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SIMPLE SHEAR OF A STRAIN-HARDENING ELASTOPLASTIC HOLLOW CIRCULAR CYLINDER

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Abstract—Stress and deformation analysis of the simple shear at finite strain of a strain-hardening elastoplastic hollow circular cylinder is given. Both isotropic and anisotropic hardening models are considered. In the case of isotropic hardening, there is a closed form analytical solution. No normal stresses exist in this case. Purely kinematic hardening with a Mises-type yield condition is utilized as a model of anisotropic hardening. Conventional (average) spin is taken to construct the objective Jaumann derivative needed in the structure of the corresponding constitutive laws. Governing partial differential equations are derived and solved numerically to give stress and deformation distribution following the advance of plastic flow. The extent or range of the appropriateness of the considered constitutive model is also established.

I. INTRODUCTION

This article deals with the problem of a simple shear of a strain-hardening elastoplastic hollow circular cylinder, which can be considered to be the analogue in the cylindrical coordinates to the simple shear problem of the rectangular block, treated in the literature by NAGTEGAAL & DE JONG [1982], LEE et al. [1983] and others, but which is more involved due to the nonhomogeneous stress and deformation distribution that takes place. After we introduce kinematic and other preliminaries, and give simple, infinitesimal elasticity solution, we first consider the isotropic hardening elastoplastic behavior. It is shown that no normal stresses exist in this case. A closed form analytical solution is obtained for the deformation and shear stress distribution at any stage of deformation. Rigid-plastic case follows from the elastoplastic solution when the shear modulus becomes infinitely large. An interesting observation is made with respect to the lack of the residual stresses on unloading, in spite of the previous nonhomogeneous plastic deformation in loading.

The anisotropic hardening behavior is subsequently analyzed in the case of purely kinematic hardening with a Mises-type yield condition, and with the conventional (average) spin used to define the objective Jaumann derivative needed in the structure of the corresponding constitutive laws. All three nonvanishing normal stresses (in cylindrical coordinates) exist in this case. The governing partial differential equations are derived, whose solution gives the deformation and stress distribution in the course of deformation. Equations are solved numerically by adequate finite difference step by step procedure, following the advance of the plastic region. An analytical explanation for the qualitatively limited acceptability of the considered model is given, which is fully in agreement with obtained numerical solution.

II. KINEMATIC AND OTHER PRELIMINARIES

Consider a long hollow circular cylinder with inner radius a and outer radius b, which is bounded by two rigid casings attached to its curved surfaces. The inner casing is fixed, while the outer casing rotates about its axis through a prescribed angle $\Omega = \Omega(t)$ (Fig. 1). Using the cylindrical coordinates (r, ϕ, z) and in view of symmetry of the problem, displacement components are given by:

$$u_\phi = r\omega(r,t), \qquad u_r = \mu_z = 0, \tag{1}$$

where $\omega = \omega(r,t)$ is the angle of rotation at radius r and time t. Velocity components are then:

$$v_\phi = r\,\frac{\partial\omega(r,t)}{\partial t}, \qquad v_r = \nu_z = 0, \tag{2}$$

so that velocity gradient matrix in cylindrical coordinates (see, for example, MALVERN [1969]) is

$$\mathbf{L} = \begin{bmatrix} 0 & -\dfrac{\partial\omega}{\partial t} & 0 \\ \dfrac{\partial\omega}{\partial t} + r\,\dfrac{\partial^2\omega}{\partial r\partial t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

having its symmetric part as the rate of deformation or velocity strain

$$\mathbf{D} = \begin{bmatrix} 0 & \dfrac{1}{2}r\,\dfrac{\partial^2\omega}{\partial r\partial t} & 0 \\ \dfrac{1}{2}r\,\dfrac{\partial^2\omega}{\partial r\partial t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

and its antisymmetric part as a spin matrix

$$\mathbf{W} = \begin{bmatrix} 0 & -\left(\dfrac{\partial\omega}{\partial t} + \dfrac{1}{2}r\,\dfrac{\partial^2\omega}{\partial r\partial t}\right) & 0 \\ \dfrac{\partial\omega}{\partial t} + \dfrac{1}{2}r\,\dfrac{\partial^2\omega}{\partial r\partial t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

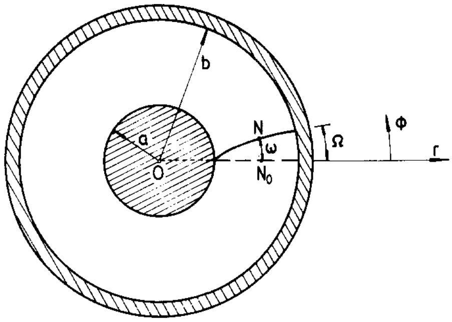

Fig. 1. Hollow circular cylinder bounded by two rigid casings.

We wish to determine, for the assumed range of deformation and known material properties, the stresses in the cylinder and the angle change $\omega = \omega(r, t)$ if the angle of rotation $\Omega = \Omega(t)$ of the outer casing is prescribed. Boundary conditions at radii $r = a$ and $r = b$ are, therefore,

$$\omega(a, t) = 0, \qquad \omega(b, t) = \Omega(t). \tag{6}$$

Neglecting the inertial effects and in view of the symmetries (no dependence on ϕ and z, with shear stresses $\sigma_{zr} = \sigma_{z\phi} = 0$), equilibrium equations reduce to:

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\phi}{r} &= 0 \\ \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{2}{r}\,\sigma_{r\phi} &= 0. \end{aligned} \tag{7}$$

Regardless of the material properties, from eqn $(7)_2$ we obtain

$$\sigma_{r\phi} = \frac{C_1(t)}{r^2}. \tag{8}$$

Determination of the function $C_1(t)$, the angle change $\omega = \omega(r, t)$ and possibly non-vanishing normal stresses σ_r, σ_ϕ and σ_z demands naturally the specification of the material properties (i.e., the corresponding constitutive equations of the problem).

III. INFINITESIMAL ELASTIC BEHAVIOR

We first consider infinitesimal elastic deformation, since if the cylinder is metallic it can be deformed in an elastic manner only infinitesimally. If the cylinder is made of rubber, for example, large elastic deformation can take place, and finite elasticity analysis applies, as discussed in Green & Zerna [1968]. Restricting, therefore, to metals and small enough values of the angle Ω deformation to be elastic, Hooke's law applies, which in the rate-type form gives for the dilatational strain rates:

$$D_r = \frac{1}{E}\,[\dot{\sigma}_r - \nu(\dot{\sigma}_\phi + \dot{\sigma}_z)] = 0$$

$$D_\phi = \frac{1}{E}\,[\dot{\sigma}_\phi - \nu(\dot{\sigma}_z + \dot{\sigma}_r)] = 0 \tag{9}$$

$$D_z = \frac{1}{E}\,[\dot{\sigma}_z - \nu(\dot{\sigma}_r + \dot{\sigma}_\phi)] = 0,$$

where E and ν are the elastic constants, and the superposed dot stands for the time derivative. No corotational (objective) derivatives are needed since the elastic moduli of metals are far greater than applied stresses. Equations (9) give $\dot{\sigma}_r = \dot{\sigma}_\phi = \dot{\sigma}_z = 0$, and since initially (when $\Omega = \Omega(0) = 0$) no stresses are present, therefore, $\sigma_r = \sigma_\phi = \sigma_z = 0$. For the shear strain rate we have

$$D_{r\phi} = \frac{1}{2}\,r\,\frac{\partial^2\omega}{\partial r\partial t} = \frac{1}{2\mu}\,\dot{\sigma}_{r\phi}, \tag{10}$$

μ being the shear modulus. On integrating this

$$\sigma_{r\phi} = \mu r\,\frac{\partial\omega}{\partial r}, \tag{11}$$

which together with eqn (8) defines the differential equation for ω,

$$\frac{\partial\omega}{\partial r} = \frac{1}{\mu}\,\frac{C_1(t)}{r^3}, \tag{12}$$

whose solution, after satisfying boundary conditions (6), is

$$\omega(r,t) = \frac{\Omega(t)}{1 - \dfrac{a^2}{b^2}}\left(1 - \frac{a^2}{r^2}\right). \tag{13}$$

The stress is given by

$$\sigma_{r\phi}(r,t) = 2\mu\,\frac{\Omega(t)}{1 - \dfrac{a^2}{b^2}}\,\frac{a^2}{r^2}. \tag{14}$$

This is greatest at inner radius $r = a$, and if the yield stress in shear is K_0, plastic deformation starts when

$$\Omega = \Omega_0 = \left(1 - \frac{a^2}{b^2}\right)\frac{K_0}{2\mu}. \tag{15}$$

The angle and stress distribution at that instant (t_0) are:

$$\omega(r,t_0) = \frac{K_0}{2\mu}\left(1 - \frac{a^2}{r^2}\right) \tag{16}$$

$$\sigma_{r\phi}(r,t_0) = K_0 \frac{a^2}{r^2}. \tag{17}$$

IV. ISOTROPIC HARDENING ELASTOPLASTIC BEHAVIOR

If the material is nonhardening (i.e., modeled as ideally plastic ($K = K_0 = \text{const.}$)), angle Ω_0 when plasticity first starts is also the ultimate (collapse) angle, since as infinitesimally thin plastic ring forms at $r = a$, the whole system becomes a mechanism ready to rotate as a rigid body about the z axis. Further deformation is therefore possible only if the cylinder is of a strain-hardening material. Let us first consider the isotropic hardening assumption with a relationship

$$\bar{\sigma} = \sigma_0(1 + m\bar{E}_p)^n \tag{18}$$

betweeen the generalized (equivalent) stress $\bar{\sigma}$ and plastic strain $\bar{E}_p$ defined by HILL [1950]:

$$\bar{\sigma} = \sqrt{3/2}(\mathbf{S}:\mathbf{S})^{1/2} \tag{19}$$

$$\bar{E}_p = \int_{t_0}^{t} \sqrt{2/3}(\mathbf{D}^p:\mathbf{D}^p)^{1/2}dt, \tag{20}$$

where $\mathbf{S}$ is the deviatoric part of the stress tensor $\boldsymbol{\sigma}$, $\mathbf{D}^p$ is the plastic part of the velocity strain matrix, while ":" stands for the trace of the matrix product. In eqn (18), σ_0 is the initial yield stress in the simple tension test, while m and n are the appropriate constants. Consider now the instant of deformation when the plastic region has spread from radius $r = a$ to $r = c(t) < b$. In this region, we have velocity strain to be the sum of elastic and plastic part $\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p$, so that by using eqn (4) and Hooke's law:

$$\begin{aligned}
D_r^p &= -D_r^e = -\frac{1}{E}[\dot{\sigma}_r - \nu(\dot{\sigma}_\phi + \dot{\sigma}_z)] \\
D_\phi^p &= -D_\phi^e = -\frac{1}{E}[\dot{\sigma}_\phi - \nu(\dot{\sigma}_z + \dot{\sigma}_r)] \\
D_z^p &= -D_z^e = -\frac{1}{E}[\dot{\sigma}_z - \nu(\dot{\sigma}_r + \dot{\sigma}_\phi)] \\
D_{r\phi}^p &= \frac{1}{2} r \frac{\partial^2 \omega}{\partial r \partial t} - \frac{1}{2\mu}\dot{\sigma}_{r\phi}.
\end{aligned} \tag{21}$$

But, due to the incompressibility of plastic deformation $D_r^p + D_\phi^p + D_z^p = 0$, hence from eqns (21), $\dot{\sigma}_r + \dot{\sigma}_\phi + \dot{\sigma}_z = 0$ and therefore $\sigma_r + \sigma_\phi + \sigma_z = 0$, as it was so initially, at the beginning of plasticity. The result that the hydrostatic part of stress is identically zero is, of course, clear since otherwise it would produce (elastic) volume change, which is impossible due to the isochoric nature of the overall deformation. The state of stress is therefore deviatoric and the Prandtl–Reuss equations become:

$$\frac{D_r^p}{\sigma_r} = \frac{D_\phi^p}{\sigma_\phi} = \frac{D_z^p}{\sigma_z} = \frac{D_{r\phi}^p}{\sigma_{r\phi}}, \tag{22}$$

i.e.,

$$\frac{\dot{\sigma}_r}{\sigma_r} = \frac{\dot{\sigma}_\phi}{\sigma_\phi} = \frac{\dot{\sigma}_{r\phi} - \mu r \dfrac{\partial^2 \omega}{\partial r \partial t}}{\sigma_{r\phi}}, \tag{23}$$

since from eqn (21):

$$D_r^p = -\frac{1}{2\mu}\dot{\sigma}_r, \qquad D_\phi^p = -\frac{1}{2\mu}\dot{\sigma}_\phi, \qquad D_z^p = \frac{1}{2\mu}(\dot{\sigma}_r + \dot{\sigma}_\phi). \tag{24}$$

From eqn (23) we conclude that $\sigma_r = \sigma_\phi = 0$ throughout the course of deformation. Indeed, denoting the right-hand side of eqn (23) by $g(r,t)$ and intergrating for σ_r, we get

$$\sigma_r = A(r)\exp\left[\int_{t_0}^{t} g(r,t)dt\right]. \tag{25}$$

But initially (at $t = t_0$) $\sigma_r = 0$, hence $A(r) = 0$ and $\sigma_r \equiv 0$. Similarly, $\sigma_\phi \equiv 0$ and $\sigma_z \equiv 0$. From eqn (22) then also $D_r^p = D_\phi^p = D_z^p \equiv 0$ throughout the course of deformation, and from eqns (19) and (20) we have:

$$\bar{\sigma} = \sqrt{3}\sigma_{r\phi}, \qquad \bar{E}_p = \frac{2}{\sqrt{3}}\int_{t_0}^{t} D_{r\phi}^p\, dt. \tag{26}$$

At the instant when the plastic region has spread to radius $c = c(t)$, the stress is

$$\sigma_{r\phi} = K_0 \frac{c^2(t)}{r^2}, \qquad K_0 = \frac{1}{\sqrt{3}}\sigma_0 \tag{27}$$

so that, after using eqn $(21)_4$, the generalized plastic strain can be expressed as

$$\bar{E}_p = \frac{1}{\sqrt{3}} r \frac{\partial \omega}{\partial r} - \frac{1}{3}\frac{\sigma_0}{\mu}\frac{c^2(t)}{r^2}. \tag{28}$$

This, as easily can be checked, satisfies the requirement $\bar{E}_p(r, t_0) = 0$, when $c(t_0) = a$. Restricting for simplicity further analysis to linear hardening ($n = 1$) (for example, the strain hardening of an aluminum alloy can be appropriately modeled by eqn (18) with $\sigma_0 = 207$ MPa (30 ksi), $m = 1.5$ and $n = 1$), substitution of eqns (27) and (28) into eqn (18), gives

$$\frac{\partial \omega}{\partial r} = \left(\frac{\sqrt{3}}{m} + \frac{1}{\sqrt{3}}\frac{\sigma_0}{\mu}\right)\frac{c^2(t)}{r^3} - \frac{\sqrt{3}}{m}\frac{1}{r}, \tag{29}$$

or, on integration,

$$\omega(r,t) = -\frac{1}{2}\left(\frac{\sqrt{3}}{m} + \frac{1}{\sqrt{3}}\frac{\sigma_0}{\mu}\right)\frac{c^2(t)}{r^2} - \frac{\sqrt{3}}{m}\ln\frac{r}{A_1(t)}, \tag{30}$$

where $A_1(t)$ is the integration function to be determined. Notice that from eqns (28) and (30), $\bar{E}_p(c,t) = 0$, as it should be.

In the elastic region $c(t) \le r \le b$ we have Hooke's law

$$\dot{\sigma}_{r\phi} = 2\mu D_{r\phi}, \tag{31}$$

from where, by using eqn (27) for stress and eqn (4) for strain rate,

$$\frac{\partial^2 \omega}{\partial r \partial t} = \frac{2}{\sqrt{3}}\frac{\sigma_0}{\mu}c(t)\frac{dc}{dt}\frac{1}{r^3}, \tag{32}$$

that is, by integrating

$$\omega(r,t) = -\frac{1}{2\sqrt{3}}\frac{\sigma_0}{\mu}\frac{c^2(t)}{r^2} + B_1(t) + B_2(r). \tag{33}$$

As at $t = t_0$, when $c(t_0) = a$, $\omega(r,t_0)$ is given by eqn (16) with $K_0 = \frac{1}{\sqrt{3}}\sigma_0$, we must have in eqn (33)

$$B_1(t_0) = \frac{1}{2\sqrt{3}}\frac{\sigma_0}{\mu}, \qquad B_2(r) = 0. \tag{34}$$

Further conditions for determining integration functions and the relationship between the extent of the plastic region c and the angle Ω are the boundary conditions (6), and condition at the interface of elastic and plastic region

$$\omega(c^-,t) = \omega(c^+,t). \tag{35}$$

Substitution of eqn (30) into eqn (6) gives

$$\ln\frac{a}{A_1(t)} = -\frac{m}{2\sqrt{3}}\left(\frac{\sqrt{3}}{m} + \frac{1}{\sqrt{3}}\frac{\sigma_0}{\mu}\right)\frac{c^2(t)}{a^2}, \tag{36}$$

hence

$$\omega(r,t) = \frac{1}{2}\left(\frac{\sqrt{3}}{m} + \frac{1}{\sqrt{3}}\frac{\sigma_0}{\mu}\right)c^2(t)\left(\frac{1}{a^2} - \frac{1}{r^2}\right) - \frac{\sqrt{3}}{m}\ln\left(\frac{r}{a}\right), \qquad a \le r \le c(t). \tag{37}$$

Substitution of eqns (33) and (37) into eqn (35) determines the expression for integration function B_1

$$B_1(t) = \frac{1}{2}\left(\frac{\sqrt{3}}{m} + \frac{1}{\sqrt{3}}\frac{\sigma_0}{\mu}\right)\frac{c^2(t)}{a^2} - \frac{\sqrt{3}}{m}\ln\frac{c(t)}{a} - \frac{\sqrt{3}}{2m}. \tag{38}$$

Clearly, $B_1(t_0) = \frac{1}{2\sqrt{3}}\sigma_0/\mu$, as it should be according to eqn (34). Therefore, in the elastic region the angle change is given by

$$\omega(r,t) = \frac{1}{2\sqrt{3}}\frac{\sigma_0}{\mu}\left(\frac{1}{a_2} - \frac{1}{r^2}\right)c^2(t) + \frac{\sqrt{3}}{2m}\left[\frac{c^2(t)}{a^2} - 1\right] - \frac{\sqrt{3}}{m}\ln\frac{c(t)}{a}, \qquad c(t) \le r \le b. \tag{39}$$

Finally, the remaining condition (6) at radius $r = b$ gives the relationship between the angle $\Omega(t)$ and radius $c(t)$ which defines the corresponding extent of the plastic zone

$$\frac{1}{2\sqrt{3}}\frac{\sigma_0}{\mu}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)c^2(t) + \frac{\sqrt{3}}{2m}\left[\frac{c^2(t)}{a^2} - 1\right] - \frac{\sqrt{3}}{m}\ln\frac{c(t)}{a} = \Omega(t). \tag{40}$$

Note that having this, eqn (39) can be rewritten as

$$\omega(r,t) = \Omega(t) + \frac{1}{2\sqrt{3}}\frac{\sigma_0}{\mu}\left(1 - \frac{b^2}{r^2}\right)\frac{c^2(t)}{b^2}, \qquad c(t) \le r \le b. \tag{41}$$

The angle $\Omega = \Omega_1$ at which plasticity first spreads through all the cylinder is obtained from eqn (40) by setting $c(t) = b$

$$\Omega_1 = \frac{1}{2}\left(\frac{\sqrt{3}}{m} + \frac{1}{\sqrt{3}}\frac{\sigma_0}{\mu}\right)\left(\frac{b^2}{a^2} - 1\right) - \frac{\sqrt{3}}{m}\ln\left(\frac{b}{a}\right). \tag{42}$$

Note that at this instant the generalized plastic strain is

$$\bar{E}_P = \frac{1}{m}\left(\frac{b^2}{r^2} - 1\right). \tag{43}$$

For example, if $b = 2a$, $m = 1.5$ and neglecting the σ_0/μ term, which is for most metals usually of the order 10^{-3} to 10^{-2}, we obtain from eqn (42) the value $\Omega_1 \cong 0.93$ rad for the angle at which plasticity spreads through all the cylinder (assuming no instability has occurred). Note that neglect of the σ_0/μ term in eqns (37) and (39) in fact reduces these expressions to ones that correspond to rigid-plastic behavior of the cylinder. Indeed, from eqn (39), for example, we see that neglecting the first term on the right-hand side makes the angle ω independent of r outside the plastic region, which means that the region rotates as a rigid body about the z axis. Since for b greater than a, plastic strain soon becomes much larger than elastic in most of the plastic region, rigid-plastic analysis can be appropriately applied for large angles $\Omega(t)$. For example, the generalized plastic strain at inner radius $r = a$ at the instant when $\Omega = \Omega_1$ is from eqn (43) for $b = 2a$, $\bar{E}_p = 2$.

For $\Omega > \Omega_1$ the whole cylinder is plastic and is strain hardening. At this stage of deformation, the angle change is given by

$$\omega(r,t) = \frac{\sqrt{3}}{m}\frac{a^2}{b^2 - a^2}\ln\left(\frac{b}{a}\right)\left(1 - \frac{b^2}{r^2}\right) + \frac{b^2}{b^2 - a^2}\Omega(t)\left(1 - \frac{a^2}{r^2}\right) - \frac{\sqrt{3}}{m}\ln\left(\frac{r}{b}\right), \quad (44)$$

as can be established in a similar manner as done previously, while the stress distribution is

$$\sigma_{r\phi} = \frac{2\sigma_0}{\dfrac{3}{m} + \dfrac{\sigma_0}{\mu}}\frac{a^2b^2}{b^2 - a^2}\left[\Omega(t) + \frac{\sqrt{3}}{m}\ln\left(\frac{b}{a}\right)\right]\frac{1}{r^2}. \quad (45)$$

For example, for $b = 2a$ and $m = 1.5$, neglecting again the σ_0/μ term, this reduces to

$$\sigma_{r\phi} = \frac{4}{3}\sigma_0[\Omega(t) + 0.8]\frac{a^2}{r^2}. \quad (46)$$

V. THE UNLOADING BEHAVIOR

The simple shear of an isotropic hardening elastoplastic hollow circular cylinder presents an example that shows that the statement often made in plasticity literature (see, for example, KACHANOV [1974], p. 110), that a body which undergoes a nonhomogeneous deformation in loading is left in the state of residual stresses after unloading, is not correct in general. Indeed, in the problem analyzed in the previous section we have nonhomogeneous plastic deformation and stress in loading, yet on unloading no residual stresses remain. This is obviously so due to the equilibrium requirement, since when we remove the moment corresponding to rotation of amount Ω (i.e., when we make the stress at outer radius $r = b$ equal to zero), no stresses can exist at any other radius r. Elastic strains on unloading are therefore zero throughout the cylinder, and residual (plastic) deformation presents a compatible strain field (i.e., no residual stresses are needed to make it compatible via the elastic strains). Consider, for example, unloading performed at the instant when $\Omega = \Omega_1$, as given by eqn (42), while

$$\omega(r,t_1) = \frac{1}{2}\left(\frac{\sqrt{3}}{m} + \frac{\sigma_0}{\sqrt{3}\mu}\right)\frac{b^2}{a^2}\left(1 - \frac{a^2}{r^2}\right) - \frac{\sqrt{3}}{m}\ln\left(\frac{r}{a}\right), \quad (47)$$

which follows from eqn (37) for $c(t) = b$. The stress distribution at this instant is

$$\sigma_{r\phi}(r,t_1) = \frac{\sigma_0}{\sqrt{3}}\frac{b^2}{r^2}, \quad (48)$$

so that the required moment (per unit length of the cylinder) for this state of stress and deformation is $M_1 = \frac{2\pi}{\sqrt{3}} b^2\sigma_0$. Removing this moment, elastic destressing is taking place and, as easily can be shown, we have:

$$\omega(r,t) = \frac{b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)\Omega(t) + \frac{\sqrt{3}}{m}\frac{a^2}{b^2 - a^2}\ln\left(\frac{b}{a}\right)\left(1 - \frac{b^2}{r^2}\right) - \frac{\sqrt{3}}{m}\ln\left(\frac{r}{b}\right) \tag{49}$$

$$\sigma_{r\phi}(r,t) = 2\mu\frac{a^2}{b^2 - a^2}\left[\Omega(t) + \frac{\sqrt{3}}{m}\ln\left(\frac{b}{a}\right) - \frac{\sqrt{3}}{2m}\left(\frac{b^2}{a^2} - 1\right)\right]\frac{b^2}{r^2}. \tag{50}$$

Residual angle $\Omega = \Omega_r$ at which unloading is completed follows from eqn (50) by setting $\sigma_{r\phi} = 0$

$$\Omega_r = \frac{\sqrt{3}}{2m}\left(\frac{b^2}{a^2} - 1\right) - \frac{\sqrt{3}}{m}\ln\left(\frac{b}{a}\right). \tag{51}$$

This, of course, can also be obtained directly from eqn (42) by substructing the elastic part (i.e., the term proportional to σ_0/μ). On substituting eqn (51) into eqn (49) we get the residual deformation

$$\omega_r = \frac{\sqrt{3}}{2m}\frac{b^2}{a^2}\left(1 - \frac{a^2}{r^2}\right) - \frac{\sqrt{3}}{m}\ln\left(\frac{r}{a}\right). \tag{52}$$

Clearly this is same as eqn (47) if material is elastically rigid, or if we neglect the small term proportional to σ_0/μ.

VI. ANISOTROPIC HARDENING BEHAVIOR

The nature of the problem and the stress-deformation behavior is quite different when a model of anisotropic hardening is utilized. Consider, for example, purely kinematic hardening with a Mises-type yield condition

$$(\mathbf{S} - \boldsymbol{\alpha}) : (\mathbf{S} - \boldsymbol{\alpha}) = \frac{2}{3}\sigma_0^2, \tag{53}$$

where $\boldsymbol{\alpha}$ is the back stress whose growth (as an internal variable) is prescribed by the evolution law

$$\frac{\mathfrak{D}\boldsymbol{\alpha}}{\mathfrak{D}t} = \Lambda : \mathbf{D}^p, \tag{54}$$

where the fourth-order operator Λ depends on $\mathbf{S}$, $\boldsymbol{\alpha}$, and variables selected to account for the history of deformation, while $\mathfrak{D}/\mathfrak{D}t$ stands for the objective Jaumann derivative associated with the appropriate spin. In the simplest case Λ is assumed to be a constant operator ($\Lambda_{ijkl} = h\delta_{ik}\delta_{jl}$, with h = const., and $\boldsymbol{\delta}$ the Kronecker delta), corresponding to linear hardening in tension with constant tangent modulus (equal to $\frac{3}{2}h$). Using the Jaumann derivative of the back stress with respect to the spin $\mathbf{W}$, given in our problem by eqn (5), which is the average of the angular velocities over all directions in the current configuration, the change of $\boldsymbol{\alpha}$ is given by

$$\dot{\boldsymbol{\alpha}} = h\mathbf{D}^p + \mathbf{W}\boldsymbol{\alpha} - \boldsymbol{\alpha}\mathbf{W}, \tag{55}$$

where the superimposed dot denotes material time derivative. The first term on the right-hand side of eqn (55) gives contribution to the growth of $\boldsymbol{\alpha}$ due to the current plastic flow, while the last two are due to the rotation of $\boldsymbol{\alpha}$.

In view of small elastic strains, rigid-plastic theory can be adopted and the flow law associated with the yield condition (53) becomes

$$\mathbf{D} \sim (\mathbf{S} - \boldsymbol{\alpha}). \tag{56}$$

In the considered problem of simple shear, this gives

$$\frac{0}{S_r - \alpha_r} = \frac{0}{S_\phi - \alpha_\phi} = \frac{0}{S_z - \alpha_z} = \frac{\frac{1}{2}\, r\, \frac{\partial^2 \omega}{\partial r \partial t}}{\sigma_{r\phi} - \alpha_{r\phi}}. \tag{57}$$

Hence, $S_r = \alpha_r$, $S_\phi = \alpha_\phi$ and $S_z = \alpha_z = -(\alpha_r + \alpha_\phi)$, so that unless $\alpha_r = \alpha_\phi = 0$, there are normal stresses developed in the cylinder. From the yield condition (53) we further have

$$\sigma_{r\phi} = \frac{\sigma_0}{\sqrt{3}} + \alpha_{r\phi}, \tag{58}$$

while from the evolution law (55)

$$\dot{\alpha}_r = -2\alpha_{r\phi}\, W_{\phi r} = -\dot{\alpha}_\phi, \tag{59}$$

(so that $\alpha_r = -\alpha_\phi$, i.e., $S_r = -S_\phi$ and $S_z = \alpha_z = 0$), and

$$\dot{\alpha}_{r\phi} = 2\alpha_r W_{\phi r} + \frac{1}{2}\, hr\, \frac{\partial^2 \omega}{\partial r \partial t}. \tag{60}$$

But, from the equilibrium equations (7):

$$\frac{\partial S_r}{\partial r} + 2\frac{S_r}{r} = \frac{\partial p}{\partial r} \tag{61}$$

$$\sigma_{r\phi} = \frac{H(t)}{r^2}, \tag{62}$$

where $p = -\frac{1}{3}(\sigma_r + \sigma_\phi + \sigma_z)$, while $H(t)$ is the integration function. Equation (61) serves to determine the unknown pressure $p = p(r,t)$, within an arbitrary constant, after deviatoric stress S_r is found. The pressure distribution is, of course, needed to get the overall stress distribution: $\sigma_r = S_r - p$, $\sigma_\phi = S_\phi - p$, $\sigma_z = -p$. To determine the stresses S_r and $\sigma_{r\phi}$ (i.e., α_r and $\alpha_{r\phi}$) we, therefore, have on disposal eqns (59), (60) with (58), (62) and the appropriate boundary conditions at $r = a$ and $r = c$, where the plastic front has just arrived. Recall that at $r = a$ we have $\partial\omega/\partial t = 0$, as $\omega(a,t) = 0$, while at $r = c$ we have $\alpha_r = \alpha_\phi = \alpha_{r\phi} = 0$ and $\sigma_{r\phi} = \frac{\sigma_0}{\sqrt{3}}$, as no plastic deformation yet occurred at interface $r = c$. The function $H(t)$ in eqn (62) can then clearly be expressed as $H(t) = \frac{\sigma_0}{\sqrt{3}} c^2(t)$. Of course, at $r = c$ we also have $\omega = \Omega$.

In the integration procedure of solving eqns (59) and (60), it is convenient to take monotonically increasing angle Ω as the measure of "time" t, so that the governing equations of the problem in the region $a \leq r \leq c$ become:

$$\frac{\partial S_r}{\partial \Omega} = -2\left[\frac{H(\Omega)}{r^2} - \frac{\sigma_0}{\sqrt{3}}\right]\left(\frac{\partial\omega}{\partial\Omega} + \frac{1}{2} r \frac{\partial^2\omega}{\partial r \partial\Omega}\right) \tag{63}$$

$$\frac{1}{r^2}\frac{dH}{d\Omega} = 2S_r\left(\frac{\partial\omega}{\partial\Omega} + \frac{1}{2} r \frac{\partial^2\omega}{\partial r \partial\Omega}\right) + \frac{1}{2} hr \frac{\partial^2\omega}{\partial r \partial\Omega}. \tag{64}$$

These equations are solved numerically by an adequate finite difference, step-by-step procedure following advance of the plastic region. The important feature of these equations and hence of the utilized kinematic hardening model is, however, that after a certain amount of plastic deformation, the instant is reached when further plastic deformation is accompanied by decreasing load, which is physically an unacceptable behavior. Indeed, from eqn (64) at $r = a$, $\mathrm{d}H/\mathrm{d}\Omega$ becomes equal to zero when

$$S_r + \frac{1}{2} h = 0, \tag{65}$$

hence shear stress $\sigma_{r\phi}$ at that instant reaches its extremum (maximum), and further deformation would be followed by decreasing stress, clearly an unrealistic behavior. That condition (65) will be attained after some amount of plastic deformation is certain since from eqns (59) and (60) at $r = a$, upon integration

$$\alpha_{r\phi}^2 = -\alpha_r^2 - h\alpha_r, \tag{66}$$

hence $-h \leq \alpha_r \leq 0$, and $S_r (\equiv \alpha_r)$ is negative. It turns out, therefore, that the considered model of kinematic hardening gives qualitatively acceptable results only in the limited range of stress and deformation (until condition (65) is reached). This is fully confirmed by numerical solution of eqns (63) and (64). For example, for $h = \sigma_0$, the plastic zone reaches the radius $c = 1.16a$, when condition (65) is attained, angle Ω being 0.040 rad, whereas the corresponding plastic strain at $r = a$ is $\bar{E}_p = 0.35$. For higher rates of hardening, a higher extent of plastic region is attained before the critical instant is reached (for $h = 1.5\sigma_0$: $c = 1.29a$, $\Omega = 0.084$ rad, $\bar{E}_p(r = a) = 0.42$, while for $h = 2\sigma_0$: $c = 1.42a$, $\Omega = 0.13$ rad, $\bar{E}_p(r = a) = 0.52$). Note that the angle change $\omega = \omega(r,\Omega)$ is independent of the (initial) yield stress σ_0, as is clear from eqns (63) and (64), since S_r, H, and h are all proportional to σ_0 (of course, the generalized plastic strain $\bar{E}_p$ is also independent of σ_0). Figures 2 to 5 present numerically obtained stress and defor-

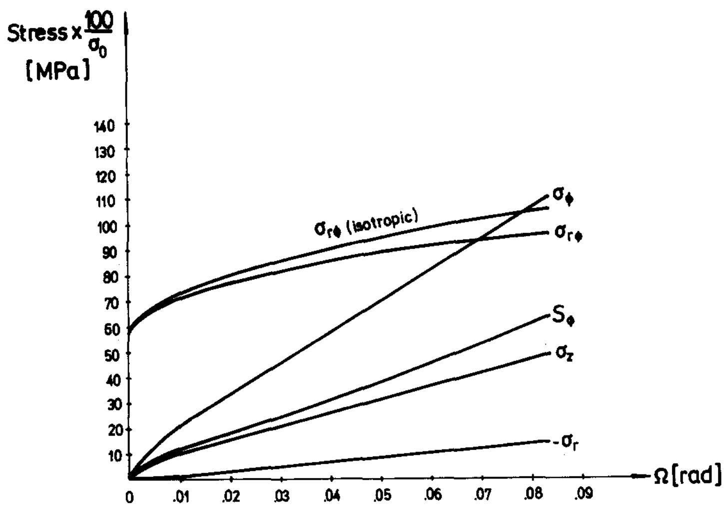

Fig. 2. Stress variation at $r = a$.

mation distribution when $h = 1.5\sigma_0$. For comparison, corresponding isotropic hardening solution is also plotted (eqns (27), (37), and (40) with $\sigma_0/\mu = 0$ and $m = 2.25$). Figure 2 gives the stress variation at inner radius $r = a$, as the angle of rotation Ω is being increased from zero to critical value $\Omega_c = 0.084$ rad, as previously discussed. Figure 3 shows the stress distribution in the plastic region ($a \leq r \leq c$) at the instant when $\Omega = \Omega_c$, while Fig. 4 gives the extent of the plastic region c as a function of the angle Ω. Finally, Fig. 5 presents the plastically deformed shape of the initially straight line ($\phi =$ const.), i.e. the function $\omega = \omega(r)$ at the instant $\Omega = \Omega_c$. Outside the plastic region ($c \leq r \leq b$), the angle $\omega =$ const. $\equiv \Omega_c$ as that region is not deformed but only rigidly rotated for the angle Ω_c. No normal stresses exist there, whereas shear stress is simply determined from the moment equilibrium equation. Note that the kinematic hardening plasticity model gives softer response than the isotropic hardening, which is due to the presence of growing back stress which, as a residual stress on microlevel, helps macroscopic stress in producing a given amount of plastic deformation.

VII. DISCUSSION

A quite analogous situation with anomalous stress-deformation behavior after a certain amount of prior plastic deformation was observed by NAGTEGAAL & JONG [1982] in the problem of large simple shear of rectangular block. LEE et al. [1983] consequently suggested that, in place of the average spin **W**, the spin of the specific lines of material elements, such as are directions of the eigenvectors of $\boldsymbol{\alpha}$ (which may be considered to carry the major influence of the back stress), should be used as an appropriate spin in the evolution law (55). This choice of the spin $\mathbf{W}^*$ eliminates the spurious oscillation of stresses obtained at large finite strain by using the conventional (average) spin **W**. The use of the average spin within the considered anisotropic model is indeed the reason for the onset of unrealistic behavior after a certain amount of plastic deformation. If a modified spin has been used, that does not make the right hand side of eqn (64) equal to zero at any (however large) stage of deformation; steady growth of the stress distri-

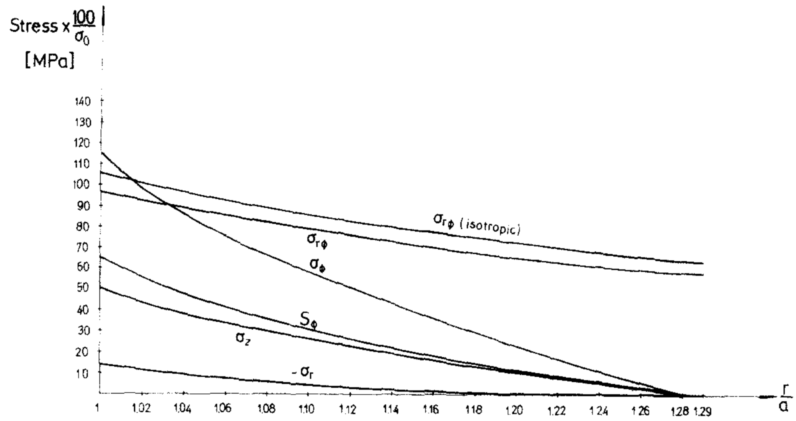

Fig. 3. Stress distribution when $\Omega = 0.084$ rad.

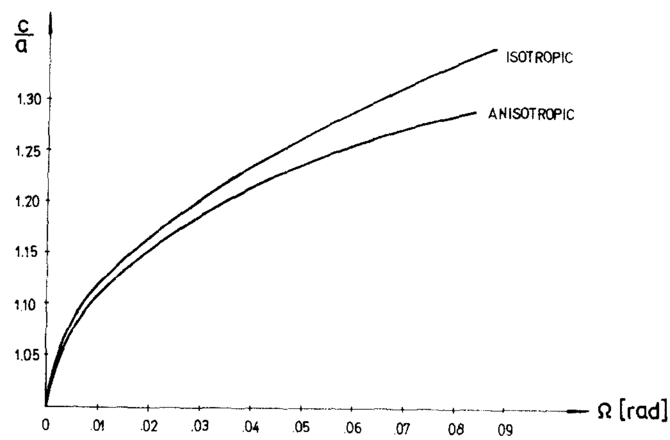

Fig. 4. Extent of the plastic region $c = c\ (\Omega)$.

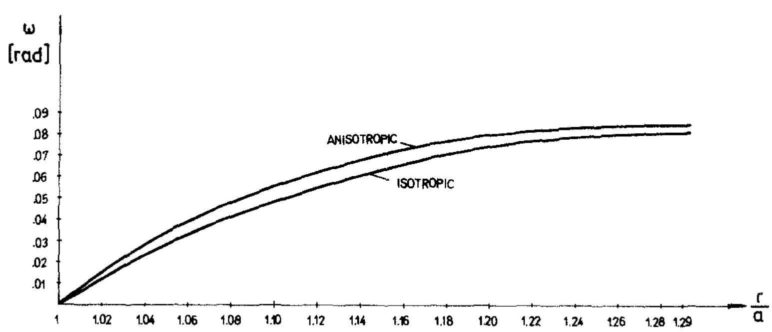

Fig. 5. Angle change $\omega = \omega(r)$ when $\Omega = 0.084$ rad.

bution would be achieved corresponding to continuing plastic deformation. At moderate finite strain, however, where the generalized plastic strain is below a certain limit, the use of either spin leads to approximately the same results (for example, LEE et al. [1981] obtained this in the range of $\bar{E}_p < \sim 0.5$), which agrees with findings obtained here that oscillations start approximately at that amount of (maximum) generalized plastic strain. Further development of the kinematic and, more generally, anisotropic hardening plasticity theory is needed to obtain satisfactory results at arbitrary strain with respect to both qualitatively and quantitatively correct prediction of stress-deformation distribution. For example, FARDSHISHEH & ONAT [1974] suggested, in place of eqn (54), a more general structure of the evolution law

$$\frac{\mathfrak{D}\boldsymbol{\alpha}}{\mathfrak{D}t} = \mathbf{f}(\boldsymbol{\alpha}, \mathbf{D}^p), \tag{67}$$

where $\mathbf{f}$ is an isotropic tensor function of $\boldsymbol{\alpha}$ and $\mathbf{D}^p$, while $\mathfrak{D}/\mathfrak{D}t$ is the Jaumann derivative with respect to total spin $\mathbf{W}$. Further development of the evolution law structure based on the representation theorems of isotropic tensor functions is given by AGAH-TEHRANI et al. [1986] and LUBARDA [1987]. This, however, still corresponds to the simplest model of the plastic anisotropy, the shift of the yield surface in stress space. The theory that includes the shape changes of the yield surface or other more involved complexities introduced to adequately represent experimentally observed anisotropic hardening effects is inevitably needed in order to accurately predict the complicated stress-deformation behavior that occurs in many technological processes of plastic forming, such as extrusion or drawing.

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