the two quantities are examined, and the connection with the plastic part of the rate of the Lagrangian strain is given. Some

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PII: S0749-6419(99)00014-5

new kinematic and kinetic relationships are derived in the latter case, in addition to those previously obtained by Hill and Havner (1982). The framework of rate-independent crystal plasticity is assumed, in which crystallographic slip is the sole mechanism of plastic deformation. The plastic stress and strain rates are expressed in terms of the slip rates, the total rate of deformation, and the stress rates. The corresponding constitutive inequalities are discussed. The present work is an extension of an earlier study by the author (Lubarda, 1994), given in the framework of phenomenological plasticity, and based on the multiplicative decomposition of the deformation gradient into its elastic and plastic parts (Lee, 1969).

2. Kinematic preliminaries

The kinematic representation of elastic–plastic deformation of single crystals, in which crystallographic slip is assumed to be the only cause of plastic deformation, is described by Asaro (1983). The material flows through the crystal lattice via dislocation motion, while the lattice itself, with the material embedded to it, undergoes elastic deformation and rotation. The total deformation gradient $\mathbf{F}$ is decomposed as

$$\mathbf{F} = \mathbf{F}_* \mathbf{F}_p, \tag{1}$$

where $\mathbf{F}_p$ is the part of $\mathbf{F}$ due to slip only, while $\mathbf{F}_*$ is due to lattice stretching and rotation. The following Lagrangian strains can be introduced:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \qquad \mathbf{E}_p = \frac{1}{2}(\mathbf{F}_p^T\mathbf{F}_p - \mathbf{I}), \qquad \mathbf{E}_* = \frac{1}{2}(\mathbf{F}_*^T\mathbf{F}_* - \mathbf{I}), \tag{2}$$

where $\mathbf{I}$ is the identity tensor. The relationship holds

$$\mathbf{E} = \mathbf{F}_p^T\mathbf{E}_*\mathbf{F}_p + \mathbf{E}_p. \tag{3}$$

By introducing the decomposition (1), the velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ can be expressed as

$$\mathbf{L} = \dot{\mathbf{F}}_*\mathbf{F}_*^{-1} + \mathbf{F}_*(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_*^{-1}, \tag{4}$$

while the rate of the Lagrangian strain is

$$\dot{\mathbf{E}} = \mathbf{F}_p^T\dot{\mathbf{E}}_*\mathbf{F}_p + \frac{1}{2}\mathbf{F}_p^T\Big[\mathbf{C}_*(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}) + (\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})^T\mathbf{C}_*\Big]\mathbf{F}_p, \tag{5}$$

where $\mathbf{C}_* = \mathbf{F}_*^T\mathbf{F}_*$.

Denote the slip direction and the normal to the corresponding slip plane in the undeformed configuration by $\mathbf{s}_0^\alpha$ and $\mathbf{m}_0^\alpha$, where α designates the slip system. The vector $\mathbf{s}^\alpha$ is embedded in the lattice, so that it becomes $\mathbf{s}^\alpha = \mathbf{F}_* \cdot \mathbf{s}_0^\alpha$ in the deformed configuration. The normal to the slip plane in the deformed configuration is defined by the reciprocal vector $\mathbf{m}^\alpha = \mathbf{m}_0^\alpha \cdot \mathbf{F}_*^{-1}$. The velocity gradient in the intermediate configuration is a consequence of the slip rates γ^α over n active slip systems, such that

$$\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \sum_{\alpha=1}^{n} \dot{\gamma}^\alpha \mathbf{s}_0^\alpha \otimes \mathbf{m}_0^\alpha \tag{6}$$

where $\otimes$ designates a dyadic product. The corresponding tensor in the deformed configuration is

$$\mathbf{F}_* (\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}) \mathbf{F}_*^{-1} = \sum_{\alpha=1}^{n} \dot{\gamma}^\alpha \mathbf{s}^\alpha \otimes \mathbf{m}^\alpha = \sum_{\alpha=1}^{n} (\mathbf{P}^\alpha + \mathbf{W}^\alpha) \dot{\gamma}^\alpha. \tag{7}$$

For convenience, the second-order tensors are introduced:

$$\mathbf{P}^\alpha = \frac{1}{2} (\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha + \mathbf{m}^\alpha \otimes \mathbf{s}^\alpha), \qquad \mathbf{W}^\alpha = \frac{1}{2} (\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha - \mathbf{m}^\alpha \otimes \mathbf{s}^\alpha), \tag{8}$$

as commonly done in crystal plasticity (e.g. Asaro, 1983).

The rate of the Lagrangian strain is

$$\dot{\mathbf{E}} = \mathbf{F}_p^T \dot{\mathbf{E}}_* \mathbf{F}_p + \sum_{\alpha=1}^{n} \mathbf{P}_0^\alpha \dot{\gamma}^\alpha, \tag{9}$$

with the symmetric second-order tensor $\mathbf{P}_0^\alpha$, dual to the tensor $\mathbf{P}^\alpha$, defined by

$$\mathbf{P}_0^\alpha = \frac{1}{2} \mathbf{F}_p^T \big[\mathbf{C}_* (\mathbf{s}_0^\alpha \otimes \mathbf{m}_0^\alpha) + (\mathbf{m}_0^\alpha \otimes \mathbf{s}_0^\alpha) \mathbf{C}_* \big] \mathbf{F}_p. \tag{10}$$

It can be easily verified that the connection holds

$$\mathbf{P}_0^\alpha = \mathbf{F}^T \mathbf{P}^\alpha \mathbf{F}. \tag{11}$$

For the later purposes, it is useful to rewrite the tensor $\mathbf{P}_0^\alpha$ in the following equivalent form

$$\mathbf{P}_0^\alpha = (\mathbf{E} - \mathbf{E}_p) \mathbf{B}_0^\alpha + \mathbf{B}_0^{\alpha T} (\mathbf{E} - \mathbf{E}_p) + \frac{1}{2} \mathbf{F}_p^T (\mathbf{s}_0^\alpha \otimes \mathbf{m}_0^\alpha + \mathbf{m}_0^\alpha \otimes \mathbf{s}_0^\alpha) \mathbf{F}_p, \tag{12}$$

where

$$\mathbf{B}_0^\alpha = \mathbf{F}_p^{-1} (\mathbf{s}_0^\alpha \otimes \mathbf{m}_0^\alpha) \mathbf{F}_p. \tag{13}$$

The second-order tensors $\mathbf{P}_0^\alpha$ and $\mathbf{B}_0^\alpha$ were originally introduced by Hill and Havner (1982) ($\tilde{\nu}$ and $\tilde{\mathbf{C}}$ in their notation).

3. Kinetic preliminaries

It will be assumed that elastic properties of the crystal are unaffected by the crystallographic slip. Since slip is an isochoric deformation process, the elastic strain energy per unit initial volume can be written as

$$\psi = \psi(\mathbf{E}_*) = \psi\Big[\mathbf{F}_p^{-T}(\mathbf{E} - \mathbf{E}_p)\mathbf{F}_p^{-1}\Big], \tag{14}$$

The symmetric Piola–Kirchoff stress tensors with respect to lattice and total deformation are:

$$\mathbf{S}_* = \frac{\partial\psi}{\partial\mathbf{E}_*}, \qquad \mathbf{S} = \frac{\partial\psi}{\partial\mathbf{E}}. \tag{15}$$

The two stresses and their rates are related to each other by:

$$\mathbf{S}_* = \mathbf{F}_p\mathbf{S}\mathbf{F}_p^T, \tag{16}$$

$$\dot{\mathbf{S}}_* = \mathbf{F}_p\dot{\mathbf{S}}\mathbf{F}_p^T + (\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_p\mathbf{S}\mathbf{F}_p^T + \mathbf{F}_p\mathbf{S}\mathbf{F}_p^T(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})^T, \tag{17}$$

In terms of the Kirchhoff stress $\tau = (\det\mathbf{F})\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ denotes the Cauchy stress, the relationships are:

$$\mathbf{S}_* = \mathbf{F}_*^{-1}\tau\mathbf{F}_*^{-T}, \qquad \mathbf{S} = \mathbf{F}^{-1}\tau\mathbf{F}^{-T}, \tag{18}$$

$$\dot{\mathbf{S}}_* = \mathbf{F}_*^{-1}\overset{\nabla *}{\boldsymbol{\tau}}\mathbf{F}_*^{-T}, \qquad \dot{\mathbf{S}} = \mathbf{F}^{-1}\overset{\nabla}{\boldsymbol{\tau}}\mathbf{F}^{-T}, \tag{19}$$

The convected derivatives of the Kirchhoff stress with respect to lattice and total deformation are:

$$\overset{\nabla *}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{L}_*\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{L}_*^T, \qquad \overset{\nabla}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{L}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{L}^T, \tag{20}$$

where $\mathbf{L}_* = \dot{\mathbf{F}}_*\mathbf{F}_*^{-1}$. By using Eq. (7), the two rates can be related by

$$\overset{\nabla *}{\boldsymbol{\tau}} = \overset{\nabla}{\boldsymbol{\tau}} + \sum_{\alpha=1}^{n}[(\mathbf{P}^\alpha + \mathbf{W}^\alpha)\boldsymbol{\tau} + \boldsymbol{\tau}(\mathbf{P}^\alpha - \mathbf{W}^\alpha)]\dot{\gamma}^\alpha, \tag{21}$$

Similarly, for the rates of the Piola–Kirchhoff stresses we have

$$\mathbf{F}_p^{-1}\dot{\mathbf{S}}_*\mathbf{F}_p^{-T} = \dot{\mathbf{S}} + \sum_{\alpha=1}^{n}(\mathbf{B}_0^\alpha\mathbf{S} + \mathbf{S}\mathbf{B}_0^{\alpha T})\dot{\gamma}^\alpha. \tag{22}$$

If the response is purely elastic ($\dot{\mathbf{F}}_p = 0$), the following rate-type elasticity equations hold:

$$\dot{\mathbf{S}}_* = \mathcal{L}_* : \dot{\mathbf{E}}_*, \qquad \dot{\mathbf{S}} = \mathcal{L}_0 : \dot{\mathbf{E}}, \qquad \overset{\nabla *}{\boldsymbol{\tau}} = \mathcal{L} : \mathbf{D}_*. \tag{23}$$

The trace product is denoted by (:), and the lattice rate of deformation $\mathbf{D}_*$ is the symmetric part of $\mathbf{L}_*$. The instantaneous elastic stiffness tensors are:

$$\mathcal{L}_* = \frac{\partial^2\psi}{\partial\mathbf{E}_*\otimes\partial\mathbf{E}_*}, \qquad \mathcal{L}_0 = \frac{\partial^2\psi}{\partial\mathbf{E}\otimes\partial\mathbf{E}}. \tag{24}$$

The easily established connections between the stiffness tensors are:

$$\mathcal{L} = \mathbf{F}_*\mathbf{F}_*\mathcal{L}_*\mathbf{F}_*^T\mathbf{T}_*^T, \qquad \mathcal{L}_0 = \mathbf{F}^{-1}\mathbf{F}^{-1}\mathcal{L}\mathbf{F}^{-T}\mathbf{F}^{-T}, \tag{25}$$

The products in Eq. (25) are defined such that, for example,

$$\mathcal{L}_{ijkl} = F_{im}^*F_{jn}^*\mathcal{L}_{mnpq}^*F_{pk}^{*T}F_{ql}^{*T}. \tag{26}$$

4. Partition of the stress rates

In the case of an ongoing plastic deformation, by combining Eqs. (4), (21) and (23) we obtain

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathcal{L} : \mathbf{D} - \sum_{\alpha=1}^{n}\boldsymbol{\Lambda}^\alpha\dot{\gamma}^\alpha. \tag{27}$$

The rate of deformation tensor $\mathbf{D}$ is the symmetric part of $\mathbf{L}$, and

$$\boldsymbol{\Lambda}^\alpha = \mathcal{L} : \mathbf{P}^\alpha + (\mathbf{P}^\alpha + \mathbf{W}^\alpha)\boldsymbol{\tau} + \boldsymbol{\tau}(\mathbf{P}^\alpha - \mathbf{W}^\alpha) \tag{28}$$

is the symmetric second-order tensor, originally introduced in a more general context by Hill and Rice (1972). The constitutive equation (27) can be rewritten in terms of the Jaumann derivative $\overset{\circ}{\boldsymbol{\tau}} = \overset{\nabla}{\boldsymbol{\tau}} + \boldsymbol{\tau}\mathbf{D} + \mathbf{D}\boldsymbol{\tau}$, as

$$\overset{\circ}{\boldsymbol{\tau}} = \hat{\mathcal{L}} : \mathbf{D} - \sum_{\alpha=1}^{n} \boldsymbol{\Lambda}^{\alpha} \dot{\gamma}^{\alpha}. \tag{29}$$

The associated instantaneous elastic stiffness is $\hat{\mathcal{L}} = \mathcal{L} + \boldsymbol{J}$, where the fourth-order tensor $\boldsymbol{J}$ is defined by $\boldsymbol{J} : \mathbf{D} = \boldsymbol{\tau}\mathbf{D} + \mathbf{D}\boldsymbol{\tau}$. The tensor $\boldsymbol{\Lambda}^{\alpha}$ can be conveniently recast, in connection with Eq. (29), in the following form

$$\boldsymbol{\Lambda}^{\alpha} = \hat{\mathcal{L}} : \mathbf{P}^{\alpha} + \mathbf{W}^{\alpha}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W}^{\alpha}. \tag{30}$$

This representation of $\boldsymbol{\Lambda}^{\alpha}$ was used in most of the work in crystal plasticity, e.g. Asaro (1983). The rate of the Kirchhoff stress is similarly

$$\dot{\mathbf{S}} = \mathcal{L}_0 : \dot{\mathbf{E}} - \sum_{\alpha=1}^{n} \boldsymbol{\Lambda}_0^{\alpha} \dot{\gamma}^{\alpha} \tag{31}$$

where

$$\boldsymbol{\Lambda}_0^{\alpha} = \mathcal{L}_0 : \mathbf{P}_0^{\alpha} + \mathbf{B}_0^{\alpha}\mathbf{S} + \mathbf{S}\mathbf{B}_0^{\alpha T}. \tag{32}$$

The tensor $\boldsymbol{\Lambda}_0^{\alpha}$ is a symmetric second-order tensor, dual to $\boldsymbol{\Lambda}^{\alpha}$, such that:

$$\boldsymbol{\Lambda}_0^{\alpha} = \mathbf{F}^{-1}\boldsymbol{\Lambda}^{\alpha}\mathbf{F}^{-T}, \qquad \boldsymbol{\Lambda}_0^{\alpha} : \mathbf{P}_0^{\alpha} = \boldsymbol{\Lambda}^{\alpha} : \mathbf{P}^{\alpha}. \tag{33}$$

The elastic parts of the stress rates $\overset{\triangledown}{\boldsymbol{\tau}}$, $\overset{\circ}{\boldsymbol{\tau}}$ and $\dot{\mathbf{S}}$ are:

$$(\overset{\triangledown}{\boldsymbol{\tau}})_e = \mathcal{L} : \mathbf{D}, \qquad (\overset{\circ}{\boldsymbol{\tau}})_e = \hat{\mathcal{L}} : \mathbf{D}, \qquad (\dot{\mathbf{S}})_e = \mathcal{L}_0 : \dot{\mathbf{E}}. \tag{34}$$

The remaining, plastic parts of the stress rates follow from Eqs. (27), (29) and (31). They are:

$$(\overset{\triangledown}{\boldsymbol{\tau}})_p = (\overset{\circ}{\boldsymbol{\tau}})_p = -\sum_{\alpha=1}^{n} \boldsymbol{\Lambda}^{\alpha} \dot{\gamma}^{\alpha}, \qquad (\dot{\mathbf{S}})_p = -\sum_{\alpha=1}^{n} \boldsymbol{\Lambda}_0^{\alpha} \dot{\gamma}^{\alpha} \tag{35}$$

Evidently,

$$(\dot{\mathbf{S}})^p = \mathbf{F}^{-1}(\overset{\triangledown}{\boldsymbol{\tau}})^p\mathbf{F}^{-T}, \tag{36}$$

as it should be.

5. Partition of the rate of deformation tensor

According to Hill and Rice's (1972) general framework, the elastic part of the strain rates, associated with the introduced stress rates, are:

$$\mathbf{D}^e = \boldsymbol{\mathcal{M}} : \overset{\triangledown}{\boldsymbol{\tau}}, \qquad \hat{\mathbf{D}}^e = \hat{\boldsymbol{\mathcal{M}}} : \overset{\circ}{\tau}, \qquad (\dot{\mathbf{E}})^e = \boldsymbol{\mathcal{M}}_0 : \dot{\mathbf{S}}, \tag{37}$$

where $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{L}}^{-1}$, $\hat{\boldsymbol{\mathcal{M}}} = \hat{\boldsymbol{\mathcal{L}}}^{-1}$ and $\boldsymbol{\mathcal{M}}_0 = \boldsymbol{\mathcal{L}}_0^{-1}$ are the corresponding instantaneous elastic compliances of the crystal. Thus, by substituting Eqs. (27), (29) and (31) for the stress rates, we obtain:

$$\mathbf{D}^e = \mathbf{D} - \sum_{\alpha=1}^{n} \boldsymbol{\mathcal{M}} : \boldsymbol{\Lambda}^\alpha \dot{\gamma}^\alpha, \tag{38}$$

$$\hat{\mathbf{D}}^e = \mathbf{D} - \sum_{\alpha=1}^{n} \hat{\boldsymbol{\mathcal{M}}} : \boldsymbol{\Lambda}^\alpha \dot{\gamma}^\alpha, \tag{39}$$

$$(\dot{\mathbf{E}})^e = \dot{\mathbf{E}} - \sum_{\alpha=1}^{n} \boldsymbol{\mathcal{M}}_0 : \boldsymbol{\Lambda}_0^\alpha \dot{\gamma}^\alpha. \tag{40}$$

This identifies the plastic parts of the strain rates as:

$$\mathbf{D}^p = \sum_{\alpha=1}^{n} \mathbf{M}^\alpha \dot{\gamma}^\alpha, \qquad \hat{\mathbf{D}}^p = \sum_{\alpha=1}^{n} \hat{\mathbf{M}}^\alpha \dot{\gamma}^\alpha, \qquad (\dot{\mathbf{E}})^p = \sum_{\alpha=1}^{n} \mathbf{M}_0^\alpha \dot{\gamma}^\alpha, \tag{41}$$

where:

$$\mathbf{M}^\alpha = \boldsymbol{\mathcal{M}} : \boldsymbol{\Lambda}^\alpha, \qquad \hat{\mathbf{M}}^\alpha = \hat{\boldsymbol{\mathcal{M}}} : \boldsymbol{\Lambda}^\alpha, \qquad \mathbf{M}_0^\alpha = \boldsymbol{\mathcal{M}}_0 : \boldsymbol{\Lambda}_0^\alpha. \tag{42}$$

For example, $(\dot{\mathbf{E}})^p = \mathbf{F}^T \mathbf{D}^P \mathbf{F}$. More specifically, the three plastic strain rates are:

$$\mathbf{D}^p = \sum_{\alpha=1}^{n} \left\{ \mathbf{P}^\alpha + \boldsymbol{\mathcal{M}} : [(\mathbf{P}^\alpha + \mathbf{W}^\alpha)\boldsymbol{\tau} + \boldsymbol{\tau}(\mathbf{P}^\alpha - \mathbf{W}^\alpha)] \right\} \dot{\gamma}^\alpha, \tag{43}$$

$$\hat{\mathbf{D}}^p = \sum_{\alpha=1}^{n} \left[\mathbf{P}^\alpha + \hat{\boldsymbol{\mathcal{M}}} : (\mathbf{W}^\alpha \boldsymbol{\tau} - \boldsymbol{\tau} \mathbf{W}^\alpha) \right] \dot{\gamma}^\alpha, \tag{44}$$

$$(\dot{\mathbf{E}})^p = \sum_{\alpha=1}^{n} \left[\mathbf{P}_0^\alpha + \boldsymbol{\mathcal{M}}_0 : (\mathbf{B}_0^\alpha \mathbf{S} + \mathbf{S} \mathbf{B}_0^{\alpha T}) \right] \dot{\gamma}^\alpha. \tag{45}$$

Physically, $\mathbf{D}^p \mathrm{d}t$ represents the residual strain increment left in the crystal upon an infinitesimal loading/unloading cycle associated with the stress increment $(\overset{\triangledown}{\boldsymbol{\tau}} \mathrm{d}t)$. This is proportional to the increment of the symmetric Piola–Kirchhoff stress, when the reference configuration is taken to momentarily coincide with the current configuration. On the other hand, the increment $\hat{\mathbf{D}}^p \mathrm{d}t$ represents the residual strain increment left in the crystal upon an infinitesimal loading/unloading cycle associated with the stress increment $\mathring{\boldsymbol{\tau}}\mathrm{d}t$. This is proportional to the increment of the stress conjugate to the logarithmic strain, when the reference configuration is taken to momentarily coincide with the current configuration (Hill, 1978). Finally, $(\dot{\mathbf{E}})^p \mathrm{d}t$ represents the residual strain increment left in the crystal upon an infinitesimal loading/unloading cycle associated with the stress increment $\dot{\mathbf{S}}\mathrm{d}t$.

6. Normality structure

For rate-independent materials it is commonly assumed that plastic flow occurs on a slip system when the resolved shear stress on that system reaches the critical value ($\tau^\alpha = \tau^\alpha_{\mathrm{cr}}$). The Schmid stress τ^α can be defined as the work conjugate to the slip rate $\dot{\gamma}^\alpha$, such that

$$\sum_{\alpha=1}^{n} \tau^\alpha \dot{\gamma}^\alpha = \mathbf{S} : \sum_{\alpha=1}^{n} \mathbf{P}_0^\alpha \dot{\gamma}^\alpha = \boldsymbol{\tau} : \sum_{\alpha=1}^{n} \mathbf{P}^\alpha \dot{\gamma}^\alpha. \tag{46}$$

Therefore,

$$\tau^\alpha = \mathbf{P}_0^\alpha : \mathbf{S} = \mathbf{P}^\alpha : \boldsymbol{\tau}. \tag{47}$$

The plastic rate $(\dot{\mathbf{E}})^p$ lies within a pyramid of outward normals to the yield surface at $\mathbf{S}$, each normal being associated with an active slip system (Havner, 1992). The direction of the normal to the yield plane $\tau^\alpha = \tau^\alpha_{\mathrm{cr}}$ at $\mathbf{S}$ is determined from

$$\frac{\partial \tau^\alpha}{\partial \mathbf{S}} = \mathbf{P}_0^\alpha + \frac{\partial \mathbf{P}_0^\alpha}{\partial \mathbf{S}} : \mathbf{S} = \mathbf{P}_0^\alpha + \frac{\partial \mathbf{P}_0^\alpha}{\partial \mathbf{E}} : \boldsymbol{\mathcal{M}}_0 : \mathbf{S}. \tag{48}$$

From Eq. (12), it follows that

$$\mathbf{S} : \frac{\partial \mathbf{P}_0^\alpha}{\partial \mathbf{E}} = \mathbf{B}_0^\alpha \mathbf{S} + \mathbf{S} \mathbf{B}_0^{\alpha T}. \tag{49}$$

Thus, Eq. (48) becomes

$$\frac{\partial \tau^\alpha}{\partial \mathbf{S}} = \mathbf{P}_0^\alpha + \boldsymbol{\mathcal{M}}_0 : (\mathbf{B}_0^\alpha \mathbf{S} + \mathbf{S} \mathbf{B}_0^{\alpha T}), \tag{50}$$

which proves the normality structure of $(\dot{\mathbf{E}})^p$, inherent in Eq. (45), i.e.

$$(\dot{\mathbf{E}})^p = \sum_{\alpha=1}^{n} \frac{\partial \tau^\alpha}{\partial \mathbf{S}} \dot{\gamma}^\alpha. \tag{51}$$

The normality holds with respect to conjugate measures $(\mathbf{E}, \mathbf{S})$, or with respect to any other conjugate measures of stress and strain (Hill and Havener, 1982).

7. Relationships between D_p and $\hat{D}_p$

From Eqs. (43) and (44), clearly, $\mathbf{D}^p \neq \hat{\mathbf{D}}^p$. This was also evident from the outset, since the reversible parts of the rate of deformation, corresponding to two stress rates, are different, $\mathbf{D}^e \neq \hat{\mathbf{D}}^e$, while $\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \equiv \hat{\mathbf{D}}^e + \hat{\mathbf{D}}^p$. To establish the relationships between $\mathbf{D}^p$ and $\hat{\mathbf{D}}^p$, we use the identity $(\overset{\nabla}{\boldsymbol{\tau}})^p \equiv (\overset{\circ}{\boldsymbol{\tau}})_p$, and the connections:

$$(\overset{\nabla}{\boldsymbol{\tau}})^p = -\boldsymbol{\mathcal{L}} : \mathbf{D}^p, \qquad (\overset{\circ}{\tau})^p = -\hat{\boldsymbol{\mathcal{L}}} : \hat{\mathbf{D}}^p. \tag{52}$$

This gives:

$$\mathbf{D}^p = \hat{\mathbf{D}}^p + \boldsymbol{\mathcal{M}} : \boldsymbol{J} : \hat{\mathbf{D}}^p, \tag{53}$$

$$\hat{\mathbf{D}}^p = \mathbf{D}^p - \hat{\boldsymbol{\mathcal{M}}} : \boldsymbol{J} : \mathbf{D}^p. \tag{54}$$

The elastic parts of the rate of deformation are, accordingly, related by:

$$\mathbf{D}^e = \hat{\mathbf{D}}^e - \boldsymbol{\mathcal{M}} : \boldsymbol{J} : \hat{\mathbf{D}}^p, \tag{55}$$

$$\hat{\mathbf{D}}^e = \mathbf{D}^e + \hat{\boldsymbol{\mathcal{M}}} : \boldsymbol{J} : \mathbf{D}^p. \tag{56}$$

Furthermore, from Eqs. (43) and (44) we have

$$\mathbf{D}^p - \hat{\mathbf{D}}^p = \sum_{\alpha=1}^{n} (\mathbf{M}^\alpha - \hat{\mathbf{M}}^\alpha)\dot{\gamma}^\alpha = \sum_{\alpha=1}^{n} (\boldsymbol{\mathcal{M}} - \hat{\boldsymbol{\mathcal{M}}}) : \boldsymbol{\Lambda}^\alpha \dot{\gamma}^\alpha, \tag{57}$$

which indicates that the difference between the two plastic rates of deformation depends on the difference between instantaneous elastic compliances, associated with the Jaumann and convected rates of the Kirchhoff stress.

8. Hardening rules and slip rates

The rate of change of the critical value of the resolved shear stress on a given slip system is defined by the hardening law (Asaro, 1983; Havner, 1985, 1992)

$$\dot{\tau}_{cr}^{\alpha} = \sum_{\beta=1}^{n} h_{\alpha\beta} \dot{\gamma}^{\beta} \tag{58}$$

where $h_{\alpha\beta}$ are the slip-plane hardening rates. The diagonal terms represent self-hardening on a given slip system, while off-diagonal terms represent latent hardening. The consistency condition for slip on the system α is

$$\dot{\tau}^{\alpha} = \sum_{\beta=1}^{n} h_{\alpha\beta} \dot{\gamma}^{\beta}. \tag{59}$$

The rate of Schmid stress is obtained by differentiation from Eq. (47),

$$\dot{\tau}^{\alpha} = \dot{\mathbf{P}}_0^{\alpha} : \mathbf{S} + \mathbf{P}_0^{\alpha} : \dot{\mathbf{S}}. \tag{60}$$

After a somewhat lengthy derivation, the following remarkable expression for the rate $\dot{\mathbf{P}}_0^{\alpha}$ is found from Eq. (10)

$$\dot{\mathbf{P}}_0^{\alpha} = \dot{\mathbf{E}} \mathbf{B}_0^{\alpha} + \mathbf{B}_0^{\alpha T} \dot{\mathbf{E}}. \tag{61}$$

The rate of Schmid stress, therefore, becomes

$$\dot{\tau}^{\alpha} = (\mathbf{S} \mathbf{B}_0^{\alpha T} + \mathbf{B}_0^{\alpha} \mathbf{S}) : \dot{\mathbf{E}} + \mathbf{P}_0^{\alpha} : \dot{\mathbf{S}}. \tag{62}$$

This is also equal to $\boldsymbol{\Lambda}_0^{\alpha} : (\mathbf{F}_p^T \dot{\mathbf{E}}_* \mathbf{F}_p)$, or $\boldsymbol{\Lambda}^{\alpha} : \mathbf{D}_*$, the latter expression being used in Asaro's (1983) analysis. The slip rates can, accordingly, be determined from either of:

$$\dot{\gamma}^{\alpha} = \sum_{\beta=1}^{n} g_{\alpha\beta}^{-1} \boldsymbol{\Lambda}_0^{\beta} : \dot{\mathbf{E}} = \sum_{\beta=1}^{n} g_{\alpha\beta}^{-1} \boldsymbol{\Lambda}^{\beta} : \mathbf{D}, \tag{63}$$

provided that the matrix

$$g_{\alpha\beta} = h_{\alpha\beta} + \boldsymbol{\Lambda}_0^{\alpha} : \mathbf{P}_0^{\beta} = h_{\alpha\beta} + \boldsymbol{\Lambda}^{\alpha} : \mathbf{P}^{\beta} \tag{64}$$

is positive-definite, as originally shown by Hill and Rice (1972).

Substituting Eq. (64) into Eqs. (27), (29) and (31) gives the three alternative constitutive structures for the elastic-plastic single crystals. The plastic parts of the corresponding stress rates are:

$$(\overset{\nabla}{\boldsymbol{\tau}})^p = (\overset{\circ}{\boldsymbol{\tau}})^p = -\left(\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \boldsymbol{\Lambda}^{\alpha} g_{\alpha\beta}^{-1} \boldsymbol{\Lambda}^{\beta} \right) : \mathbf{D}, \tag{65}$$

$$(\dot{\mathbf{S}})^p = -\left(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\mathbf{\Lambda}_0^\alpha g_{\alpha\beta}^{-1}\mathbf{\Lambda}_0^\beta\right) : \dot{\mathbf{E}} \tag{66}$$

9. Plastic parts of the rate of deformation in terms of stress rates

Since

$$\mathbf{D}_* = \boldsymbol{\mathcal{M}} : \overset{\nabla *}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{M}} : \left[\overset{\nabla}{\boldsymbol{\tau}} + \sum_{\beta=1}^{n}(\mathbf{\Lambda}^\beta - \boldsymbol{\mathcal{L}} : \mathbf{P}^\beta)\dot{\gamma}^\beta\right], \tag{67}$$

the substitution into $\dot{\boldsymbol{\tau}}^\alpha = \mathbf{\Lambda}^\alpha : \mathbf{D}_*$ yields

$$\mathbf{M}^\alpha : \overset{\nabla}{\boldsymbol{\tau}} = \sum_{\beta=1}^{n} k_{\alpha\beta}\dot{\gamma}^\beta, \qquad k_{\alpha\beta} = g_{\alpha\beta} - \mathbf{\Lambda}^\alpha : \boldsymbol{\mathcal{M}} : \mathbf{\Lambda}^\beta. \tag{68}$$

If $\mathbf{k}$ is an invertible matrix, from Eq. (68), it follows that the slip rates are

$$\dot{\gamma}^\alpha = \sum_{\beta=1}^{n} k_{\alpha\beta}^{-1}\mathbf{M}^\beta : \overset{\nabla}{\boldsymbol{\tau}}. \tag{69}$$

Similarly, we have:

$$\hat{\mathbf{M}}^\alpha \overset{\circ}{\tau} = \sum_{\beta=1}^{n} \hat{k}_{\alpha\beta}\dot{\gamma}^\beta, \qquad \hat{k}_{\alpha\beta} = g_{\alpha\beta} - \mathbf{\Lambda}^\alpha : \hat{\boldsymbol{\mathcal{M}}} : \mathbf{\Lambda}^\beta, \tag{70}$$

$$\mathbf{M}_0^\alpha \dot{\mathbf{S}} = \sum_{\beta=1}^{n} k_{\alpha\beta}^0\dot{\gamma}^\beta, \qquad k_{\alpha\beta}^0 = g_{\alpha\beta} - \mathbf{\Lambda}_0^\alpha : \boldsymbol{\mathcal{M}}_0 : \mathbf{\Lambda}_0^\beta. \tag{71}$$

Clearly, $\hat{k}_{\alpha\beta} \neq k_{\alpha\beta} \neq k_{\alpha\beta}^0$. In the context of general strain and its conjugate stress measures, this has been extensively discussed by Hill and Rice (1972), Hill and Havner (1982) and Havner (1992). For the prescribed stress rates $\overset{\circ}{\boldsymbol{\tau}}$ or $\dot{\boldsymbol{S}}$, the slip rates $\dot{\gamma}^\alpha$ are uniquely determined provided that $\hat{\mathbf{k}}$ and $\mathbf{k}_0$ are positive-definite, respectively. The slip rates are then determined from

$$\dot{\gamma}^\alpha = \sum_{\beta=1}^{n} \hat{k}_{\alpha\beta}^{-1}\hat{\mathbf{M}}^\beta \overset{\circ}{\tau} = \sum_{\beta=1}^{n} k_{\alpha\beta}^{0-1}\mathrm{M}_0^\beta\dot{\mathbf{S}}. \tag{72}$$

The plastic strain rates in three cases are, consequently:

$$\mathbf{D}^p = \left(\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \mathbf{M}^\alpha k_{\alpha\beta}^{-1} \mathbf{M}^\beta \right) : \overset{\nabla}{\boldsymbol{\tau}}, \tag{73}$$

$$\hat{\mathbf{D}}^p = \left(\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \hat{\mathbf{M}}^\alpha \hat{k}_{\alpha\beta}^{-1} \hat{\mathbf{M}}^\beta \right) : \overset{\circ}{\boldsymbol{\tau}}, \tag{74}$$

$$(\dot{\mathbf{E}})^p = \left(\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \mathbf{M}_0^\alpha k_{\alpha\beta}^{0-1} \mathbf{M}_0^\beta \right) : \dot{\mathbf{S}}. \tag{75}$$

10. Constitutive inequalities

If the matrix $\mathbf{g}$ is positive definite, $\mathbf{g}^{-1}$ is as well, and

$$(\dot{\mathbf{S}})^p : \dot{\mathbf{E}} = (\overset{\nabla}{\boldsymbol{\tau}})^p : \mathbf{D} = (\overset{\circ}{\boldsymbol{\tau}})^p : \mathbf{D} = -\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \mathbf{D} : \boldsymbol{\Lambda}^\alpha g_{\alpha\beta}^{-1} \boldsymbol{\Lambda}^\beta : \mathbf{D} < 0. \tag{76}$$

The inequality holds regardless of whether the crystal is in the state of overall hardening or softening. On the other hand,

$$\dot{\mathbf{S}} : (\dot{\mathbf{E}})^p = \overset{\nabla}{\boldsymbol{\tau}} : \mathbf{D}^p \neq \overset{\circ}{\boldsymbol{\tau}} : \hat{\mathbf{D}}^p, \tag{77}$$

since from Eqs. (73) and (74), and in view of Eq. (37),

$$\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \mathbf{D}^e : \boldsymbol{\Lambda}^\alpha k_{\alpha\beta}^{-1} \boldsymbol{\Lambda}^\beta : \mathbf{D}^e \neq \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \hat{\mathbf{D}}^e : \boldsymbol{\Lambda}^\alpha \hat{k}_{\alpha\beta}^{-1} \boldsymbol{\Lambda}^\beta : \hat{\mathbf{D}}^e. \tag{78}$$

In particular, it may happen that $\hat{\mathbf{k}}^{-1}$ is positive-definite, so that $\overset{\circ}{\boldsymbol{\tau}} : \hat{\mathbf{D}}^p > 0$ (implying hardening), while $\mathbf{k}^{-1}$ is negative definite, so that $\overset{\nabla}{\boldsymbol{\tau}} : \mathbf{D}^p < 0$ (implying softening). In a more general context, these possibilities were discussed by Havner (1992). In fact, it can be shown that:

$$\overset{\nabla}{\boldsymbol{\tau}} : \mathbf{D}^p = \sum_{\alpha=1}^{n} \dot{\tau}^\alpha \dot{\gamma}^\alpha + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} (\boldsymbol{\Lambda}^\alpha : \mathbf{P}^\beta - \boldsymbol{\Lambda}^\alpha : \boldsymbol{\mathcal{M}} : \boldsymbol{\Lambda}^\beta) \dot{\gamma}^\alpha \dot{\gamma}^\beta, \tag{79}$$

while

$$\overset{\circ}{\boldsymbol{\tau}} : \hat{\mathbf{D}}^p = \sum_{\alpha=1}^{n} \dot{\tau}^\alpha \dot{\gamma}^\alpha + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} (\boldsymbol{\Lambda}^\alpha : \mathbf{P}^\beta - \boldsymbol{\Lambda}^\alpha : \hat{\boldsymbol{\mathcal{M}}} : \boldsymbol{\Lambda}^\beta) \dot{\gamma}^\alpha \dot{\gamma}^\beta. \tag{80}$$

Their difference is, thus,

$$\overset{\triangledown}{\boldsymbol{\tau}} : \mathbf{D}^p - \overset{\circ}{\boldsymbol{\tau}} : \hat{\mathbf{D}}^p = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Big[\boldsymbol{\Lambda}^{\alpha} : (\hat{\boldsymbol{\mathcal{M}}} - \boldsymbol{\mathcal{M}}) : \boldsymbol{\Lambda}^{\beta}\Big]\dot{\gamma}^{\alpha}\dot{\gamma}^{\beta}, \tag{81}$$

which can be either positive or negative, depending on the positive-definiteness of the tensor $(\hat{\boldsymbol{\mathcal{M}}} - \boldsymbol{\mathcal{M}})$.

In contrast to (77), we have the equality

$$\mathbf{F}_p^{-1}\dot{\mathbf{S}}_*\mathbf{F}_p^{-T} : (\dot{\mathbf{E}})^p = \overset{\triangledown *}{\boldsymbol{\tau}} : \mathbf{D}^p = \overset{\circ *}{\boldsymbol{\tau}} : \hat{\mathbf{D}}^p = \sum_{\alpha=1}^{n}\dot{\tau}^{\alpha}\dot{\gamma}^{\alpha}. \tag{82}$$

For example, if $\hat{\mathbf{D}}^p$ in Eq. (82) is plastic part of the rate of deformation during plastic loading, while $\overset{\circ *}{\boldsymbol{\tau}}$ is the stress rate during elastic unloading, it follows that $\overset{\circ *}{\boldsymbol{\tau}} : \mathbf{D}^p < 0$, provided that during elastic unloading $\dot{\tau}^{\alpha} < 0$ for each α. The slip rates $\dot{\gamma}^{\alpha}$ are assumed to be always positive during plastic loading, so that opposite senses of slip in the same glide plane are represented by distinct α. Thus, in this case $\overset{\triangledown *}{\boldsymbol{\tau}} : \mathbf{D}^p = \overset{\circ *}{\boldsymbol{\tau}} : \hat{\mathbf{D}}^p < 0$, which is measure invariant and in accord with Ilyushin's postulate, regardless of the hardening or softening of the crystal at the considered instant of deformation (Hill, 1978; Havner, 1992).

11. Rigid–plastic behavior

Although the partition of the rate of deformation is not an issue in the rigid-plastic idealization, the previous results can be specialized when the elastic compliances $\boldsymbol{\mathcal{M}}$ and $\hat{\boldsymbol{\mathcal{M}}}$, and the lattice rate of deformation $\mathbf{D}_*$, all vanish. The rate of deformation and the spin tensor are then:

$$\mathbf{D} = \sum_{\alpha=1}^{n}\mathbf{P}^{\alpha}\dot{\gamma}^{\alpha}, \qquad \mathbf{W} = \mathbf{W}_* + \sum_{\alpha=1}^{n}\mathbf{W}^{\alpha}\dot{\gamma}^{\alpha}, \tag{83}$$

where $\mathbf{W}_* = \dot{\mathbf{R}}_*\mathbf{R}_*^{-1}$ is the lattice spin, and $\mathbf{R}_*$ is the lattice rotation.

If convected derivative of the Kirchhoff stress is used, the slip rates can be expressed as

$$\dot{\gamma}^{\alpha} = \sum_{\beta=1}^{n}k_{\alpha\beta}^{-1}\mathbf{P}^{\beta} : \overset{\triangledown}{\boldsymbol{\tau}}, \qquad k_{\alpha\beta} = h_{\alpha\beta} - \mathbf{P}^{\alpha} : \big[(\mathbf{P}^{\beta} + \mathbf{W}^{\beta})\boldsymbol{\tau} + \boldsymbol{\tau}(\mathbf{P}^{\beta} - \mathbf{W}^{\beta})\big]. \tag{84}$$

If the Jaumann derivative is used, the same slip rates are

$$\dot{\gamma}^{\alpha} = \sum_{\beta=1}^{n}\hat{k}_{\alpha\beta}^{-1}\mathbf{P}^{\beta} : \overset{\circ}{\boldsymbol{\tau}}, \qquad \hat{k}_{\alpha\beta} = h_{\alpha\beta} - \mathbf{P}^{\alpha} : (\mathbf{W}^{\beta}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W}^{\beta}). \tag{85}$$

The last expression is in accord with the corresponding expression given by Khan and Huang (1995). The matrix of hardening moduli $h_{\alpha\beta}$, related to $\hat{k}_{\alpha\beta}$ as in Eq. (85), was originally introduced by Havner and Shalaby (1977) in their study of latent hardening of single crystals. Evidently,

$$\overset{\nabla}{\boldsymbol{\tau}} : \mathbf{D} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \overset{\nabla}{\boldsymbol{\tau}} : \mathbf{P}^{\alpha} k_{\alpha\beta}^{-1} \mathbf{P}^{\beta} : \overset{\nabla}{\boldsymbol{\tau}}, \tag{86}$$

$$\overset{\circ}{\boldsymbol{\tau}} : \mathbf{D} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \overset{\circ}{\boldsymbol{\tau}} : \mathbf{P}^{\alpha} \hat{k}_{\alpha\beta}^{-1} \mathbf{P}^{\beta} : \overset{\circ}{\boldsymbol{\tau}}. \tag{87}$$

The sign of these clearly depends on the positive-definiteness of the matrices $\mathbf{k}$ and $\hat{\mathbf{k}}$, respectively. In particular, one can be positive, the other can be negative.

12. Discussion

We have demonstrated in this paper the non-uniqueness of the partition of the rate of deformation into its elastic and plastic parts, by providing explicit expressions associated with two different partitions. The first one is well-known partition based on the Jaumann rate of the Kirchhoff stress. The second one is less known, and is based on the convected rate of the Kirchhoff stress. Since this rate involves the total rate of deformation, the latter partition appears to be less appealing than the partition based on the Jaumann rate, which involves the spin part of the velocity gradient only. The partition based on the convected rate, however, is inherent in the formulation of the theory based on the Lagrangian strain and its conjugate Piola–Kirchhoff stress. Thus, it can be deduced from the partition of the Lagrangian strain rate by adequate transformation formulas, or it can be obtained directly, by an independent derivation, as done in this paper. The two partitions are particularly suitable to discuss the nature of the constitutive inequalities in the plastic regime of the material response.

Although the presented analysis was restricted to single crystal plasticity, some of the results can be easily extended to polycrystalline plasticity. In that case, we may use the multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$, with appropriate interpretations of its constituents $\mathbf{F}_e$ and $\mathbf{F}_p$ (Lee, 1969). For example, if $\boldsymbol{\mathcal{M}}_0$ is the instantaneous elastic compliance of a polycrystalline aggregate relative to the Lagrangian strain, the elastic and plastic parts of the strain rate become:

$$(\dot{\mathbf{E}})^{e} = \mathbf{F}_p^T \dot{\mathbf{E}}_e \mathbf{F}_p - \boldsymbol{\mathcal{M}}_0 : (\mathbf{B}_p\mathbf{S} + \mathbf{S}\mathbf{B}_p^T), \tag{88}$$

$$(\dot{\mathbf{E}})^{p} = \frac{1}{2}\mathbf{F}_p^T \Big[\mathbf{C}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}) + (\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})^T\mathbf{C}_e\Big]\mathbf{F}_p + \boldsymbol{\mathcal{M}}_0 : (\mathbf{B}_p\mathbf{S} + \mathbf{S}\mathbf{B}_p^T), \tag{89}$$

where $\mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e$, $\mathbf{E}_e = \frac{1}{2}(\mathbf{C}_e - \mathbf{I})$, and $\mathbf{B}_p = \mathbf{F}_p^{-1}(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_p$. The first part of $(\dot{\mathbf{E}})^p$ on the right-hand side of Eq. (89) can also be written as $(\mathbf{C}\mathbf{B}_p)_s$, where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and the subscript s denotes the symmetric part. The plastic strain rate $(\dot{\mathbf{E}})^p$ is codirectional with the outward normal to a locally smooth yield surface in the stress space $\mathbf{S}$, and is within the cone of outward normals at the vertex point of the yield surface.

The elastic and plastic constituents of the rate of deformation, corresponding to the convected stress rate $\overset{\nabla}{\boldsymbol{\tau}}$, are:

$$\mathbf{D}_e = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s - \boldsymbol{\mathcal{M}} : \left\{\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]\boldsymbol{\tau} + \boldsymbol{\tau}\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]^T\right\} \tag{90}$$

$$\mathbf{D}_p = \left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]_s + \boldsymbol{\mathcal{M}} : \left\{\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]\boldsymbol{\tau} + \boldsymbol{\tau}\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]^T\right\}. \tag{91}$$

Finally, the elastic and plastic parts of $\mathbf{D}$, corresponding to the Jaumann stress rate $\overset{\circ}{\boldsymbol{\tau}}$, are given by:

$$\hat{\mathbf{D}}_e = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s - \hat{\boldsymbol{\mathcal{M}}} : \left\{\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]_a\boldsymbol{\tau} - \boldsymbol{\tau}\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]_a\right\}, \tag{92}$$

$$\hat{\mathbf{D}}_p = \left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]_s + \hat{\boldsymbol{\mathcal{M}}} : \left\{\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]_a\boldsymbol{\tau} - \boldsymbol{\tau}\left[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}\right]_a\right\}. \tag{93}$$

It is easily verified that $(\dot{\mathbf{E}})^p = \mathbf{F}^T\mathbf{D}^p\mathbf{F}$. The relationship between $\mathbf{D}^p$ and $\hat{\mathbf{D}}^p$ is examined by Lubarda (1994). See also a related discussion by Lubarda and Shih (1994) regarding the independence of the final results on the rotation superimposed to the intermediate configuration. The incorporation of anisotropic elasticity in the considered constitutive framework was studied by Lubarda (1991), and of evolving elastic properties by Lubarda and Krajcinovic (1995).

Acknowledgements

Research support provided by the Alcoa Center is kindly acknowledged.

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