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SOME COMMENTS ON PLASTICITY POSTULATES AND NON-ASSOCIATIVE FLOW RULES

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Abstract—The plasticity postulates of Drucker and Ilyushin do not in general apply to materials whose mechanical behavior is described by non-associative flow rules, although for many stress and strain cycles and under many loading conditions the corresponding work inequalities defining the postulates are not violated. This is illustrated in this paper by considering pressure-dependent dilatant materials subjected to combined shear-pressure, and triaxial compression loadings. Both, hardening and softening ranges of material response are analyzed. It is shown that, depending on the relationship between the dilatancy factor and the internal friction coefficient, the net work of added stresses in some typical closed stress cycles can be either positive or negative, while the net work in the corresponding closed strain cycles is in each case positive. Constitutive implications and the relationship between plasticity postulates and actual material stability are also discussed.

NOTATION

W_D	Drucker's work
W_I	Ilyushin's work
σ_{ij}	stress tensor
$d\varepsilon_{ij}$	strain increment tensor
$d\varepsilon_{ij}^p$	plastic strain increment tensor
f	yield function
g	plastic potential function
$d\lambda$	loading parameter
h	history dependent parameter
I_1, J_2	stress invariants
S_{ij}	deviatoric stress tensor
$\bar{\sigma}$	yield stress in uniaxial tension
α, β	material parameters
k	yield size parameter
h^p	plastic modulus
$\bar{\varepsilon}^p$	generalized plastic strain
τ	shear stress
p	pressure
$d\gamma^p$	plastic shear strain increment
$d\varepsilon^p$	plastic dilatation increment
G, K	elastic shear and bulk moduli

1. INTRODUCTION

One of the central ingredients in the constitutive modeling of plastic behavior is a concept of the yield surface. This represents a locus of stress states within which the material response is purely elastic. For materials that obey either Drucker's or Ilyushin's postulate, given by the appropriate stress or strain cycle inequalities [1–3], it follows that the plastic strain increment must be codirectional with the outward normal to a locally smooth yield surface, while at a vertex it must lie within or on the cone of limiting outward normals. It is also known that Ilyushin's requirement of positive work in every strain cycle is a weaker restriction on the material behavior than Drucker's requirement of positive work of added stresses in every stress cycle. Hence, the class of materials obeying Ilyushin's postulate is

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broader than the class of materials obeying Drucker's postulate. Neither of the two postulates is applicable to materials which dissipate energy by friction. Although both postulates represent sufficient conditions for the normality of the plastic strain increment to a locally smooth part of the yield surface, neither one is a necessary condition. In other words, a consitutive model for plastic deformation can be governed by an associative flow rule, with the plastic strain increment normal to the yield surface, even though there may be cycles of stress and strain for which Drucker's and Ilyushin's work inequalities are both violated.

Regardless of the plasticity postulates, it has been experimentally observed that for some materials associative flow rules do not accurately predict certain essential features of the mechanical response. For example, they tend to largely overestimate the inelastic volume changes in geomaterials like rocks and soils. Consequently, non-associative flow rules have been employed for such materials. The plastic strain increment is then normal to the plastic potential surface, which is distinct from the yield surface. With regard to plasticity postulates it follows that for materials described by a non-associative flow rule, there are at least some cycles of stress and strain violating Drucker's and Ilyushin's work inequalities. Hence, neither postulate in general applies to such materials. Although the postulates do not apply to every cycle, there are many cycles and loading conditions for which the respective work inequalities are not violated. This fact is illustrated in this paper by considering a combined shear-pressure, and confined compression loading of the pressure-dependent dilatant material, which is governed by a non-associative flow rule and the Drucker–Prager yield condition. Both, hardening and softening ranges of the material response are analyzed. Depending on the relationship between the dilatancy factor and the internal friction coefficient, it is shown that the net work of added stresses in many closed stress cycles can be either positive or negative, while the net work in the corresponding closed strain cycles is in each case positive. It is further found in the case of an associative flow rule that the net work of added stresses in an infinitesimal stress cycle is independent of the direction of the cycle. However, the net work is strongly dependent on the direction of the cycle in the case of a non-associative flow rule. The relationship between plasticity postulates and actual material stability is also discussed.

The issues discussed in this paper are addressed in the context of an infinitesimal strain formulation. The implications of the two plasticity postulates and the representation of the corresponding work inequalities in the domain of finite deformations have been examined elsewhere (see Hill [4]). Second-order tensors are expressed by their Cartesian components, with the summation convention implied over repeated indices.

2. PLASTICITY POSTULATES

The two well-known plasticity postulates, one by Drucker [1, 2] and the other by Ilyushin [3], both lead to normality of the plastic strain increment to a locally smooth part of the yield surface in stress space. Drucker's postulate states that the net work of added stresses in a *stress* cycle originating and terminating at some initial stress state σ_{ij}^0 is non-negative

$$W_D = \oint (\sigma_{ij} - \sigma_{ij}^0)\, d\varepsilon_{ij} \geqslant 0. \tag{1}$$

Ilyushin defines a class of materials for which the net work in an arbitrary *strain* cycle is non-negative

$$W_I = \oint \sigma_{ij} d\varepsilon_{ij} \geqslant 0. \tag{2}$$

Ilyushin has shown that $W_I > W_D$, hence the class of materials obeying Ilyushin's postulate is broader than that obeying Drucker's postulate. For example, it may happen that a material behavior is such that over some stress cycles $W_D < 0$, while $W_I > 0$ for every strain cycle. Since Ilyushin's postulate is a sufficient condition for the normality of plastic strain increment to a locally smooth part of the yield surface f (Ilyushin [3], Hill [4]), above indicates that the normality and associative flow rule

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{3}$$

can hold although the material does not satisfy Drucker's postulate. In other words, Drucker's postulate is not a necessary condition for the normality of the plastic strain increment to the yield surface. This is not surprising because the Drucker postulate is here considered in its broad sense referring to any closed stress cycle, involving infinitesimal or large additional plastic straining. This is certainly a strong restriction or constraint on the admissible material behavior. If Drucker's postulate is restricted to stress cycles that involve only infinitesimal additional plastic straining, the inequality (1) becomes, in view of elastic recovery in closed stress cycles,

$$W_D = (\sigma_{ij} - \sigma_{ij}^0)\, d\varepsilon_{ij}^{p} + \tfrac{1}{2}\, d\sigma_{ij} d\varepsilon_{ij}^{p} \geqslant 0, \tag{4}$$

to terms of second order. If the stress state σ_{ij}^0 is well inside the current yield surface or on the yield surface far away from the current stress state σ_{ij}, the term proportional to $d\sigma_{ij}$ vanishes in the limit as $d\varepsilon_{ij}^{p} \to 0$, and (4) becomes

$$W_D = (\sigma_{ij} - \sigma_{ij}^0)\, d\varepsilon_{ij}^{p} \geqslant 0. \tag{5}$$

The inequality (5) is often referred to as the principle of maximum plastic work (Hill [5], Johnson and Mellor [6], Lubliner [7]). It assures convexity of the yield surface and normality of the plastic strain increment to a locally smooth part of the yield surface. It also leads directly to certain general principles, such as uniqueness and variational theorems. For convex yield surfaces, (5) is a necessary and sufficient condition for the normality of the plastic strain increment to the yield surface. This holds for both hardening and softening ranges of material response, i.e. regardless of whether the current yield surface locally expands or shrinks during an ongoing increment of plastic deformation (Lubarda [8]). When elastic response is nonlinear and is altered by plastic deformation, which is often the case in geomaterials, Drucker's postulate still implies the normality, but not necessarily the convexity of the yield surface (Palmer *et al.* [9]).

Consider next the initial stress state σ_{ij}^0 to be on the current yield surface. In hardening material an infinitesimal stress cycle of adding and removing the stress increment $d\sigma_{ij}$, involving plastic deformation $d\varepsilon_{ij}^{p}$ during the loading segment, can always be performed and (4) gives

$$W_D = \tfrac{1}{2}\, d\sigma_{ij} d\varepsilon_{ij}^{p} \geqslant 0, \quad \text{i.e. } d\sigma_{ij} d\varepsilon_{ij}^{p} \geqslant 0. \tag{6}$$

The inequality (6) is a necessary and sufficient condition for the normality of the plastic strain increment to the yield surface if material is in a hardening range. If material is in a softening range, the stress cycle originating from the stress state on the current yield surface and involving plastic deformation is physically not possible. Consequently, consider a stress cycle starting from the stress state $\sigma_{ij} + d\sigma_{ij}$ within the current yield surface, infinitesimally close to the stress state σ_{ij} on the yield surface. A cycle consists of elastic loading from $\sigma_{ij} + d\sigma_{ij}$ to σ_{ij}, followed by plastic loading from σ_{ij} back to $\sigma_{ij} + d\sigma_{ij}$, which is now a stress state on the new locally shrinked yield surface. The corresponding work is obtained from (4) as

$$W_D = -\tfrac{1}{2}\, d\sigma_{ij} d\varepsilon_{ij}^{p} \geqslant 0, \quad \text{i.e. } d\sigma_{ij} d\varepsilon_{ij}^{p} \leqslant 0. \tag{7}$$

Since in a softening range the plastic strain increment is produced by a stress decrement pointing inward to the current yield surface, the inequality (7) is a necessary and sufficient condition for the plastic strain increment to be codirectional with the outward normal to the yield surface in a softening range of the material response.

Returning to Ilyushin's postulate, we note that although it imposes less restrictions on the material behavior than Drucker's postulate applied to all possible cycles of stress, Ilyushin's postulate is also not a necessary condition for the normality of the plastic strain increment. For example, Palmer *et al.* [9] provide an example of negative work in certain stress and strain cycles for materials that have experienced enormous cyclic work-softening. Yet, an associated flow rule can be employed in describing their material response, with the plastic strain increment normal to the current yield surface.

Many materials of engineering importance, particularly geomaterials such as rocks and soils, which exhibit pressure dependence, plastic volumetric changes and frictional effects, generally obey neither Drucker's nor Ilyushin's postulate [9, 10]. To describe their mechanical response, the non-associative flow rules are commonly employed utilizing the plastic potential function g, different from the yield function f, such that

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial g}{\partial \sigma_{ij}}. \tag{8}$$

In this case, the plastic strain increment $d\varepsilon_{ij}^{p}$ is not normal to the yield surface and none of the inequalities (4–7) in general apply. Indeed, from the consistency condition for the continuing plastic deformation it follows that the loading parameter is $d\lambda = h^{-1}(\partial f/\partial\sigma_{kl})d\sigma_{kl}$, where h is a history dependent scalar function of stress and other state variables introduced in the considered material model. Therefore,

$$d\sigma_{ij}d\varepsilon_{ij}^{p} = \frac{1}{h}\left(\frac{\partial f}{\partial \sigma_{kl}}\, d\sigma_{kl}\right)\left(\frac{\partial g}{\partial \sigma_{ij}}\, d\sigma_{ij}\right). \tag{9}$$

This clearly becomes negative in the hardening range ($h > 0$) whenever the increment of stress $d\sigma_{ij}$ makes an obtuse angle with the plastic potential gradient $\partial g/\partial\sigma_{ij}$, violating the Drucker inequality (6). It also violates the inequality (7) by becoming positive in the softening range ($h < 0$) whenever the decrement of stress $d\sigma_{ij}$ makes an acute angle with the gradient $\partial g/\partial\sigma_{ij}$. Consequently, the positive value of $d\sigma_{ij}d\varepsilon_{ij}^{p}$ is neither a necessary nor a sufficient condition for actual stability of the material response. The relationship between material stability and plasticity postulates in the case of non-associative flow rules have been studied by Mróz [11], Mandel [12], Maier [13], and others. Mandel [12] proposed as a necessary condition for stability that all eigenvalues of the acoustic tensor be real and positive, based on the assumption that in stable material a small disturbance can propagate in the form of waves. More recently, there has been a series of papers by Lade and his coworkers [14–16], who examined the conditions for stability on the basis of experimental observations of the behavior of granular materials in triaxial tests, and found that $d\sigma_{ij}d\varepsilon_{ij}^{p}$ can be negative even in the hardening range of material response, without any physical instability actually occurring.

3. NON-ASSOCIATIVE FLOW RULE WITH DRUCKER–PRAGER YIELD CRITERION

To describe the pressure-dependent yield of soils, Drucker and Prager [17] suggested a yield condition

$$\alpha I_1 + J_2^{1/2} = k, \tag{10}$$

where $\alpha > 0$ is a material parameter, $I_1 = \sigma_{kk}$ is the first stress invariant, $J_2 = \frac{1}{2}S_{ij}S_{ij}$ is the second invariant of the deviatoric stress $S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$, and k is the parameter that defines the size of the yield surface. If the yield stress in uniaxial tension is $\bar{\sigma}$, the parameter k is equal to $(\alpha + 1/\sqrt{3})\bar{\sigma}$. For $\alpha = 0$, Drucker–Prager condition reduces to the (pressure-independent) von Mises yield condition. More general structures of the pressure-dependent yield condition, involving also the third invariant of the deviatoric stress, have been suggested by various authors, as discussed by Chen and Han [18], and Lubarda *et al.* [19].

It has been experimentally observed that the associative flow rules tend to overestimate inelastic volume changes in geomaterials like rocks and soils. Consequently, the non-associative flow rules have been commonly utilized for such materials, with the plastic or flow potential function different from the yield function. A simple modification of the Drucker–Prager yield function often used as a plastic potential is

$$g = \beta I_1 + J_2^{1/2}, \tag{11}$$

where β is a material parameter, generally different from α. The plastic strain increment is obtained from (8) and (11) as

$$d\varepsilon_{ij}^{p} = d\lambda(\beta\delta_{ij} + \tfrac{1}{2}J_2^{-1/2}S_{ij}). \tag{12}$$

In the case of isotropically hardening Drucker–Prager material, the parameter k, which defines the size of the yield surface, can be taken as a function of the generalized plastic strain given by the integral of

$$d\bar{\varepsilon}^{p} = \frac{1 + \sqrt{3}\beta}{(1 + 6\beta^2)^{1/2}}(\tfrac{2}{3}d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p})^{1/2}. \tag{13}$$

The coefficient on the right-hand side of (13) is chosen such that in a uniaxial tension test $\bar{\varepsilon}^{p}$ coincides with the longitudinal plastic strain. In view of (12), the increment of the generalized plastic strain (13) can shortly be written as

$$d\bar{\varepsilon}^{p} = \left(\beta + \frac{1}{\sqrt{3}}\right)d\lambda. \tag{14}$$

The loading parameter $d\lambda$ is obtained from the consistency condition for continuing plastic flow ($df = dk$), f being the yield function defined by the left-hand side of (10). This gives

$$d\lambda = \frac{1}{h}(\alpha\delta_{kl} + \tfrac{1}{2}J_2^{-1/2}S_{kl})d\sigma_{kl}, \tag{15}$$

where

$$h = \left(\alpha + \frac{1}{\sqrt{3}}\right)\left(\beta + \frac{1}{\sqrt{3}}\right)h^{p} \tag{16}$$

is a history dependent parameter, equal for all stress states on a given yield surface, and $h^{p} = d\bar{\sigma}/d\bar{\varepsilon}^{p}$ is the plastic modulus (positive for hardening, and negative for softening range of the material response). Note also that $h = dk/d\lambda$. Substitution of (15) into (12) gives the plastic strain increment in terms of the stress increment. Whenever $\alpha \neq \beta$, the corresponding plastic compliance tensor does not possess a reciprocal symmetry, and the variational integral can not be constructed for a material with a non-associative flow rule.

In the case of loading by shear stress τ under confining pressure p, the non-vanishing plastic strain increments are the engineering shear strain $d\gamma^{p}$ and the volumetric strain $d\varepsilon^{p}$. From (12) they are

$$d\gamma^{p} = d\lambda, \quad d\varepsilon^{p} = 3\beta d\lambda. \tag{17}$$

Hence, the ratio $d\varepsilon^{p}/d\gamma^{p}$ is equal to 3β, which is often called the dilatancy factor (Rudnicki and Rice [20], Nemat-Nasser and Shokooh [21]). The Drucker–Prager yield condition (10) becomes $\tau - 3\alpha p = k$, so that the parameter 3α can be interpreted as the internal friction coefficient. (According to [20], the representative values of the friction coefficient and the dilatancy factor of fissured rocks are $3\alpha = 0.3$–1.0, and $3\beta = 0.1$–0.5). The inner product of the increments of stress and plastic strain is easily found to be

$$d\sigma_{ij}d\varepsilon_{ij}^{p} = \frac{1}{h}(d\tau - 3\alpha dp)(d\tau - 3\beta dp), \tag{18}$$

where h is defined by (16). In a hardening range of the material response $d\tau > 3\alpha dp$, because the yield surface expands during plastic deformation ($dk > 0$), and from (18) it follows that $d\sigma_{ij}d\varepsilon_{ij}^{p}$ is positive when $d\tau > 3\beta dp$, and negative when $d\tau < 3\beta dp$. The range of plastic loading directions for the stress increment which make $d\sigma_{ij}d\varepsilon_{ij}^{p}$ negative in the hardening range is depicted by the shaded area in Fig. 1. If the material is in a softening range, the yield surface shrinks during plastic loading ($dk < 0$) and $d\tau < 3\alpha dp$. From (18), therefore, follows that $d\sigma_{ij}d\varepsilon_{ij}^{p}$ is negative when $d\tau < 3\beta dp$, and positive when $d\tau > 3\beta dp$. The range of plastic loading directions for the stress decrement which make $d\sigma_{ij}d\varepsilon_{ij}^{p}$ positive in the softening range is depicted by the dotted area in Fig. 1. The violation of the usual inequalities $d\sigma_{ij}d\varepsilon_{ij}^{p} > 0$ in the hardening, and $d\sigma_{ij}d\varepsilon_{ij}^{p} < 0$ in the softening range of material response is

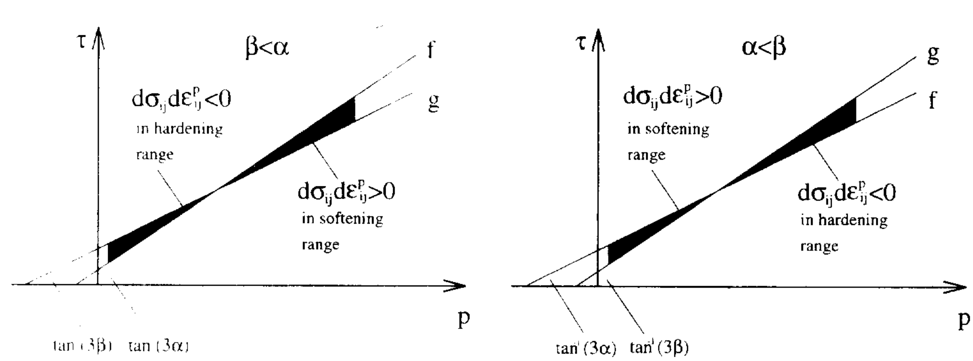

Fig. 1. The shaded regions define directions of the stress increment which make $d\sigma_{ij}d\varepsilon_{ij}^{p}$ negative in the hardening range of material response. The dotted regions define directions of the stress decrement which make $d\sigma_{ij}d\varepsilon_{ij}^{p}$ positive in the softening range of material response.

a consequence of the non-associative flow rule, since in the case of associative flow rule ($\alpha = \beta$), the sign of the right-hand side of (18) is determined solely by the sign of the parameter h. The fact that $d\sigma_{ij}d\varepsilon_{ij}^{p}$ can be negative in the hardening range of the material response, of course, does not necessarily imply that material becomes unstable in such cases. Whether an instability will occur at the given stress and state of the material can only be answered by a separate, bifurcation type analysis, such as one used in [20].

4. INFINITESIMAL CYCLES OF STRESS AND STRAIN

4.1. *Stress cycles*

To further illustrate the connection between plasticity postulates and plasticity flow rules, consider again the combined shear-pressure loading. Assume the material to be in the hardening range and to be taken through an infinitesimal stress cycle ABCA (Fig. 2). The cycle originates at the stress state $A(\tau, p)$ on the current yield surface. It consists of the neutral loading along the yield surface to the neighboring state $B(\tau - d\tau, p - d\tau/3\alpha)$, followed by plastic loading under constant pressure to state $C(\tau, p - d\tau/3\alpha)$ on the expanded yield surface. The elastic unloading under constant shear stress closes the cycle by returning the stress to its original state $A(\tau, p)$. This stress cycle is of the type considered by Drucker [1] in his discussion of plasticity postulates and material stability. The net work of added stresses in this cycle (relative to stress state at A) can be calculated by using (4), i.e.

$$W_D = (\sigma_{ij}^{B} - \sigma_{ij}^{A})d\varepsilon_{ij}^{p} + \tfrac{1}{2}d\sigma_{ij}d\varepsilon_{ij}^{p}, \tag{19}$$

which gives

$$W_D = -\tfrac{1}{2}d\tau d\gamma^{p} + dp\,d\varepsilon^{p}, \tag{20}$$

where $dp = d\tau/3\alpha$. In view of the expression (17) for the plastic shear and volumetric strain increments, (20) becomes

$$W_D = \left(-\frac{1}{2} + \frac{\beta}{\alpha}\right)d\tau d\lambda, \tag{21}$$

with the loading parameter $d\lambda = d\tau/h$. If the flow rule is associative ($\alpha = \beta$), the net work W_D is clearly positive. It is also positive for the non-associative flow rules with β greater than $\alpha/2$, while it becomes negative for β less than $\alpha/2$. No net work of added stresses is done in the cycle if $\beta = \alpha/2$.

If the stress cycle is traversed in the opposite direction, starting with the plastic segment AC, the net work of added stresses is calculated from (6) as

$$W_D = \tfrac{1}{2}dp\,d\varepsilon^{p} = \tfrac{3}{2}\beta\,dp\,d\lambda, \tag{22}$$

which is positive for the positive dilatancy factor. When the flow rule is associative, (21) and (22) coincide and the net work over the cycle is the same regardless of the direction of the cycle. This is not so for the non-associative flow rule. For example, if $\beta = \alpha/3$, (21) gives

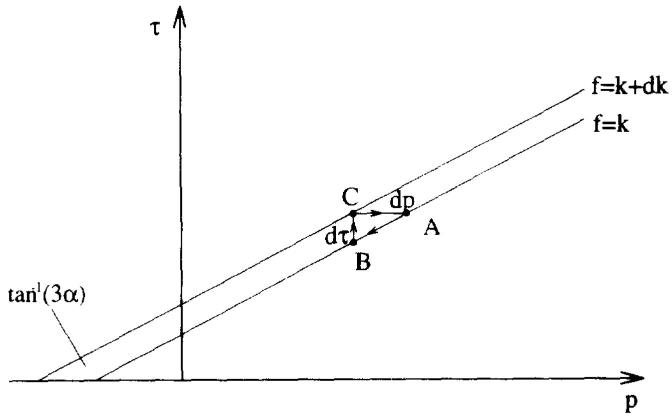

Fig. 2. An infinitesimal closed stress cycle ABCA of a hardening Drucker–Prager material in the shear-pressure test. The segment AB is the neutral loading segment. Plastic deformation occurs along constant pressure segment BC, while CA constitutes elastic unloading under constant shear stress.

negative work of magnitude $\mathrm{d}\tau\mathrm{d}\lambda/6$, while (22) gives positive work of the same magnitude. The reason for this is easily understood by observing that in the case of an associative flow rule, the component of the stress increment directed tangentially to the yield surface does not do any work on the plastic strain increment, which is directed normal to the yield surface. In the case of a non-associative flow rule this is not the case, and the component of the stress increment tangential to the yield surface does work on the plastic strain increment, which is normal to the plastic potential surface, distinct from the yield surface.

Consider next the stress cycle BCAB. The new work of added stresses is from (6)

$$W_{\mathrm{D}} = \tfrac{1}{2}\mathrm{d}\tau\mathrm{d}\gamma^{\mathrm{p}} = \tfrac{1}{2}\mathrm{d}\tau\mathrm{d}\lambda, \tag{23}$$

which is always positive. If direction of the cycle is reversed, i.e. if the cycle is traversed in the direction BACB, the net work of added stresses (relative to the stress state at B) is

$$W_{\mathrm{D}} = \mathrm{d}\tau\mathrm{d}\gamma^{\mathrm{p}} - \frac{1}{2}\mathrm{d}p\mathrm{d}\varepsilon^{\mathrm{p}} = \left(1 - \frac{1}{2}\frac{\beta}{\alpha}\right)\mathrm{d}\tau\mathrm{d}\lambda. \tag{24}$$

Again, for the associative flow rule this is positive and equal to W_{D} calculated by (23). Expression (24) also gives positive work for the non-associative flow rules with the parameter β less than 2α, although these values of β correspond to a dilatancy factor that is probably unrealistically large for most materials.

If the material is in the softening range, the corresponding infinitesimal cycles originate from the stress state A inside the current yield surface, and consist of the elastic loading segment AB to stress state B on the yield surface, plastic loading segment BC accompanied by the shrinkage of the yield surface, and the neutral loading segment CA along the new yield surface, back to the original stress state at A. The two cycles depicted in Fig. 3 give that the net work of added stresses is again positive provided that $\alpha/2 < \beta < 2\alpha$.

4.2. *Strain cycles*

The strain state corresponding to the stress state at A after the stress cycle ABCA in Fig. 2 differs from the strain state before the cycle by the shear strain increment $\mathrm{d}\gamma^{\mathrm{p}}$, and the volumetric strain increment $\mathrm{d}\varepsilon^{\mathrm{p}}$. Since the loading parameter in the considered stress cycle is $\mathrm{d}\lambda = \mathrm{d}\tau/h$, from (17) it follows that

$$\mathrm{d}\gamma^{\mathrm{p}} = \frac{\mathrm{d}\tau}{h}, \quad \mathrm{d}\varepsilon^{\mathrm{p}} = 3\beta\frac{\mathrm{d}\tau}{h}. \tag{25}$$

To complete the strain cycle and return the strain state to that which existed before the stress cycle, an elastic stress segment AD is needed upon the stress cycle ABCA, so that the

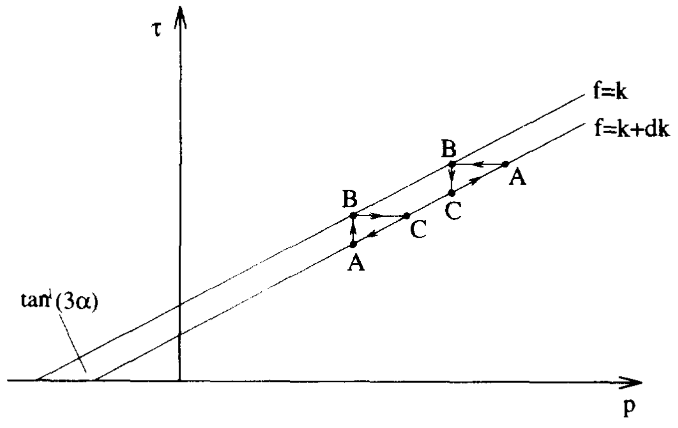

Fig. 3. Two closed stress cycles considered in a softening range of material response. The cycles originate at the stress state A inside the yield surface $f = k$, since for softening materials there are no stress cycles involving plastic deformation, which originate at the stress states on the yield surface.

added stress $\sigma_{ij}^D - \sigma_{ij}^A$ cancels the residual strains (25). The added stress components are clearly

$$d\tau^* = G d\gamma^p, \quad dp^* = K d\varepsilon^p \tag{26}$$

where G and K are the elastic shear and bulk modulus, respectively. The Ilyushin work in this strain cycle is

$$W_I = W_D + \int_A^D (\sigma_{ij} - \sigma_{ij}^A) d\varepsilon_{ij}, \tag{27}$$

since the work of constant stress σ_{ij}^A over the closed strain cycle is zero. In the considered case

$$\int_A^D (\sigma_{ij} - \sigma_{ij}^A) d\varepsilon_{ij} = \tfrac{1}{2}(d\tau^* d\gamma^p + dp^* d\varepsilon^p), \tag{28}$$

and in view of (21), (25) and (26), the Ilyushin work becomes

$$W_I = \left[h\left(-\frac{1}{2} + \frac{\beta}{\alpha}\right) + \frac{1}{2}(G + 9\beta^2 K)\right]\left(\frac{d\tau}{h}\right)^2. \tag{29}$$

Since the elastic moduli G and K are usually far greater than the hardening modulus h, from (29) follows that $W_I > 0$, for all reasonable values of the parameters α and β. Physically, this is the case because the (positive) elastic work (28) gives a dominant contribution to W_I, since elastic stress increments $d\tau^*$ and dp^* exceed the increments $d\tau$ and dp by the large factor equal to the ratio of elastic and hardening modulus.

In order to compare W_D and W_I, consider first the associative flow rule ($\alpha = \beta$). From (21) and (29)

$$W_D = \frac{1}{2}h\left(\frac{d\tau}{h}\right)^2, \quad W_I = \frac{1}{2}(h + G + 9\alpha^2 K)\left(\frac{d\tau}{h}\right)^2 \tag{30}$$

which are both positive. However, in the case of the non-associative flow rule, $W_D < 0$ for all values of $\beta < \alpha/2$, while $W_I > 0$. For example, for a non-dilatant material ($\beta = 0$)

$$W_D = -\frac{1}{2}h\left(\frac{d\tau}{h}\right)^2, \quad W_I = \frac{1}{2}(G - h)\left(\frac{d\tau}{h}\right)^2. \tag{31}$$

Hence, the Drucker work in the considered closed stress cycle is negative, while the Ilyushin work in the corresponding closed strain cycle is positive.

A non-dilatant pressure-dependent plasticity model can be approximately used to describe the behavior of certain high-strength steels exhibiting the strength-differential effect, by which the yield strength is higher in compression than in tension. For example, Spitzig *et al.* [22] found that the plastic volume change predicted by associative flow rule is nearly 15 times greater than the experimentally observed value. Hence, the non-associative flow rule with β approximately equal to zero can be used for such materials. It should also be pointed out that for progressively cavitating porous metals, the associative flow rule is commonly used with a pressure-dependent yield condition, such as one proposed by Gurson [23].

5. TRIAXIAL COMPRESSION TEST

A commonly used experiment to determine the mechanical properties of rocks and soils is a triaxial (confined) compression test, in which the specimen is subjected to longitudinal compressive stress of magnitude $\sigma + p$, under confining lateral pressure of magnitude p. The stress difference between the longitudinal compressive stress and lateral confinement is denoted by σ. The two stress invariants are in this case $I_1 = -(\sigma + 3p)$, and $J_2 = \sigma^2/3$, so that the Drucker–Prager yield criterion (10) becomes

$$\left(\frac{1}{\sqrt{3}} - \alpha\right)\sigma - 3\alpha p = k. \qquad (32)$$

The deviatoric and volumetric parts of the plastic strain increments are readily obtained from the flow rule (12)

$$\mathrm{d}e^{\mathrm{p}}_{\mathrm{long}} = -\frac{1}{\sqrt{3}}\mathrm{d}\lambda, \quad \mathrm{d}e_{\mathrm{lat}} = \frac{1}{2\sqrt{3}}\mathrm{d}\lambda, \quad \mathrm{d}\varepsilon^{\mathrm{p}} = 3\beta\mathrm{d}\lambda. \qquad (33)$$

Consider an infinitesimal stress cycle ABCA in the hardening range of material response. The segment AB represents a neutral loading along the current yield surface, BC is the plastic loading under constant pressure p, while the segment CA represents elastic unloading under constant stress difference σ (Fig. 4). The Drucker work of added stresses in this cycle is calculated from (4), which is here rewritten in terms of the deviatoric and spherical components as

$$W_{\mathrm{D}} = [(S^B_{ij} - S^A_{ij}) + \tfrac{1}{2}\mathrm{d}S_{ij}]\,\mathrm{d}e^{\mathrm{p}}_{ij} + \tfrac{1}{3}[(\sigma^B_{kk} - \sigma^A_{kk}) + \tfrac{1}{2}\mathrm{d}\sigma_{kk}]\,\mathrm{d}\varepsilon^{\mathrm{p}}_{kk}. \qquad (34)$$

This gives

$$W_{\mathrm{D}} = -\frac{1}{2\sqrt{3}}\left(1 + \sqrt{3}\beta - 2\frac{\beta}{\alpha}\right)\mathrm{d}\sigma\mathrm{d}\lambda, \qquad (35)$$

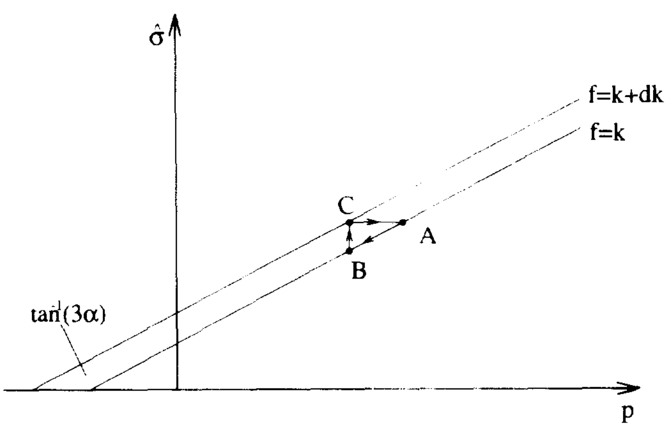

Fig. 4. A closed stress cycle ABCA in the confined compression test, involving plastic deformation along the constant pressure segment BC. The ordinate $\hat{\sigma}$ denotes the product $(1/\sqrt{3} - \alpha)\sigma$, where σ is the stress difference between longitudinal compressive stress and confining pressure.

which is positive for $\beta > \alpha/(2-\sqrt{3}\alpha)$. If the cycle BACB is considered, the work is

$$W_D = \frac{1}{2\sqrt{3}}\left(2-\sqrt{3}\beta-\frac{\beta}{\alpha}\right)\mathrm{d}\sigma\mathrm{d}\lambda, \tag{36}$$

which is positive for $\beta < 2\alpha/(1+\sqrt{3}\alpha)$. Since this upper bound for β must be greater than the previously obtained lower bound for β, defining an open interval in which W_D is positive, it follows that in the considered cycles α has to be less than $1/\sqrt{3}$. This can be explained by considering the stress cycles in the $(I_1, J^{1/2})$ plane (Fig. 5). The cycle ABCA, involving the plastic loading segment BC, is clearly possible only if $\alpha < 1/\sqrt{3}$, since then the segment BC is directed outside the current yield surface. If $\alpha > 1/\sqrt{3}$. the segment BC is directed inside the yield surface and constitutes an elastic unloading. The corresponding segment BC in Fig. 4 is then directed downward. This also follows directly from the plastic loading condition and the structure of the yield condition (32). For $\alpha = 1/\sqrt{3}$, the cycle ABCA degenerates into elastic, neutral loading–unloading path ABA along the yield surface, during which the pressure p remains constant. Observe also that since $\alpha > 0$, the length of the β interval $\alpha/(2-\sqrt{3}\alpha) < \beta < 2\alpha/(1+\sqrt{3}\alpha)$, rendering W_D positive in the cycles under confined compression, is smaller than the length of the β interval $\alpha/2 < \beta < 2\alpha$, rendering W_D positive in the cycles under combined shear-pressure loading. For example, for $\alpha = 0.3$ the respective β intervals are $0.2 < \beta < 0.4$, and $0.15 < \beta < 0.6$.

To complete the strain cycle and return the strain state to that which existed before the stress cycle ABCA, an elastic stress segment AD must be additionally applied. The corresponding Ilyushin work is easily found to be

$$W_I = W_D + \frac{1}{2}\left(\frac{1}{\sqrt{3}}-\alpha\right)^2 (G+9\beta^2 K)\left(\frac{\mathrm{d}\sigma}{h}\right)^2. \tag{37}$$

For example, in the case of a non-dilatant material

$$W_D = -ah\left(\frac{\mathrm{d}\sigma}{h}\right)^2, \quad W_I = a[(1-\sqrt{3}\alpha)G - h]\left(\frac{\mathrm{d}\sigma}{h}\right)^2 \tag{38}$$

where the coefficient $a = (1/\sqrt{3}-\alpha)/2\sqrt{3}$. Therefore, the Drucker work in the considered closed stress cycle is negative, while the Ilyushin work in the corresponding closed strain cycle is again positive.

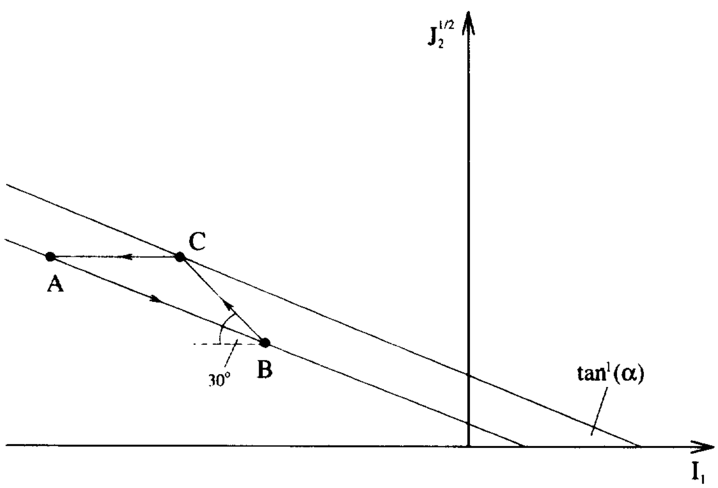

Fig. 5. A closed stress cycle ABCA from Fig. 4 represented in the space of stress invariants $I_1 = -(\sigma + 3p)$, and $J_2^{1/2} = \sigma/\sqrt{3}$. Segment AB represents neutral loading along the yield surface. Segment BC is plastic loading under constant pressure, while segment CA constitutes elastic unloading under constant stress difference σ.

6. CONCLUSION

The existence of inelastic potentials has been a longstanding issue of great importance in the phenomenological constitutive theory of inelastic material response. For time independent plasticity, the plastic potential has been most often assumed to coincide with the yield surface, which separates the elastic stress states from the plastic stress states. The normality of the plastic strain increment to a locally smooth yield surface, and the associative flow rule then hold. A sufficient condition for this constitutive structure is that a considered material obeys the cycle inequalities of Drucker or Ilyushin, which are known as plasticity postulates. However, many pressure-dependent dilatant materials with internal-microscopic frictional effects are not well described by the associative flow rules. For such materials, notably geomaterials like rocks and soils, the plastic strain increment is assumed to be normal to the plastic potential surface, which is not coincident with the yield surface. The constitutive structure with a non-associative flow rule then results. In this case, neither of the two plasticity postulates in general apply. Nonetheless, there are always many cycles and loading conditions for which the postulate inequalities are not violated. For example, by considering the pressure-dependent dilatant material which is governed by a non-associative flow rule and the Drucker–Prager yield condition, it is found that the net work of added stresses in many closed stress cycles can be either positive or negative depending on the relationship between the dilatancy factor and the internal friction coefficient. In this paper two typical loading conditions were considered, which are commonly employed in experimental determination of the material parameters. Furthermore, it was found that in the case of associative flow rule the net work in the infinitesimal stress cycles is independent of the direction of the cycle, while it is strongly dependent on such direction in the case of non-associative flow rule. A study of this and other features of the mechanical response predicted by the non-associative flow rules is important to the understanding of the nature and constitutive implications of these flow rules on phenomena such as onset of localization and inception of rupture. A positive value of the trace product between the stress increment and the plastic strain increment is neither necessary nor sufficient condition for stability of such materials. As discussed in [24], materials with non-associative flow rules are more inclined to instability by localization of deformation than materials with associative flow rules. For example, in materials with a non-associative flow rule, the non-uniqueness and localization can occur for certain stress states even in the range of positive hardening rates. This is never the case in materials described by an associative flow rule, where the localization is possible only at the non-positive values of the hardening modulus. There is also a fundamental approach to the study of the existence of inelastic potentials for materials exhibiting inelasticity as a consequence of specific structural rearrangements on the microscale. Rice [25] has shown that the normality structure in macroscopic laws arises when each local microstructural rearrangement proceeds at a rate governed by its associated thermodynamic force. In metal plasticity this is the case when the slip rate at a particular slip system is governed by the resolved shear stress on that system. Normality is not expected for systems with Coulomb type frictional resistance to slip.

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