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SOME ASPECTS OF ELASTO-PLASTIC CONSTITUTIVE ANALYSIS OF ELASTICALLY ANISOTROPIC MATERIALS

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Abstract — By utilizing the multiplicative decomposition of deformation gradient, we precisely identify the elastic and plastic contribution to the velocity strain of elasto-plastically deformed anisotropic materials. Corresponding elastic stiffness and compliance tensors are obtained to within the unknown rotation tensor, to be additionally determined. Orthotropic materials are considered in particular and constitutive structure for their plastic strain-rate established. Various kinematic aspects and relationships between different plastic strain-rate and spin tensors, as well as comparison to related work, are also given.

I. INTRODUCTION

The constitutive analysis of large deformation elasto-plastic behaviour of metals has been extensively treated in the literature, from various standpoints. Within the model of multiplicative decomposition of deformation gradient into its elastic and plastic parts, extensive research was done since the early work by LEE [1969] (see, for example, recent papers by AGAH-TEHRANI *et al.* [1987] and DAFALIAS [1987]). Many aspects of the theory, however, were treated differently, in particular the definition of elastic strain-rate and partition of total strain-rate into its elastic and plastic parts. These, and other relevant issues, were extensively discussed by LUBARDA [1991]. In this paper we confine attention to constitutive behavior of elastically anisotropic materials, assuming that anisotropy remains unaltered during the plastic flow (for example, that orthotropic material remains orthotropic, with the same elasticity constants). This case was little treated in the literature by using the multiplicative decomposition and, as a matter of fact, caused some skepticism concerning the extent of decomposition utility in the constitutive analysis. Some important work has, nevertheless, been done by MANDEL [1973, 1981, 1982], in the case of polycrystalline materials whose elastic properties are influenced by previous plastic flow, and by PIERCE *et al.* [1982] and ASARO [1983a, 1983b], in the case of a single crystal plasticity. In a series of papers by Lee and his coworkers (LEE [1969]; LUBARDA & LEE [1981]; AGAH-TEHRANI *et al.* [1987]), elastic anisotropy, either initial or by plastic flow induced, was not considered. Consequently, we analyze here the case of persistent elastic anisotropy, leaving for future investigation the more involved and, perhaps, significant problem of varying and induced elastic anisotropy.

II. KINEMATIC AND KINETIC PRELIMINARIES

Let $\mathcal{B}_0$ be the initial configuration of elastically anisotropic (say orthotropic) material and let $\mathbf{a}_i^0 (i = 1,2,3)$ define the axes of anisotropy with respect to which the strain energy has the representation $w = w(C_e)$, C_e being the right Cauchy-Green deformation tensor. Let $\mathcal{B}_t$ be the current configuration obtained from $\mathcal{B}_0$ by elasto-plastic deformation, with the corresponding gradient being F. Finally, let $\mathcal{P}_t$ be the imagined, intermediate-purely plastically deformed configuration, obtained from $\mathcal{B}_t$ by (possibly virtual) elastic destressing to zero stress. The multiplicative decomposition $F = F_e F_p$ then holds, with F_e being the elastic and F_p the plastic part of deformation gradient F. Elastic rotation on destressing is taken to be arbitrary, say R_e, so that by polar decomposition $F_e = V_e R_e$, V_e being the elastic stretch tensor. Assume, further, that the material in its intermediate configuration has the same type of elastic anisotropy as in its initial state: the axes of anisotropy $\mathbf{a}_i^0$ in $\mathcal{B}_0$ are then just rotated to axes $\mathbf{a}_i^+$ in the intermediate state, that is, $\mathbf{a}_i^+ = \mathcal{R}\mathbf{a}_i^0$, where $\mathcal{R}$ is the orthogonal tensor. The strain energy in the intermediate state can consequently be represented by $w(\mathcal{R}^T C_e \mathcal{R})$, that is, the same function w, introduced for initial state, but of the arguments which are by $\mathcal{R}$ rotated-axes components of deformation tensor C_e. As discussed by MANDEL [1973, 1981], in view of the discontinuities of displacements and rotations of elements at microscale, the rotation matrix $\mathcal{R}$ is independent of the (overall) plastic part of deformation gradient F_p. Finally, as the elastic deformation takes material from intermediate to current state, the unit vectors $\mathbf{a}_i^+$ are considered to convect with material, that is, $\mathbf{a}_i = F_e \mathbf{a}_i^+$ (elastic embedding), hence

$$\dot{\mathbf{a}}_i = [\dot{F}_e F_e^{-1} + F_e(\dot{\mathcal{R}}\mathcal{R}^{-1})F_e^{-1}]\mathbf{a}_i, \tag{1}$$

where the superimposed dot stands for the material derivative and -1 for the inverse. The (Cauchy) stress response is accordingly given by

$$(\det F_e)\sigma = 2F_e \frac{\partial w(\mathcal{R}^T C_e \mathcal{R})}{\partial C_e} F_e^T, \tag{2}$$

or, in view of the polar decomposition $F_e = V_e R_e$ and the relationship between left (B_e) and right deformation tensor $C_e = R_e^T B_e R_e$,

$$(\det V_e)\sigma = 2V_e \frac{\partial w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e} V_e. \tag{3}$$

The rotation tensor $\mathcal{R}_e$ appearing in the transition from eqn (2) to eqn (3) is clearly $\mathcal{R}_e = R_e \mathcal{R}$. This is also the rotation which appears in transition $\mathbf{a}_i = F_e \mathbf{a}_i^+ = F_e \mathcal{R}\mathbf{a}_i^0 = V_e R_e \mathcal{R}\mathbf{a}_i^0 \equiv V_e \mathcal{R}_e \mathbf{a}_i^0$, $\mathcal{R}_e = R_e \mathcal{R}$. For example, if we define the intermediate configuration by destressing without rotation ($F_e \equiv V_e$, $R_e \equiv I$), $\mathcal{R}_e$ is the rotation of elastic anisotropy axes from $\mathcal{B}_0$ to $\mathcal{P}_t$ (that is, $\mathcal{R} \equiv \mathcal{R}_e$); if the intermediate configuration is isoclinic ($\mathbf{a}_i^+ \equiv \mathbf{a}_i^0$, that is $\mathcal{R} \equiv I$), $\mathcal{R}_e$ represents the elastic rotation during the loading from $\mathcal{P}_t$ to $\mathcal{B}_t : F_e = V_e R_e$, $R_e \equiv \mathcal{R}_e$. In any case, $\mathcal{R}_e$ is the unique quantity and the stress response (3) is independent of selected intermediate configuration. Finally, in view of the decomposition $F = F_e F_p = V_e R_e F_p$, the velocity gradient $L = \dot{F}F^{-1}$ in the current configuration can be expressed as

$$L = \dot{V}_e V_e^{-1} + V_e(\dot{R}_e R_e^{-1})V_e^{-1} + V_e[R_e(\mathfrak{D}_p + \mathcal{W}_p)R_e^T]V_e^{-1}, \tag{4}$$

where $\mathfrak{D}_p$ and $\mathcal{W}_p$ are the plastic velocity and spin of the chosen intermediate state, that is, the symmetric and antisymmetric part of $\dot{F}_p F_p^{-1}$.

III. OBJECTIVITY REQUIREMENTS

On superimposing the time dependent rigid-body rotation Q in the current configuration $\mathcal{B}_t$, the deformation gradient F changes to $F^* = QF$, while elastic and plastic parts F_e and F_p change to $F_e^* = QF_e\hat{Q}^T$ and $F_p^* = \hat{Q}F_p$, where the orthogonal (rotation) tensor $\hat{Q}$, defining the corresponding rotation of intermediate state, depends on that state: for example, if intermediate state is obtained by destressing without rotation, we have $\hat{Q} \equiv Q$; if intermediate configuration is isoclinic, then $\hat{Q} \equiv I$. To simultaneously treat all possible selections of the intermediate configuration, we denote the corresponding rotation, associated with objectivity requirements, by $\hat{Q}$ and show in the body of the paper that final results (that is, elastic and plastic strain-rates and their constitutive expressions) are independent of $\hat{Q}$. In this way we essentially fulfill the invariance requirements advocated by Naghdi and his colleagues (see, for example, CASEY & NAGHDI [1981]). The introduced kinematic quantities accordingly change to:

$$V_e^* = QV_eQ^T, \qquad B_e^* = QB_eQ^T, \qquad C_e^* = \hat{Q}C_e\hat{Q}^T$$
$$\mathfrak{R}^* = \hat{Q}\mathfrak{R}, \qquad R_e^* = QR_e\hat{Q}^T \qquad \mathfrak{R}_e^* = Q\mathfrak{R}_e. \tag{5}$$

Introduce now two spins (Ω and Ω_p), LUBARDA [1988,1991], the first associated with the current and the second with the intermediate state, such that under introduced frame change they behave according to:

$$\Omega^* = \dot{Q}Q^{-1} + Q\Omega Q^T$$
$$\Omega_p^* = \dot{\hat{Q}}\hat{Q}^{-1} + \hat{Q}\Omega_p\hat{Q}^T. \tag{6}$$

The following (Jaumann type) derivatives, associated with the spins Ω and Ω_p, can then be defined, together with the rules they obey under introduced frame change:

$$\mathring{F}_e = \dot{F}_e - \Omega F_e + F_e\Omega_p, \qquad (\mathring{F}_e)^* = Q\mathring{F}_e\hat{Q}^T \tag{7.1}$$

$$\mathring{F}_p = \dot{F}_p - \Omega_p F_p, \qquad (\mathring{F}_p)^* = \hat{Q}\mathring{F}_p \tag{7.2}$$

$$\mathring{F} = \dot{F} - \Omega F, \qquad (\mathring{F})^* = Q\mathring{F} \tag{7.3}$$

$$\mathring{V}_e = \dot{V}_e - \Omega V_e + V_e\Omega, \qquad (\mathring{V}_e)^* = Q\mathring{V}_eQ^T \tag{7.4}$$

$$\mathring{R}_e = \dot{R}_e - \Omega R_e + R_e\Omega_p, \qquad (\mathring{R}_e)^* = Q\mathring{R}_e\hat{Q}^T \tag{7.5}$$

$$\mathring{\mathfrak{R}} = \dot{\mathfrak{R}} - \Omega_p\mathfrak{R}, \qquad (\mathring{\mathfrak{R}})^* = \hat{Q}\mathring{\mathfrak{R}} \tag{7.6}$$

$$\mathring{\mathfrak{R}}_e = \dot{\mathfrak{R}}_e - \Omega\mathfrak{R}_e, \qquad (\mathring{\mathfrak{R}}_e)^* = Q\mathring{\mathfrak{R}}_e. \tag{7.7}$$

For example, $\mathring{F}_e$ gives the change of elastic deformation gradient F_e observed in (or with respect to) the coordinate systems that rotate with spin Ω in the current and Ω_p in the intermediate configuration. To be compatible with introduced frame change rules (5), Ω_p depends on the selected intermediate configuration: for example, if destressing is without rotation, then $\Omega_p \equiv \Omega$. The plastic strain-rate and spin of intermediate configuration change to:

$$\mathfrak{D}_p^* = \hat{Q}\mathfrak{D}_p\hat{Q}^T, \qquad \mathfrak{W}_p^* = \dot{\hat{Q}}\hat{Q}^{-1} + \hat{Q}\mathfrak{W}_p\hat{Q}^T. \tag{8}$$

The spin Ω will be specified in the next section such as to obtain an explicit (and exact) expression for the elastic contribution to total velocity strain.

IV. ELASTIC STRAIN-RATE

As utilized in LUBARDA [1991], following HILL and RICE [1973], the elastic contribution to total strain-rate is defined by $D_e = \mathcal{M}_e : \mathring{\tau}$, where $\mathring{\tau}$ is the Jaumann rate of Kirchhoff stress $\tau = (\det F)\sigma$ at the current state as a reference ($F \equiv I$), while $\mathcal{M}_e$ is the associated instantaneous elastic compliance tensor. The rest of strain-rate ($D_p = D - D_e$) is plastic part, governed by the plastic potential and codirectional with the outward normal to the yield surface in the Cauchy stress space. To elaborate on this more explicitly, apply the Jaumann derivative, defined in the previous section, to the anisotropic constitutive law (3). We obtain

$$\begin{aligned}\mathring{\tau} = (\mathring{V}_e V_e^{-1})\sigma + \sigma(V_e^{-1}\mathring{V}_e) + \frac{2}{\det V_e} V_e\left[\frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e \otimes \partial B_e} : \mathring{B}_e\right] V_e \\ + \frac{2}{\det V_e} V_e\left[\frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e \otimes \partial \mathcal{R}_e} : \mathring{\mathcal{R}}_e\right] V_e,\end{aligned} \tag{9}$$

where $\mathring{\tau} = \mathring{\sigma} + (\operatorname{tr} D)\sigma$, : is double contraction and $\otimes$ the tensor product. Plastic incompressibility assumption has been used in arriving at eqn (9), that is, $(\det V_e)^{\cdot} = (\det F)^{\cdot} = (\det V_e)(\operatorname{tr} D)$. On the other hand, from eqn (7.4), symmetric and antisymmetric parts of the matrix $\mathring{V}_e V_e^{-1}$ are:

$$(\mathring{V}_e V_e^{-1})_s = (\dot{V}_e V_e^{-1} + V_e \Omega V_e^{-1})_s \equiv \tfrac{1}{2} V_e^{-1} \mathring{B}_e V_e^{-1} \tag{10}$$

and

$$(\mathring{V}_e V_e^{-1})_a = (\dot{V}_e V_e^{-1} + V_e \Omega V_e^{-1})_a - \Omega \equiv \omega - \Omega, \tag{11}$$

with the obvious representation of the spin ω. Therefore, by defining Ω to be such that $\omega \equiv W$ (total spin), it follows that

$$(\mathring{V}_e V_e^{-1})_s = D_e + \Delta_e, \tag{12}$$

that is, the sum of the elastic strain-rate (D_e) and an additional term (Δ_e), to be subsequently specified. Indeed, substitution of eqns (10)–(12) into eqn (9) gives

$$\begin{aligned}\mathring{\tau} = D_e\sigma + \sigma D_e + \frac{4}{\det V_e} V_e\left[\frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e \otimes \partial B_e} : (V_e D_e V_e)\right] V_e \\ + \Delta_e\sigma + \sigma\Delta_e + \frac{4}{\det V_e} V_e\left[\frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e \otimes \partial B_e} : (V_e \Delta_e V_e)\right] V_e \\ + \frac{2}{\det V_e} V_e\left[\frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e \otimes \partial \mathcal{R}_e} : (\dot{\mathcal{R}}_e \mathcal{R}_e^{-1} - \Omega)\mathcal{R}_e\right] V_e,\end{aligned} \tag{13}$$

or, in the shorter form, by introducing the fourth-order operator $\mathcal{L}_e$

$$\mathring{\tau} = \mathcal{L}_e : D_e + \mathcal{L}_e : \Delta_e + \frac{2}{\det V_e} V_e\left[\frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e \otimes \partial \mathcal{R}_e} : (\dot{\mathcal{R}}_e \mathcal{R}_e^{-1} - \Omega)\mathcal{R}_e\right] V_e. \tag{14}$$

It is now clear why Δ_e term is introduced in eqn (12): it is needed to cancel the third term on the right-hand side of eqn (14), so that this reduces to the needed elastic rate-type constitutive structure $\dot{\tau} = \mathcal{L}_e : D_e$. Hence, we define Δ_e such that

$$\mathcal{L}_e : \Delta_e = -\frac{2}{\det V_e} V_e \left[\frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_e \otimes \partial \mathcal{R}_e} : (\dot{\mathcal{R}}_e \mathcal{R}_e^{-1} - \Omega)\mathcal{R}_e \right] V_e, \tag{15}$$

which, after somewhat involving manipulation, can be written as

$$\mathcal{L}_e : \Delta_e = \sigma Z_a - Z_a \sigma + \mathcal{L}_e : Z_s, \tag{16}$$

where $Z = V_e^{-1}(\dot{\mathcal{R}}_e \mathcal{R}_e^{-1} - \Omega) V_e$ is the second-order tensor induced from the spin $(\dot{\mathcal{R}}_e \mathcal{R}_e^{-1} - \Omega)$ by deformation V_e, while s and a stand for the symmetric and antisymmetric parts. Therefore, the additional term Δ_e is given by

$$\Delta_e = \mathcal{L}_e^{-1} : (\sigma Z_a - Z_a \sigma) + Z_s. \tag{17}$$

Note that in the case of elastically isotropic characteristics, w is an isotropic function of B_e, hence independent of $\mathcal{R}_e$, and from eqn (15) we have $\Delta_e \equiv 0$, while eqn (17) gives $Z_s = \mathcal{L}_e^{-1} : (Z_a \sigma - \sigma Z_a)$.

The obtained elastic strain-rate is clearly objective and independent of the selected intermediate configuration, that is, rotation $\hat{Q}$. Indeed, the elastic moduli tensor $\mathcal{L}_e$, which in the component form reads

$$\mathcal{L}_{ijkl}^e = \frac{1}{2}\,(\delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk} + \sigma_{ik}\delta_{jl} + \sigma_{il}\delta_{jk}) + \frac{4}{\det V_e} V_{im}^e V_{jn}^e \frac{\partial^2 w(\mathcal{R}_e^T B_e \mathcal{R}_e)}{\partial B_{mn}^e \partial B_{pq}^e} V_{pk}^e V_{ql}^e, \tag{18}$$

(δ being the Kronecker delta) is independent of $\hat{Q}$, and since the stress rate $\dot{\tau}$ is also, $D_e = \mathcal{L}_e^{-1} : \dot{\tau}$ is independent of $\hat{Q}$ as well. It follows that $D_e^* = QD_eQ^T$, hence from eqn (12), or eqn (17), $\Delta_e^* = Q\Delta_e Q^T$. Also, it should be mentioned that the tensor $\mathcal{L}_e$ satisfies the usual symmetry and reciprocity relations. In the case of infinitesimal elasticity, the strain energy having the quadratic form

$$w = \tfrac{1}{2}\Lambda_e : (\mathcal{E}_e \otimes \mathcal{E}_e), \qquad \mathcal{E}_e = \tfrac{1}{2}(\mathcal{C}_e - I) \tag{19}$$

($\mathcal{C}_e = \mathcal{R}_e^T B_e \mathcal{R}_e$), eqn (18) reduces to

$$\mathcal{L}_{ijkl}^e \approx \mathcal{R}_{im}^e \mathcal{R}_{jn}^e \Lambda_{mnpq}^e \mathcal{R}_{kp}^e \mathcal{R}_{lq}^e, \tag{20}$$

so that the instantaneous elastic compliance tensor becomes

$$\mathcal{M}_{ijkl}^e = \mathcal{L}_{ijkl}^{e-1} = \mathcal{R}_{im}^e \mathcal{R}_{jn}^e \Lambda_{mnpq}^{e-1} \mathcal{R}_{kp}^e \mathcal{R}_{lq}^e, \tag{21}$$

where Λ_e^{-1} is the inverse of the initial elastic moduli tensor, defined by $\Lambda_{ijmn}^{e-1}\Lambda_{mnkl}^e = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Clearly, to know $\mathcal{M}_e$, the rotation tensor $\mathcal{R}_e$ must be determined.

V. AN ALTERNATIVE DERIVATION AND RELATIONSHIP WITH SOME EARLIER RESULTS

Since final results of the analysis are shown to be independent of selected intermediate configuration, we consider in this section more closely the so-called isoclinic intermediate configuration (MANDEL [1973,1981]), as introduced in Section II. The corresponding multiplicative decomposition of deformation gradient is $F = \mathcal{F}_e\mathcal{F}_p$, $\mathcal{F}_e = V_e\mathcal{R}_e$, while the stress response is

$$(\det V_e)\sigma = 2\mathcal{F}_e \frac{\partial w(\mathcal{C}_e)}{\partial \mathcal{C}_e} \mathcal{F}_e^T, \tag{22}$$

where $\mathcal{C}_e = \mathcal{F}_e^T\mathcal{F}_e$. Applying the material derivative to eqn (22), we have

$$\dot{\tau} = (\dot{\mathcal{F}}_e\mathcal{F}_e^{-1})\sigma + \sigma(\mathcal{F}_e^{-T}\dot{\mathcal{F}}_e^T) + \frac{2}{\det V_e} \mathcal{F}_e\left[\frac{\partial^2 w(\mathcal{C}_e)}{\partial \mathcal{C}_e \otimes \partial \mathcal{C}_e} : \dot{\mathcal{C}}_e\right]\mathcal{F}_e^T. \tag{23}$$

It then follows that

$$(\dot{\mathcal{F}}_e\mathcal{F}_e^{-1})_s = D_e + \hat{\Delta}_e, \tag{24}$$

since substitution in eqn (23), in view of

$$W = (\dot{\mathcal{F}}_e\mathcal{F}_e^{-1})_a + [\mathcal{F}_e(\dot{\mathcal{F}}_p\mathcal{F}_p^{-1})\mathcal{F}_e^{-1}]_a, \tag{25}$$

gives

$$\mathring{\tau} = \mathcal{L}_e : D_e + \mathcal{L}_e : \hat{\Delta}_e - \hat{W}_p\sigma + \sigma\hat{W}_p, \tag{26}$$

where $\hat{W}_p$ denotes the spin corresponding to the second term on the right-hand side of eqn (25). It is now clear that the additional term $\hat{\Delta}_e$ in eqn (24) has to be such that eqn (26) reduces to $\mathring{\tau} = \mathcal{L}_e : D_e$, hence

$$\hat{\Delta}_e = \mathcal{L}_e^{-1} : (\hat{W}_p\sigma - \sigma\hat{W}_p). \tag{27}$$

The comparison with expression (17) is easily made. It follows that $\hat{W}_p \equiv -Z_a$, therefore

$$\Delta_e = \hat{\Delta}_e + Z_s, \qquad Z_s = [V_e^{-1}(\dot{\mathcal{R}}_e\mathcal{R}_e^{-1} - \Omega)V_e]_s. \tag{28}$$

If elasticity is isotropic, $\Delta_e \equiv 0$ and $\hat{\Delta}_e = -Z_s$. Equation (28) also follows by comparing eqns (12) and (24).

The analysis presented here is in accord with the results of HILL and RICE [1973], HILL and HAVNER [1982] and ASARO [1983a,1983b]. Indeed, by taking the symmetric part of the velocity gradient, expressed via $\mathcal{F}_e$ and $\mathcal{F}_p$ (in the context of crystal plasticity, $\mathcal{F}_e$ would correspond to lattice contribution to F, associated with stretching and rotation of the lattice, whereas $\mathcal{F}_p$ would be due solely to slip), we get

$$D = (\dot{\mathcal{F}}_e\mathcal{F}_e^{-1})_s + [\mathcal{F}_e(\dot{\mathcal{F}}_p\mathcal{F}_p^{-1})\mathcal{F}_e^{-1}]_s, \tag{29}$$

hence, in view of eqn (24) and $D = D_e + D_p$, we have

$$D_p = \hat{D}_p + \mathcal{L}_e^{-1} : (\hat{W}_p \sigma - \sigma \hat{W}_p), \tag{30}$$

where $\hat{D}_p$ is the second term on the right-hand side of eqn (29). As discussed by HILL and HAVNER [1982] and ASARO [1983a]: D_p, which gives the plastic increment of strain in a stress cycle, does not come from the slip deformation ($\hat{D}_p$) alone; there is a further net elastic contribution from the lattice, which is caused by the slip-induced rotation of the lattice relative to the material stress σ. Nonetheless, it is D_p (and not $\hat{D}_p$) that is governed by the plastic potential. (The distinction between D_p and $\hat{D}_p$ is small and involves $O(\sigma/\mathcal{L}_e)$ terms in comparison to $O(1)$. However, in some applications it is necessary to retain such accuracy (ASARO & RICE [1977]).) Note, also, that the decomposition $D = D_e + D_p = \mathcal{L}_e^{-1} : \mathring{\tau} + D_p$, utilized in this paper, exactly corresponds to eqn (3.25) of ASARO [1983b], with associated normality structure established in his paper following eqn (3.30).

The elastic moduli tensor $\mathcal{L}_e$, appearing in eqn (26) via eqn (23), has the component form

$$\mathcal{L}_e^{ijkl} = \frac{1}{2}\,(\delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk} + \sigma_{ik}\delta_{jl} + \sigma_{il}\delta_{jk}) + \frac{4}{\det V_e}\,\mathcal{F}^e_{im}\mathcal{F}^e_{jn}\,\frac{\partial^2 w(\mathcal{C}_e)}{\partial \mathcal{C}^e_{mn}\partial \mathcal{C}^e_{pq}}\,\mathcal{F}^e_{kp}\mathcal{F}^e_{lq}, \tag{31}$$

so that $\mathcal{F}_e = V_e \mathcal{R}_e$ must be determined. Of course, the elastic moduli operator (31) is identical to that already obtained in eqn (18).

VI. CONSTITUTIVE EXPRESSION FOR PLASTIC STRAIN-RATE OF ORTHOTROPIC MATERIAL

In this section we consider plastic behavior of orthotropic material which hardens in an isotropic manner. If the principal axes of orthotropy (intersections of three mutually orthogonal planes of symmetry) are, say, parallel to coordinate axes in the initial configuration, in the current state they are approximately rotated by $\mathcal{R}_e$, as we assumed that elastic anisotropy remains unaltered. The yield function can accordingly be introduced as a function of the rotated-axes components of the Cauchy stress $\mathcal{T} = \mathcal{R}_e^T \sigma \mathcal{R}_e$, as

$$f = f(\mathcal{T}, \bar{\sigma}), \tag{32}$$

where $\bar{\sigma}$ is the equivalent stress which, by analogy with isotropic theory, is taken to be a function of the total plastic work, or the equivalent plastic strain

$$\bar{E}_p = \int_0^t \left(\frac{2}{3}\,D_p : D_p\right)^{1/2} dt. \tag{33}$$

Clearly, the yield function (32) is independent of the superimposed rotation Q, since $\mathcal{R}_e^* = Q\mathcal{R}_e$ and $\sigma^* = Q\sigma Q^T$, Hence $\mathcal{T}^* \equiv \mathcal{T}$. For example, in the case of Hill's orthotropic criterion

$$\begin{aligned} f = {} & [f_0(\mathcal{T}_{22} - \mathcal{T}_{33})^2 + g_0(\mathcal{T}_{33} - \mathcal{T}_{11})^2 + h_0(\mathcal{T}_{11} - \mathcal{T}_{22})^2 \\ & + 2l_0\mathcal{T}_{23}^2 + 2m_0\mathcal{T}_{31}^2 + 2n_0\mathcal{T}_{12}^2]^{1/2} - \bar{\sigma} = 0. \end{aligned} \tag{34}$$

For further analysis, we recall from HILL [1968,1978] that the work conjugate to logarithmic strain $E_0 = \ln U = E - E^2 + \cdots$, E being the Lagrange strain, is

$$T_0 = (\det F)R^T\sigma R + O(E^2), \tag{35}$$

where R is the rotation tensor from the polar decomposition of deformation gradient $F = RU$ (from the reference state). Now, if the reference state is defined such that in the current state we have $F = \mathcal{R}_e$ (reference state differs from the current state by rotation $\mathcal{R}_e$), then: $E_0 = E = 0$, $\dot{E}_0 = \dot{E} = \mathcal{R}_e^T D \mathcal{R}_e$, $T_0 = \mathcal{R}_e^T \sigma \mathcal{R}_e$ and $\dot{T}_0 = \mathcal{R}_e^T \mathring{\tau} \mathcal{R}_e$. It then follows that the normality for the plastic strain-rate $(\mathcal{R}_e^T D \mathcal{R}_e)_p \equiv \mathcal{R}_e^T D_p \mathcal{R}_e$ holds in the $\mathcal{T}$ space, that is,

$$\mathcal{R}_e^T D_p \mathcal{R}_e = \dot{\lambda}\, \frac{\partial f(\mathcal{T}, \bar{\sigma})}{\partial \mathcal{T}}, \tag{36}$$

where from

$$D_p = \dot{\lambda}\, \frac{\partial f(\mathcal{T}, \bar{\sigma})}{\partial \sigma}. \tag{37}$$

The consistency condition ($\dot{f} = 0$), on the other hand, gives

$$\frac{\partial f}{\partial \sigma} : \mathring{\tau} + \frac{\partial f}{\partial \bar{\sigma}}\, \dot{\bar{\sigma}} = 0, \tag{38}$$

hence, assuming known the relationship $\bar{\sigma} = \bar{\sigma}(\bar{E}_p)$, we obtain the loading index

$$\dot{\lambda} = \frac{1}{h}\, \frac{\partial f}{\partial \sigma} : \mathring{\tau}, \tag{39}$$

with

$$h = -\frac{\partial f}{\partial \bar{\sigma}}\, \frac{d\bar{\sigma}}{d\bar{E}_p} \left(\frac{2}{3}\, \frac{\partial f}{\partial \sigma} : \frac{\partial f}{\partial \sigma} \right)^{1/2}. \tag{40}$$

Note that $\dot{T}_0 = \mathcal{R}_e^T \mathring{\tau} \mathcal{R}_e$ has been utilized as the stress rate in the consistency condition; if, instead, $\dot{\mathcal{T}}$ has been used, erroneous results would be obtained. Therefore, the plastic strain-rate is given by

$$D_p = \mathcal{M}_p : \mathring{\tau}, \qquad \mathcal{M}_p = \frac{1}{h}\, \frac{\partial f}{\partial \sigma} \otimes \frac{\partial f}{\partial \sigma}, \tag{41}$$

with the obvious symmetry and reciprocity properties of the plastic compliance tensor $\mathcal{M}_p$. In view of the yield function dependence on the rotated axes components of Cauchy stress $\mathcal{T} = \mathcal{R}_e^T \sigma \mathcal{R}_e$, rotation $\mathcal{R}_e$ is needed in the structure of plastic compliance, as well as elastic compliance. MANDEL [1982] has discussed the question of developing constitutive expression for the spin $\dot{\mathcal{R}}_e \mathcal{R}_e^{-1}$, which would give, by integration, the rotation $\mathcal{R}_e$. In the next section we derive some additional relationships between various measures of strain-rate and spin tensors.

VII. ADDITIONAL KINEMATIC ANALYSIS

VII.1. *Relationship between plastic strain-rates in the current and intermediate configurations*

First, consider the obvious identity

$$V_e^{-1}(\mathring{V}_e V_e^{-1}) = (\mathring{V}_e V_e^{-1})^T V_e^{-1}. \tag{42}$$

As shown by AGAH-TEHRANI *et al.* [1987], for given symmetric part of $\mathring{V}_e V_e^{-1}$ (here expressed by eqn (12)), eqn (42) can be solved for the antisymmetric part to give

$$(\mathring{V}_e V_e^{-1})_a = Z_1 H_1 + H_1 Z_1 - (\text{tr}\, Z_1) H_1, \tag{43}$$

where $Z_1 = [(\text{tr}\, V_e^{-1})I - V_e^{-1}]^{-1}$ and

$$H_1 = V_e^{-1}(D_e + \Delta_e) - (D_e + \Delta_e)V_e^{-1}. \tag{44}$$

(Indeed, eqn (42) is of the form $AX = X^T A$, A being a symmetric matrix, which for a given symmetric part of X can be solved for X.) Hence, substitution into eqn (11), gives an expression for the spin Ω

$$\Omega = W - (Z_1 H_1 + H_1 Z_1) + (\text{tr}\, Z_1) H_1. \tag{45}$$

Note that $\Omega^* = \dot{Q}Q^{-1} + Q\Omega Q^T$, as it should be, since Ω is independent of the selected intermediate configuration. If the elastic component of strain is small, we have $\Omega \approx W$.

Next, from eqns (10)–(12), we have

$$\dot{V}_e V_e^{-1} = D_e + \Delta_e + W - V_e \Omega V_e^{-1}, \tag{46}$$

and substitution into eqn (4) gives

$$D_p = \Delta_e + V_e[(\dot{R}_e R_e^{-1} - \Omega) + R_e(\mathfrak{D}_p + \mathfrak{W}_p)R_e^T]V_e^{-1}. \tag{47}$$

Hence:

$$R_e \mathfrak{D}_p R_e^T = [V_e^{-1}(D_p - \Delta_e)V_e]_s \tag{48}$$

and

$$(\dot{R}_e R_e^{-1} - \Omega) + R_e \mathfrak{W}_p R_e^T = [V_e^{-1}(D_p - \Delta_e)V_e]_a. \tag{49}$$

On the other hand, the identity

$$B_e[V_e^{-1}(D_p - \Delta_e)V_e] = [V_e^{-1}(D_p - \Delta_e)V_e]^T B_e \tag{50}$$

holds, which, by using expression (48), can be solved for the antisymmetric part

$$[V_e^{-1}(D_p - \Delta_e)V_e]_a = Z_2 H_2 + H_2 Z_2 - (\text{tr}\, Z_2) H_2, \tag{51}$$

where:

$$Z_2 = [(\operatorname{tr} B_e)I - B_e]^{-1}$$
$$H_2 = B_e(R_e\mathfrak{D}_pR_e^T) - (R_e\mathfrak{D}_pR_e^T)B_e. \tag{52}$$

Substitution of eqn (51) into eqn (49), and incorporating this into eqn (47), then gives the relationship between D_p and $\mathfrak{D}_p$

$$D_p = \Delta_e + F_e[\mathfrak{D}_p + Z_0(C_e\mathfrak{D}_p - \mathfrak{D}_pC_e) + (C_e\mathfrak{D}_p - \mathfrak{D}_pC_e)Z_0 - (\operatorname{tr} Z_0)(C_e\mathfrak{D}_p - \mathfrak{D}_pC_e)]F_e^{-1}, \tag{53}$$

with $Z_0 = [(\operatorname{tr} C_e)I - C_e]^{-1}$. Since $F_e^* = QF_e\hat{Q}^T$, $C_e^* = \hat{Q}C_e\hat{Q}^T$, $\mathfrak{D}_p^* = \hat{Q}\mathfrak{D}_p\hat{Q}^T$ and $Z_0 = \hat{Q}Z_0\hat{Q}^T$, we have $D_p^* = QD_pQ^T$, that is, D_p is independent of $\hat{Q}$. Note also that, in view of eqn (53), expression (44) can be rewritten as

$$H_1 = \{2V_e^{-1}D - R_e[2\mathfrak{D}_p + Z_0(C_e\mathfrak{D}_p)_a + (C_e\mathfrak{D}_p)_aZ_0 - (\operatorname{tr} Z_0)(C_e\mathfrak{D}_p)_a]R_e^TV_e^{-1}\}_a. \tag{54}$$

VII.2. *Relationships among various spin tensors*

By considering destressing without rotation ($R_e \equiv I$), from eqns (49) and (51) we obtain the expression for the spin Ω in terms of velocity strain $\mathfrak{D}_p^L$ and spin $\mathfrak{W}_p^L$ of the Lee (L) intermediate configuration

$$\Omega = \mathfrak{W}_p^L - \tfrac{1}{2}[Z_2(B_e\mathfrak{D}_p^L)_a + (B_e\mathfrak{D}_p^L)_aZ_2 - (\operatorname{tr} Z_2)(B_e\mathfrak{D}_p^L)_a]. \tag{55}$$

On the other hand, if we utilize the isoclinic (I) intermediate configuration, we obtain, from eqns (49) and (51), the spin ($\dot{\mathfrak{R}}_e\mathfrak{R}_e^{-1} - \Omega$) in terms of the velocity strain $\mathfrak{D}_p^I$ and spin $\mathfrak{W}_p^I$ of isoclinic intermediate configuration.

$$\dot{\mathfrak{R}}_e\mathfrak{R}_e^{-1} - \Omega = -\mathfrak{R}_e\mathfrak{W}_p^I\mathfrak{R}^T + \tfrac{1}{2}\{Z_2[B_e(\mathfrak{R}_e\mathfrak{D}_p^I\mathfrak{R}_e^T)]_a + [B_e(\mathfrak{R}_e\mathfrak{D}_p^I\mathfrak{R}_e^T)]_aZ_2 - (\operatorname{tr} Z_2)[B_e(\mathfrak{R}_e\mathfrak{D}_p^I\mathfrak{R}_e^T)]_a\}. \tag{56}$$

Note that

$$\dot{\mathfrak{R}}_e\mathfrak{R}_e^{-1} = \mathfrak{W}_p^L - \mathfrak{R}_e\mathfrak{W}_p^I\mathfrak{R}_e^T, \tag{57}$$

so that the plastic spin introduced by DAFALIAS [1985], his eqn $(5)_2$, is $W_0^P = \mathfrak{W}_p^L - \dot{\mathfrak{R}}_e\mathfrak{R}_e^{-1} = \mathfrak{R}_e\mathfrak{W}_p^I\mathfrak{R}_e^T$. Also $\mathfrak{D}_p^L = \mathfrak{R}_e\mathfrak{D}_p^I\mathfrak{R}_e^T$.

If the elastic component of strain is infinitesimal ($V_e \approx I$), from eqn (45) we have $\Omega \approx W$, eqn (55) gives $\mathfrak{W}_p^L \approx \Omega$, while from eqn (56) we have

$$\dot{\mathfrak{R}}_e\mathfrak{R}_e^{-1} \approx W - \mathfrak{R}_e\mathfrak{W}_p^I\mathfrak{R}_e^T, \tag{58}$$

which, of course, also follows directly from expression (25), by substituting $\mathfrak{F}_e = V_e\mathfrak{R}_e \approx \mathfrak{R}_e$. Clearly from eqn (58), for a given (proposed) constitutive expression for the spin of isoclinic intermediate configuration, the rotation $\mathfrak{R}_e$ follows by integration.

This has been originally suggested by MANDEL [1973,1981]. In the context of crystal plasticity, $\mathcal{W}_p^I$ is given by the antisymmetric part of the proposed expression for the velocity gradient of the intermediate, solely by slip deformed configuration (ASARO [1983a, 1983b]).

VIII. CONCLUDING REMARKS

We have developed here the continuum mechanics analysis of large elasto-plastic deformation of anisotropic materials by utilizing the multiplicative decomposition of deformation gradient into its elastic and plastic parts. This has not been previously much treated in the literature, as most of the constitutive elaborations following LEE's early [1969] paper were devoted to elastically isotropic materials. Exact kinematic analysis developed here therefore extends some of the previous results, such as those presented by LUBARDA and LEE [1981] and AGAH-TEHRANI *et al.* [1987]. Relationship to other work is also given. Several points should, however, be mentioned. In the analysis given in this paper, elastic anisotropy was assumed to remain unaltered during the plastic deformation. Elimination of this restriction and inclusion of the varying or induced elastic anisotropy is clearly an important extension. This, as well as the development of the spin expression needed to determine the anisotropy orientation and to evaluate elasto-plastic stiffness and compliance tensors, may demand development of the specific micromechanical framework, with the appropriate (averaging type) procedure to incorporate its pertinent features into the macroscopic-continuum theory. Some of these issues have been discussed, for example, by MANDEL [1982] and AIFANTIS [1987]. Important results are obtained in studying various, more specific issues. An extensive analysis of the texture development and its effect on the constitutive behaviour of the polycrystals by using the transition from the micro-response of the individual grains to the macro-response of the aggregate has been done by ASARO and NEEDLEMAN [1985]. Other issues, such as modeling of different strain hardening characteristics, are also in progress, making the theory capable of more satisfactory description of the complex elasto-plastic behaviour of deformed metals.

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