Constitutive Structure of Rate Theory of Damage in Brittle Elastic Solids

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ABSTRACT

This paper presents a rate-type constitutive analysis applicable to brittle elastic materials whose elastic properties degrade in the course of deformation due to evolution of many microcracks. Thermodynamic analysis is used to identify the appropriate damage tensors and their conjugate forces (affinities). It is shown that the introduced damage potentials are dual: for the damage fluxes in the space of affinities, and for the damage strain and stress rates in the corresponding stress and strain spaces. Rate constitutive equations are derived, providing explicit representations of the tangent stiffness and compliance tensors. General results are specialized in the case of some simple, physically appealing choices of damage functions, analogous to those used in other damage and pressure-dependent plasticity models.

1. INTRODUCTION

Analysis of brittle deformation of materials such as rocks, concrete, ceramics, etc., have lately attracted a great deal of attention. Degradation of elastic properties and accumulation of inelastic strains are primarily attributable to the evolution of many internal microcracks and other microdefects within the considered microstructure. The mode of crack nucleation

depends on the microstructure and state of stress. In very porous rocks, crack nucleation is attributed to the tensile hoop stresses generated at the surface of relatively large pores. In low porosity, compact rocks frictional sliding of crack surfaces may destabilize existing shear cracks, causing them to kink and develop wing cracks. Other causes of microcrack nucleation, such as elastic mismatch and bending mechanisms, may also take place in some rocks. Some of these mechanisms were implemented in analytical modeling of cracking and failure of brittle solids, as discussed, for example, in [1].

The mode and stability of crack growth and, therefore, behavior of brittle material strongly depends on the sign and magnitudes of applied stresses. For instance, the response of a brittle material subjected to compressive loading is strongly dependent on the magnitude of lateral confinement. An unconfined specimen fails by axial splitting, attributed to unstable growth of a single crack, at a relatively small microcrack density. As the confinement is increased, axial splitting is suppressed and at large confinement levels homogeneous microcracking prevails throughout the sample, resulting in a quasiductile overall response [2], and ultimately in localization failure mode.

Progressive degradation of mechanical macroproperties is an important feature of the brittle material behavior. Several analytical models were suggested to estimate the effective elastic properties of a solid weakened by a given distribution of many cracks or other defects. An extensive review of these models can be found in a recent treatise [3]. An appropriate choice of the mathematical form for the damage variable which approximates the microcrack distribution is one of the central issues in the process of the formulation of a continuum damage model. Contradictory requirements of facility on one and accuracy on the other hand are to a large extent responsible for the proliferation of damage models. The issue of the appropriate choice of the damage variable has been recently examined in [4]. Several frequently encountered distributions of microcracks, corresponding to uniaxial and biaxial tension and compression, were considered. In each case the microcrack distribution was developed into an infinite series of spherical functions in form of even order tensors [5, 6]. The infinite series was then truncated to one (scalar), two (scalar and second-order tensor), and three (scalar, second- and fourth-order tensor) terms of the series. The analysis demonstrated the shortcomings of the scalar and second-order tensor damage variables, and the accuracy gained by using the fourth-order tensor representation.

Both micromechanical and percolation models have supported the selection of the fourth-order effective compliance (or effective stiffness) tensor as the appropriate damage variable [7–9]. In percolation models the effec-

tive stiffness is the order parameter which exhibits universal behavior in the neighborhood of the critical state. Thus, even at failure (or critical state) effective stiffness represents an appropriate measure of damage. Finally, the change of effective stiffness can be easily measured, which is one of the important criteria for the selection of the internal variable.

During a typical deformation process a majority of microcracks nucleate and grow in planes which are roughly orthogonal to the direction of the maximum principal stress. The macro response becomes both anisotropic and path-dependent. This type of behavior can be adequately modeled only by the rate-type theories. With the current elastic compliance selected as an appropriate measure of damage, the constitutive theory in the spirit of classical rate-type plasticity theory was proposed in [9]. A similar approach, including the notion of damage surface, loading and unloading conditions, and other familiar plasticity concepts, was used in other constitutive models, such as [7, 10–12].

The present paper deals exclusively with the damage of elastic solids, assuming that residual strains vanish upon unloading from any state of deformation. Incorporation of residual (plastic) strains into the constitutive framework is left for the forthcoming study. Basic formulation of the analysis is provided in both stress and strain space. Thermodynamic analysis is presented in Section 2, where appropriate damage tensors and their conjugate thermodynamic forces are identified, and the damage flux potentials introduced. The expressions for the damage stress and strain rates are derived, and duality of the damage potentials established in Section 3. Damage surface is defined in Section 4, while the complete rate-type constitutive equations are formulated in Section 5. Explicit representations of the tangent stiffness and compliance tensors are also given in Section 5. General results are illustrated in Section 6 by considering some simple, appealing forms of the damage functions, analogous to those used in other damage and pressure-dependent plasticity studies.

Modifications and extensions of the presented formulation required to model different tensile and compressive responses of brittle materials are presented in a separate paper [13].

2. THERMODYNAMIC ANALYSIS

Consider small deformations of brittle material whose elastic properties change during a deformation process. Assume that the residual strain vanishes upon unloading initiated from an arbitrary state of deformation. The corresponding uniaxial stress-strain behavior is of the type sketched in

Figure 1a. Degradation of elastic properties is assumed to be a consequence of accumulated damage, i.e., nucleation and propagation of microcracks during a deformation process. Let ε_γ denote the surface energy of the microcrack surfaces created in the course of deformation from the initial to the current state. The Helmholtz free energy Φ and the Gibbs energy Ψ (per unit volume) can be then defined as

$$\Phi = \frac{1}{2}\mathcal{L} : (\epsilon \otimes \epsilon) + \varepsilon_\gamma \qquad (2.1)$$

$$\Psi = \frac{1}{2}\mathcal{M} : (\sigma \otimes \sigma) - \varepsilon_\gamma, \qquad (2.2)$$

where σ and ϵ are the stress and strain tensors, and $\mathcal{L}$ and $\mathcal{M}$ the current elastic stiffness and compliance tensors. The symbol (:) stands for the inner, trace product, and $\otimes$ for the outer tensor product. As discussed in the context of the thermodynamic analysis of the quasistatic growth of Griffith cracks [14], free energy at the current state is equal to the work done in transforming the body from its initial to current state along an imagined reversible and isothermal path. This path can be created by a sequence of two steps. First, new microcrack surfaces are formed separating reversibly the adjacent layers of atoms, pulling against their cohesive forces. The work needed for this step is denoted by ε_γ. Second, the microcracked solid is deformed elastically to the current state of deformation ϵ. The required work is $\frac{1}{2}\mathcal{L} : (\epsilon \otimes \epsilon)$, where $\mathcal{L}$ denotes the current elastic stiffness which accounts for the presence of all existing active cracks. A microcrack is considered to be active if it imposes a discontinuity in at least one component of the displacement vector across its surface.

As the damage evolves the elastic moduli decrease and compliances increase. Thus, it is convenient to represent the tensors $\mathcal{L}$ and $\mathcal{M}$ in additive form as

$$\mathcal{L} = \mathcal{L}^0 - \mathbf{L} \qquad (2.3)$$

$$\mathcal{M} = \mathcal{M}^0 + \mathbf{M}, \qquad (2.4)$$

where $\mathcal{L}^0$ and $\mathcal{M}^0$ denote the initial elastic stiffness and compliance tensors of the undamaged material. The fourth-order tensors $\mathbf{L}$ and $\mathbf{M}$ can be considered as being the measures of damage. These tensors change during a deformation process as a consequence of material degradation, i.e., nucleation of new and growth of existing microcracks. The rates of the Helmholtz and Gibbs energies are obtained by differentiating (2.1) and (2.2):

$$\dot{\Phi} = \mathcal{L} : (\epsilon \otimes \dot{\epsilon}) - \dot{\mathbf{L}} : \frac{1}{2}(\epsilon \otimes \epsilon) + \dot{\varepsilon}_\gamma \qquad (2.5)$$
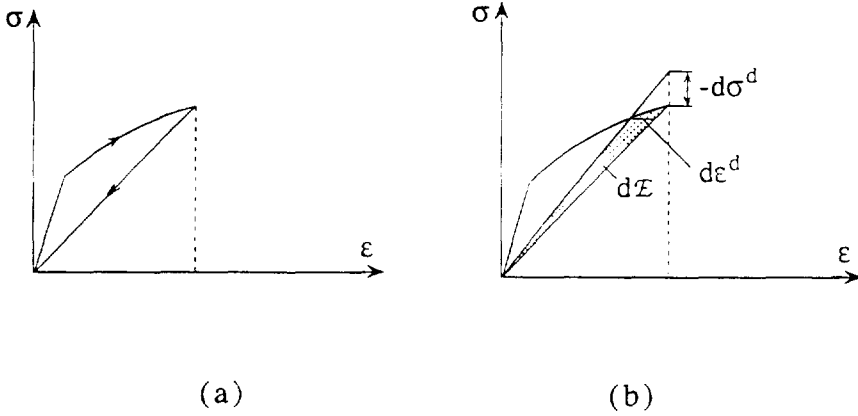
FIG. 1. (a) Loading and unloading portions of the uniaxial stress-strain curve; (b) increments of the damage stress and strain, and the energy expended on the corresponding damage progression.

$$\dot{\Psi} = \boldsymbol{\mathcal{M}} : (\boldsymbol{\sigma} \otimes \dot{\boldsymbol{\sigma}}) + \dot{\mathbf{M}} : \frac{1}{2}(\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}) - \dot{\varepsilon}_\gamma. \tag{2.6}$$

The reciprocity symmetries of the tensors $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{M}}$ are used in arriving at (2.5) and (2.6).

Assuming the deformation processes to be isothermal, from the first law of thermodynamics it follows that

$$\dot{\Phi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - T\Lambda \tag{2.7}$$

$$\dot{\Psi} = \boldsymbol{\epsilon} : \dot{\boldsymbol{\sigma}} + T\Lambda, \tag{2.8}$$

where T is the temperature, and Λ the irreversible entropy production rate. The product $T\Lambda$ represents the energy dissipation rate associated with the damage evolution. As a consequence of the second law of thermodynamics, $\Lambda \geq 0$.

Comparing (2.5) and (2.7), it follows that

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{L}} : \boldsymbol{\epsilon} \tag{2.9}$$

$$T\Lambda = \dot{\mathbf{L}} : \frac{1}{2}(\boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}) - \dot{\varepsilon}_\gamma, \tag{2.10}$$

while the comparison of (2.6) and (2.8) gives

$$\boldsymbol{\epsilon} = \boldsymbol{\mathcal{M}} : \boldsymbol{\sigma} \tag{2.11}$$

$$T\Lambda = \dot{\mathbf{M}} : \frac{1}{2}(\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}) - \dot{\varepsilon}_\gamma. \tag{2.12}$$

Equation (2.9) and its inverse (2.11) are the elasticity equations relating the current stress and strain tensors through the current elastic stiffness tensor $\mathcal{L}$ and its inverse, the elastic compliance tensor $\mathcal{M} = \mathcal{L}^{-1}$. Equations (2.10) and (2.12) are two alternative expressions for the energy dissipation rate, written in terms of strain and stress tensors, and damage fluxes $\dot{\mathbf{L}}$ and $\dot{\mathbf{M}}$. Indeed, introducing

$$\mathbf{G} = \frac{1}{2}(\epsilon \otimes \epsilon) \tag{2.13}$$

as the thermodynamic force (affinity) conjugate to the damage tensor $\mathbf{L}$, and

$$\mathbf{\Gamma} = \frac{1}{2}(\sigma \otimes \sigma) \tag{2.14}$$

as the thermodynamic force (affinity) conjugate to damage tensor $\mathbf{M}$, expressions (2.10) and (2.12) can be written as

$$T\Lambda = \mathbf{G} : \dot{\mathbf{L}} - \dot{\varepsilon}_\gamma \tag{2.15}$$

$$T\Lambda = \mathbf{\Gamma} : \dot{\mathbf{M}} - \dot{\varepsilon}_\gamma. \tag{2.16}$$

The affinities $\mathbf{G}$ and/or $\mathbf{\Gamma}$ are previously utilized in literature, for example in [7, 9, 11, 12]. The conjugacy property of $\mathbf{G}$ and $\mathbf{L}$, and $\mathbf{\Gamma}$ and $\mathbf{M}$, are further discussed in Subsection 3.1 of this paper.

2.1. Potential for damage fluxes

Suppose that for the material degradation process there is a scalar function $\Pi = \Pi(\mathbf{G})$ such that the damage flux $\dot{\mathbf{L}}$ can be expressed as

$$\dot{\mathbf{L}} = \dot{r}\frac{\partial \Pi}{\partial \mathbf{G}}, \tag{2.17}$$

where $\dot{r}$ is the rate of some monotonically increasing scalar parameter r, which can be considered as a measure of the cumulative damage at the considered instant of deformation process. Hence, by definition, $\dot{r} \geq 0$. The function Π is referred to as a damage potential for the flux $\dot{\mathbf{L}}$. Similarly, suppose that

$$\dot{\mathbf{M}} = \dot{\rho}\frac{\partial \Omega}{\partial \mathbf{\Gamma}}, \tag{2.18}$$

where $\Omega = \Omega(\mathbf{\Gamma})$ is the damage potential for the flux $\dot{\mathbf{M}}$, and $\dot{\rho} \geq 0$ the rate of the cumulative damage parameter ρ. The conditions for the existence of the potential for various types of fluxes introduced to describe inelastic

behavior, and their implications on the constitutive modeling have been studied in [15].

Substituting (2.17) and (2.18) into (2.15) and (2.16), the energy dissipation rate can be expressed as

$$T\Lambda = \dot{r}\left(\mathbf{G} : \frac{\partial \Pi}{\partial \mathbf{G}}\right) - \dot{\varepsilon}_\gamma \qquad (2.19)$$

$$T\Lambda = \dot{\rho}\left(\mathbf{\Gamma} : \frac{\partial \Omega}{\partial \mathbf{\Gamma}}\right) - \dot{\varepsilon}_\gamma. \qquad (2.20)$$

If Π and Ω are homogeneous functions of their arguments (of degrees m and n, respectively), (2.19) and (2.20) reduce to

$$T\Lambda = \dot{r}\,m\Pi - \dot{\varepsilon}_\gamma \qquad (2.21)$$

$$T\Lambda = \dot{\rho}\,n\Omega - \dot{\varepsilon}_\gamma. \qquad (2.22)$$

Since the energy dissipation rate is nonnegative, from (2.21) and (2.22) follow

$$m\Pi - \frac{d\varepsilon_\gamma}{dr} \geq 0 \qquad (2.23)$$

$$n\Omega - \frac{d\varepsilon_\gamma}{d\rho} \geq 0. \qquad (2.24)$$

The equality sign in (2.23) and (2.24) applies only if it is assumed that all energy associated with the damage process is transformed into the surface energy of the created microcrack faces.

3. DAMAGE STRESS AND STRAIN RATES

The expression for the stress rate can be derived differentiating (2.9) and using (2.3),

$$\dot{\sigma} = \mathcal{L} : \dot{\epsilon} - \dot{\mathbf{L}} : \epsilon. \qquad (3.1)$$

The first term on the right-hand side of (3.1) is the stress rate that would correspond to strain rate $\dot{\epsilon}$ in absence of the damage flux $\dot{\mathbf{L}}$. This part of the stress rate will be referred to as the elastic part of the stress rate. The remaining part

$$\dot{\sigma}^d = -\dot{\mathbf{L}} : \epsilon \qquad (3.2)$$

is the part of the stress rate attributable to the change in damage, which shall be referred to as the damage stress rate (Figure 1b). Substituting

(2.17) into (3.2) the damage stress rate is

$$\dot{\boldsymbol{\sigma}}^d = -\dot{r}\frac{\partial \Pi}{\partial \mathbf{G}} : \boldsymbol{\epsilon}. \tag{3.3}$$

Since from (2.13)

$$\frac{\partial \Pi}{\partial \boldsymbol{\epsilon}} = \frac{\partial \Pi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial \boldsymbol{\epsilon}} = \frac{\partial \Pi}{\partial \mathbf{G}} : \boldsymbol{\epsilon}, \tag{3.4}$$

the expression (3.3) can be rewritten in the form

$$\dot{\boldsymbol{\sigma}}^d = -\dot{r}\frac{\partial \Pi}{\partial \boldsymbol{\epsilon}}. \tag{3.5}$$

Therefore, the damage potential Π serves as a dual potential: for the damage flux $\dot{\mathbf{L}}$ in the space of affinity $\mathbf{G}$, i.e., (2.17), and for the damage stress rate $\dot{\boldsymbol{\sigma}}^d$ in the space of strain $\boldsymbol{\epsilon}$, i.e., (3.5).

The strain rate is determined in a similar fashion. Differentiating (2.11) it follows

$$\dot{\boldsymbol{\epsilon}} = \boldsymbol{\mathcal{M}} : \dot{\boldsymbol{\sigma}} + \dot{\mathbf{M}} : \boldsymbol{\sigma} \tag{3.6}$$

The first term on the right-hand side of (3.6) is the strain rate that would correspond to stress rate $\dot{\boldsymbol{\sigma}}$ if the damage flux $\dot{\mathbf{M}}$ were zero. This part of the strain rate is referred to as the elastic part of the strain rate. The remaining part

$$\dot{\boldsymbol{\epsilon}}^d = \dot{\mathbf{M}} : \boldsymbol{\sigma} \tag{3.7}$$

is the strain rate attributable to change in damage, referred to concisely as the damage strain rate (Figure 1b). Substituting (2.18) into (3.7)

$$\dot{\boldsymbol{\epsilon}}^d = \dot{\rho}\frac{\partial \Omega}{\partial \boldsymbol{\Gamma}} : \boldsymbol{\sigma}, \tag{3.8}$$

i.e.,

$$\dot{\boldsymbol{\epsilon}}^d = \dot{\rho}\frac{\partial \Omega}{\partial \boldsymbol{\sigma}}. \tag{3.9}$$

Therefore, the damage potential Ω also serves as a dual potential: for the damage flux $\dot{\mathbf{M}}$ in the space of affinity $\boldsymbol{\Gamma}$, i.e., (2.18), and for the damage strain rate $\dot{\boldsymbol{\epsilon}}^d$ in the space of stress $\boldsymbol{\sigma}$, i.e., (3.9).

The above introduced definitions of the elastic and damage parts of the stress and strain rates are analogous to those introduced in [16], in a related study of the structure of inelastic constitutive laws.

3.1. Conjugacy relations

From Figure 1b it can be easily observed that, to within small quantities of second order, the rate of energy $\dot{\varepsilon}$ expended on the evolution of damage is

$$\dot{\varepsilon} = -\frac{1}{2}\epsilon : \dot{\sigma}^d = \frac{1}{2}\sigma : \dot{\epsilon}^d. \tag{3.10}$$

One part of this rate represents the rate of the surface energy $\dot{\varepsilon}_\gamma$. The remaining part is the rate of dissipated energy $T\Lambda$. Thus, $\dot{\varepsilon} = \dot{\varepsilon}_\gamma + T\Lambda$. Substituting (3.2) and (3.7) into (3.10), it follows

$$\dot{\varepsilon} = \frac{1}{2}(\epsilon \otimes \epsilon) : \dot{\mathbf{L}} = \frac{1}{2}(\sigma \otimes \sigma) : \dot{\mathbf{M}}, \tag{3.11}$$

i.e.,

$$\dot{\varepsilon} = \mathbf{G} : \dot{\mathbf{L}} = \mathbf{\Gamma} : \dot{\mathbf{M}}. \tag{3.12}$$

Equation (3.12) establishes the conjugacy between the affinity $\mathbf{G} = \frac{1}{2}(\epsilon \otimes \epsilon)$ and the damage tensor $\mathbf{L}$, and between the affinity $\mathbf{\Gamma} = \frac{1}{2}(\sigma \otimes \sigma)$ and the damage tensor $\mathbf{M}$, which were already utilized in Section 2.

4. DAMAGE SURFACE

To derive the rate-type constitutive equations which describe elastic response of materials that undergo path-dependent damage evolution, the constitutive expression for either the damage stress rate or damage strain rate have to be established. In the first case the damage surface is introduced in the strain space, while in the second case it is introduced in the stress space. The corresponding formulations in both spaces are presented in this section.

The range of the validity of damage surfaces used in the formulation of the analysis is an important issue which must be addressed in conjunction with plasticity and a probable failure mode (limit surface). Ordered brittle materials (having small bandwidths of microstructural rapture strengths), subjected to tension or uniaxial compression (without lateral confinement), fail as a result of unstable growth of a single microcrack at very small overall damage density. At the order end of the spectrum, the disordered (damage tolerant) and properly confined solids can support large compression without failure. Unloading reveals substantial residual strain and relatively mild change of the effective stiffness. Consequently, the deformation is on the macroscale similar to that associated with the ductile metals.

Specific choices of damage and limit surfaces were suggested in [1, 17,

18]. The damage and yield surfaces are often taken in the form suggested by the so-called capped Mohr–Coulomb plasticity model. Even though a more rigorous version of two surfaces, distinguishing between plastic and damage strain, may, indeed, come in this form, an utmost care must be paid to set this surface in proper (stress or strain) space and identify the appropriate thermodynamic fluxes. The plastic strain is reflected in residual deformation, while strain attributable to elastic damage is related to the degradation of effective stiffness. It is instructive to view the damage surface as a locus of points in which the Griffith criterion for one or more microcrack is satisfied. This is in a definite manner related to the resistance to damage growth commonly known in fracture mechanics as R-curve. A revealing study of several possible pairs of the damage driving and resisting forces, in the setting of a simple discrete artifice, is available in [19]. An underlying assumption on which the simple formulas, involving average (macro) fields, are based is that the mean field representation is justified. Recent analyses of network (lattice) models [20, 21] indicate that the mean field approximation, which ignores direct crack to crack interaction, is justifiable within the hardening regime (including the apex of the stress-strain curve). Another important task is to formulate the limit surfaces. In brittle deformation the failure can occur when a single microcrack becomes unstable and splits the specimen into two parts, or when the closely spaced interacting microcracks form thin, localized bands. Even though some data on limit surfaces exist [1], a comprehensive discussion of this problem has yet to appear within the framework of damage mechanics.

In this section a simple form of the damage surface is considered in accord with assumed absence of the residual (plastic) deformation. The accommodations needed to represent different tensile and compressive responses and elaboration on the corresponding structure of the damage surface is presented in a separate paper [13].

4.1.   Strain space

To distinguish between the unloading-elastic behavior and loading-damage behavior, introduce a damage function in strain space $\Xi(\epsilon)$, such that the locus of points

$$\Xi(\epsilon) - R(r) = 0 \tag{4.1}$$

encloses the stain region within which the response of material is purely elastic (without further damage evolution). The surface defined by (4.1) will be referred to as a damage surface in strain space. A simplified representation (4.1) is assumed, where the parameter R, which defines the size of the surface, depends on the cumulative damage measure r.

If the strain state is on the damage surface, the subsequent strain state

will remain on the damage surface if the consistency condition is satisfied

$$\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon} - \frac{dR}{dr} \dot{r} = 0. \tag{4.2}$$

From (4.2) the rate of cumulative damage measure is

$$\dot{r} = \frac{1}{h} \left(\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon} \right), \tag{4.3}$$

where $h = dR/dr$. Since $\dot{r} = 0$ implies no damage evolution, from (4.3) follows that the strain rate such that $\partial \Xi / \partial \epsilon : \dot{\epsilon} = 0$, represents the neutral (elastic) loading. Progressive damage implies $\dot{r} > 0$. Hence, the damage loading condition is

$$\mathrm{sign}(h) \left(\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon} \right) > 0. \tag{4.4}$$

The reversed direction of the inequality (4.4) corresponds to elastic unloading from the current state of strain on the damage surface (4.1).

The damage stress rate is derived substituting (4.3) into (3.5)

$$\dot{\sigma}^d = -\frac{1}{h} \left(\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon} \right) \frac{\partial \Pi}{\partial \epsilon}. \tag{4.5}$$

Hence, if the damage rule is associated ($\Pi = \Xi$), it follows that

$$\dot{\epsilon} : \dot{\sigma}^d = -\frac{1}{h} \left(\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon} \right)^2, \tag{4.6}$$

which has the sign opposite to that of h (it is negative for positive h).

4.2. Stress space

Introduce a damage function in stress space $\Sigma(\sigma)$, such that the locus of points

$$\Sigma(\sigma) - \mathcal{R}(\rho) = 0 \tag{4.7}$$

encloses the stress region within which the response of material is purely elastic. The surface defined by (4.7) will be referred to as a damage surface in stress space. A simplified representation of the damage surface is again assumed. The parameter $\mathcal{R}$, which defines the size of the surface, is assumed to be a function of the cumulative damage measure ρ. Alternatively, $\mathcal{R}$ can be assumed to depend on the equivalent damage strain or damage work, in analogy to the familiar plasticity formulations [22]. If the stress

state is on the damage surface, the subsequent stress state will remain on
the damage surface if the consistency condition is satisfied

$$\frac{\partial \Sigma}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} - \frac{d\mathcal{R}}{d\rho} \dot{\rho} = 0. \tag{4.8}$$

The rate of damage measure is from (4.8)

$$\dot{\rho} = \frac{1}{H} \left(\frac{\partial \Sigma}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} \right), \tag{4.9}$$

where $H = d\mathcal{R}/d\rho$. Since $\dot{\rho} = 0$ implies no damage evolution, from (4.9)
follows that the stress rate such that $\partial \Sigma / \partial \boldsymbol{\sigma} : \dot{\boldsymbol{\sigma}} = 0$, constitutes the neutral
(elastic) loading. Progressive damage implies $\dot{\rho} > 0$, hence the condition

$$\text{sign}(H) \left(\frac{\partial \Sigma}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} \right) > 0 \tag{4.10}$$

must hold during the damage loading. The stress rate such that $\partial \Sigma / \partial \boldsymbol{\sigma} :$
$\dot{\boldsymbol{\sigma}} > 0$ represents the damage loading in the hardening regime $H > 0$ of the
material response. In the softening regime $H < 0$, and

$$\frac{\partial \Sigma}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} < 0 \tag{4.11}$$

must hold during the damage loading. However, the condition (4.11) is
only necessary but not sufficient for the damage accumulation since (4.11)
can also be satisfied during elastic unloading.

The damage strain rate is derived substituting (4.9) into (3.9)

$$\dot{\boldsymbol{\epsilon}}^d = \frac{1}{H} \left(\frac{\partial \Sigma}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} \right) \frac{\partial \Omega}{\partial \boldsymbol{\sigma}}. \tag{4.12}$$

Hence, if the damage rule is associated ($\Omega = \Sigma$), it follows that

$$\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}}^d = \frac{1}{H} \left(\frac{\partial \Sigma}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} \right)^2, \tag{4.13}$$

which can be either positive or negative depending on the sign of H, i.e.,
it is positive in the hardening, and negative in the softening range of the
material response.

5. COMPLETE RATE-TYPE CONSTITUTIVE EQUATIONS

5.1. *Strain space*

As shown in Section 3, the stress rate allows decomposition into elastic and damage parts

$$\dot{\sigma} = \boldsymbol{\mathcal{L}} : \dot{\epsilon} + \dot{\sigma}^d. \tag{5.1}$$

Substituting (3.5) into (5.1), with $\dot{r}$ defined by (4.3), it follows that

$$\dot{\sigma} = \boldsymbol{\mathcal{L}} : \dot{\epsilon} - \frac{1}{h}\left(\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon}\right)\frac{\partial \Pi}{\partial \epsilon}, \tag{5.2}$$

i.e.,

$$\dot{\sigma} = \widehat{\boldsymbol{\mathcal{L}}} : \dot{\epsilon}, \tag{5.3}$$

where $\widehat{\boldsymbol{\mathcal{L}}}$ is the tangent stiffness at the current state of deformation defined as

$$\widehat{\boldsymbol{\mathcal{L}}} = \boldsymbol{\mathcal{L}} - \frac{1}{h}\frac{\partial \Pi}{\partial \epsilon} \otimes \frac{\partial \Xi}{\partial \epsilon}. \tag{5.4}$$

Unless $\Pi = \Xi$ (associated damage rule), $\widehat{\boldsymbol{\mathcal{L}}}$ does not possess the self-adjoint symmetry ($\widehat{\mathcal{L}}_{ijkl} \neq \widehat{\mathcal{L}}_{klij}$). Hence, no rate potential π exists for the stress rate (i.e., $\dot{\sigma} \neq \partial\pi/\partial\dot{\epsilon}$) in the case of nonassociated damage rule.

To invert the expression (5.2) and define $\dot{\epsilon}$ in terms of $\dot{\sigma}$, form a trace product of both sides of (5.2) with the elastic compliance tensor $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{L}}^{-1}$,

$$\boldsymbol{\mathcal{M}} : \dot{\sigma} = \dot{\epsilon} - \frac{1}{h}\left(\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon}\right)\left(\boldsymbol{\mathcal{M}} : \frac{\partial \Pi}{\partial \epsilon}\right). \tag{5.5}$$

Taking the trace product of (5.5) with $\partial\Xi/\partial\epsilon$, it follows that

$$\frac{\partial \Xi}{\partial \epsilon} : \dot{\epsilon} = \frac{h}{\widehat{h}}\left(\frac{\partial \Xi}{\partial \epsilon} : \boldsymbol{\mathcal{M}} : \dot{\sigma}\right), \tag{5.6}$$

where

$$\widehat{h} = h - \frac{\partial \Xi}{\partial \epsilon} : \boldsymbol{\mathcal{M}} : \frac{\partial \Pi}{\partial \epsilon}. \tag{5.7}$$

Substituting (5.6) back into (5.5) and solving for the strain rate, the inverse relation is obtained, i.e.,

$$\dot{\epsilon} = \widehat{\boldsymbol{\mathcal{M}}} : \dot{\sigma}, \tag{5.8}$$

where $\widehat{\boldsymbol{\mathcal{M}}}$ is the current tangent compliance

$$\widehat{\boldsymbol{\mathcal{M}}} = \boldsymbol{\mathcal{M}} + \frac{1}{\widehat{h}}\boldsymbol{\mathcal{M}} : \left(\frac{\partial \Pi}{\partial \epsilon} \otimes \frac{\partial \Xi}{\partial \epsilon}\right) : \boldsymbol{\mathcal{M}}. \tag{5.9}$$

The tangent compliance becomes singular at the state of strain for which $\widehat{h} = 0$. According to (5.7) this occurs when

$$\frac{\partial \Xi}{\partial \epsilon} : \mathcal{M} : \frac{\partial \Pi}{\partial \epsilon} = h. \tag{5.10}$$

5.2. Stress space

A completely analogous derivation ensues starting from the expression for the strain rate (3.6), in form of a sum of elastic and damage parts

$$\dot{\epsilon} = \mathcal{M} : \dot{\sigma} + \dot{\epsilon}^d. \tag{5.11}$$

Substituting of (3.9) into (5.11), where $\dot{\rho}$ is defined by (4.9), gives

$$\dot{\epsilon} = \mathcal{M} : \dot{\sigma} + \frac{1}{H}\left(\frac{\partial \Sigma}{\partial \sigma} : \dot{\sigma}\right)\frac{\partial \Omega}{\partial \sigma}. \tag{5.12}$$

Therefore,

$$\dot{\epsilon} = \widehat{\mathcal{M}} : \dot{\sigma}, \tag{5.13}$$

where $\widehat{\mathcal{M}}$ is the tangent compliance at the current instant of deformation, defined as

$$\widehat{\mathcal{M}} = \mathcal{M} + \frac{1}{H}\frac{\partial \Omega}{\partial \sigma} \otimes \frac{\partial \Sigma}{\partial \sigma}. \tag{5.14}$$

The tangent compliance becomes singular when $H = 0$. Beyond this point the material response enters into the softening regime.

Inverting expression (5.13), it follows that

$$\dot{\sigma} = \widehat{\mathcal{L}} : \dot{\epsilon}. \tag{5.15}$$

In (5.15) the tangent stiffness tensor is given by

$$\widehat{\mathcal{L}} = \mathcal{L} - \frac{1}{\widehat{H}}\mathcal{L} : \left(\frac{\partial \Omega}{\partial \sigma} \otimes \frac{\partial \Sigma}{\partial \sigma}\right) : \mathcal{L}, \tag{5.16}$$

where

$$\widehat{H} = H + \frac{\partial \Sigma}{\partial \sigma} : \mathcal{L} : \frac{\partial \Omega}{\partial \sigma}. \tag{5.17}$$

6.   SOME SPECIFIC FORMS OF DAMAGE FUNCTIONS

In this section, attention is focused on the results obtained for some appealing choices of the damage functions in strain and stress spaces. The

corresponding constitutive expressions can be easily incorporated into specific applications and numerical evaluations. For the discussion of certain physical and computational issues related to the strain vs. stress based formulation of the rate-type damage analysis, refer to [11, 12, 23].

6.1. Strain space

Assume the following simple form of the damage potential Π, introduced in (2.17),

$$\Pi = \frac{1}{2}\epsilon' : \epsilon' + g(e). \tag{6.1}$$

Here $\epsilon' = \epsilon - \frac{1}{3}e\delta$ is the deviatoric strain, $e = \delta : \epsilon$ the volumetric strain, and δ the second order identity tensor. The scalar function g, which depends on the volumetric strain, should be appropriately specified as suggested by experimental evidence. Other forms of the function Π can be considered as well. For example, the energy norm $(\epsilon : \mathcal{L}_0 : \epsilon)^{1/2}$ of the strain tensor has been utilized in [12], although such selection is compatible only with an isotropic damage evolution. From (6.1)

$$\frac{\partial \Pi}{\partial \mathbf{G}} = \frac{\partial \Pi}{\partial \epsilon} : \frac{\partial \epsilon}{\partial \mathbf{G}} = \epsilon : \frac{\partial \epsilon}{\partial \mathbf{G}} + \left(\frac{dg}{de} - \frac{1}{3}e\right)\frac{\partial e}{\partial \mathbf{G}}. \tag{6.2}$$

To express the gradient $\partial \Pi / \partial \mathbf{G}$ in the strain space, consider the identity

$$\mathbf{G} : \mathbf{G} = \left(\frac{1}{2}\epsilon : \epsilon\right)^2. \tag{6.3}$$

Differentiating (6.3) with respect to $\mathbf{G}$, it follows that

$$\mathbf{G} = \left(\frac{1}{2}\epsilon : \epsilon\right)\left(\epsilon : \frac{\partial \epsilon}{\partial \mathbf{G}}\right), \tag{6.4}$$

i.e.

$$\epsilon : \frac{\partial \epsilon}{\partial \mathbf{G}} = \frac{\epsilon \otimes \epsilon}{\epsilon : \epsilon}. \tag{6.5}$$

Likewise, introducing the fourth-order tensor $\mathbf{G}_0 = \frac{1}{2}(\delta \otimes \delta)$, the following relationship holds

$$\mathbf{G} : \mathbf{G}_0 = \left[\frac{1}{2}(\delta : \epsilon)\right]^2 = \frac{1}{4}e^2. \tag{6.6}$$

Differentiating (6.6) with respect to $\mathbf{G}$ gives

$$\mathbf{G}_0 = \frac{1}{2}e\frac{\partial e}{\partial \mathbf{G}}, \tag{6.7}$$

i.e.,

$$\frac{\partial e}{\partial \mathbf{G}} = \frac{\boldsymbol{\delta} \otimes \boldsymbol{\delta}}{e}. \tag{6.8}$$

Substitution of (6.5) and (6.8) into (6.2) leads to

$$\frac{\partial \Pi}{\partial \mathbf{G}} = \frac{\boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}}{\boldsymbol{\epsilon} : \boldsymbol{\epsilon}} + \left(\frac{1}{e} \frac{dg}{de} - \frac{1}{3} \right) \boldsymbol{\delta} \otimes \boldsymbol{\delta}. \tag{6.9}$$

The expression for the damage stress rate can be derived by substituting (6.9) into (3.3)

$$\dot{\boldsymbol{\sigma}}^d = -\dot{r} \left[\boldsymbol{\epsilon} + \left(\frac{dg}{de} - \frac{1}{3} e \right) \boldsymbol{\delta} \right] = -\dot{r} \left(\boldsymbol{\epsilon}' + \frac{dg}{de} \boldsymbol{\delta} \right). \tag{6.10}$$

For example, if $g = e^2/6$, (6.10) reduces to

$$\dot{\boldsymbol{\sigma}}^d = -\dot{r} \boldsymbol{\epsilon}. \tag{6.11}$$

Introduce next the damage function $\Xi(\boldsymbol{\epsilon})$, appearing in (4.1), in the form

$$\Xi = \frac{1}{2} \boldsymbol{\epsilon}' : \boldsymbol{\epsilon}' + f(e). \tag{6.12}$$

The rate of the cumulative damage measure, defined by (4.3), becomes

$$\dot{r} = \frac{1}{h} \left(\boldsymbol{\epsilon}' + \frac{df}{de} \boldsymbol{\delta} \right) : \dot{\boldsymbol{\epsilon}}. \tag{6.13}$$

In the case of the associated damage rule $(f = g)$ and for the choice of function $f = e^2/6$, (6.13) reduces to

$$\dot{r} = \frac{1}{h} (\boldsymbol{\epsilon} : \dot{\boldsymbol{\epsilon}}). \tag{6.14}$$

With this, the damage flux $\dot{\mathbf{L}}$, defined by (2.17), becomes

$$\dot{\mathbf{L}} = \frac{1}{h} \frac{\boldsymbol{\epsilon} : \dot{\boldsymbol{\epsilon}}}{\boldsymbol{\epsilon} : \boldsymbol{\epsilon}} (\boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}). \tag{6.15}$$

The current tangent stiffness (5.4) is

$$\widehat{\mathcal{L}} = \mathcal{L} - \frac{1}{h} (\boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}), \tag{6.16}$$

while the current tangent compliance (5.9) becomes

$$\widehat{\mathcal{M}} = \mathcal{M} + \frac{1}{\widehat{h}}\mathcal{M} : (\epsilon \otimes \epsilon) : \mathcal{M}, \tag{6.17}$$

where $\widehat{h} = h - \epsilon : \mathcal{M} : \epsilon$.

It is interesting to note that the model of isotropic damage accumulation can be recovered from the previous analysis, provided that an initially isotropic material is continuously subjected to hydrostatic states of stress and strain. Indeed, in this case $\mathcal{L} = \kappa_0(1 - d)\delta \otimes \delta$, where κ_0 is the initial bulk modulus, and d a scalar damage variable. Hence, $\dot{\mathbf{L}} = \dot{d}\kappa_0\delta \otimes \delta$. From (6.14) and (6.15), $\dot{\mathbf{L}} = \frac{1}{3}\dot{r}\delta \otimes \delta$, since ϵ is a spherical tensor, and therefore $\dot{d} = (1/3\kappa_0)\dot{r}$, i.e., $d = (1/3\kappa_0)r$.

6.2. Stress space

Similar analysis can be performed in the stress space, in which the damage potential Ω and the damage function Σ are assumed to be:

$$\Omega = \frac{1}{2}\sigma' : \sigma' + a(p) \tag{6.18}$$

$$\Sigma = \frac{1}{2}\sigma' : \sigma' + b(p), \tag{6.19}$$

where $\sigma' = \sigma + p\delta$ is the deviatoric stress, $p = -\frac{1}{3}(\delta : \sigma)$ the hydrostatic pressure, and a and b the scalar functions to be appropriately specified. For example, if the damage rule is associated ($a = b$), and the function a is taken to be $a = 3p^2/2$, it follows that

$$\Omega = \Sigma = \frac{1}{2}\sigma : \sigma \tag{6.20}$$

The damage flux $\dot{\mathbf{M}}$ is in this case

$$\dot{\mathbf{M}} = \frac{1}{H}\frac{\sigma : \dot{\sigma}}{\sigma : \sigma}(\sigma \otimes \sigma). \tag{6.21}$$

The damage function of the type (6.20) has been used in [9] to describe the inelastic behavior to concrete. It can be easily shown that in the case of uniaxial stress, the damage parameter ρ is equal to $E^{-1} - E_0^{-1}$, where E is the current degraded elastic (secant) modulus, appearing in the expression $\sigma = E(\epsilon)\epsilon$, and E_0 is the initial elastic modulus. The parameter $\mathcal{R}$ is equal to $\frac{1}{2}\sigma^2$. Consequently, the relationship $\mathcal{R} = \mathcal{R}(\rho)$ can be easily extracted from the given uniaxial stress-strain relationship $\sigma = \sigma(\epsilon)$.

The corresponding tangent compliance becomes

$$\widehat{\mathcal{M}} = \mathcal{M} + \frac{1}{H}(\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}), \tag{6.22}$$

while the tangent stiffness is

$$\widehat{\mathcal{L}} = \mathcal{L} - \frac{1}{\widehat{H}}\mathcal{L} : (\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}) : \mathcal{L}, \tag{6.23}$$

where $\widehat{H} = H + \boldsymbol{\sigma} : \mathcal{L} : \boldsymbol{\sigma}$.

In the softening regime the material response is determined by prescribing the strain rate and calculating the corresponding stress rate. Hence, substituting $\dot{\boldsymbol{\sigma}} = \widehat{\mathcal{L}} : \dot{\boldsymbol{\epsilon}}$, with $\widehat{\mathcal{L}}$ given by (6.23), into the rate of cumulative damage expression

$$\dot{\rho} = \frac{1}{H}(\boldsymbol{\sigma} : \dot{\boldsymbol{\sigma}}), \tag{6.24}$$

it follows

$$\dot{\rho} = \frac{1}{\widehat{H}}(\boldsymbol{\sigma} : \mathcal{L} : \dot{\boldsymbol{\epsilon}}). \tag{6.25}$$

Since during a damage evolution process $\dot{\rho} > 0$, the damage loading condition in strain space becomes

$$\text{sign}(\widehat{H})(\boldsymbol{\sigma} : \mathcal{L} : \dot{\boldsymbol{\epsilon}}) > 0. \tag{6.26}$$

6.2.1. *An alternative choice.*   Instead of (6.18) and (6.19), in certain cases it may be advantageous to select a different damage potential and damage function by assuming

$$\Omega = \left(\frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\right)^{1/2} + a(p) \tag{6.27}$$

$$\Sigma = \left(\frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\right)^{1/2} + b(p) \tag{6.28}$$

The functions (6.27) and (6.28) are of the type used in the studies of inelastic deformation of pressure-sensitive dilatant materials [24, 25]. From (6.27) it follows that

$$\frac{\partial \Omega}{\partial \boldsymbol{\Gamma}} = \frac{1}{2J}\left[\frac{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}{\boldsymbol{\sigma} : \boldsymbol{\sigma}} + \frac{1}{3}\left(\frac{2J}{3p}\frac{da}{dp} - 1\right)\boldsymbol{\delta} \otimes \boldsymbol{\delta}\right], \tag{6.29}$$

where $J = (\frac{1}{2}\sigma' : \sigma')^{1/2}$. Substitution of (6.29) into (2.18) provides the evolution equation for the damage flux $\dot{M}$. Using the notation

$$\sigma_b = \frac{\sigma'}{2J} - \frac{1}{3}\frac{db}{dp}\delta, \tag{6.30}$$

the rate of cumulative damage is from (4.9)

$$\dot{\rho} = \frac{1}{H}(\sigma_b : \dot{\sigma}). \tag{6.31}$$

The tangent compliance (5.9) takes the form

$$\widehat{\mathcal{M}} = \mathcal{M} + \frac{1}{H}(\sigma_a \otimes \sigma_b), \tag{6.32}$$

where

$$\sigma_a = \frac{\sigma'}{2J} - \frac{1}{3}\frac{da}{dp}\delta. \tag{6.33}$$

The tangent stiffness (5.16) becomes

$$\widehat{\mathcal{L}} = \mathcal{L} - \frac{1}{\widehat{H}}\mathcal{L} : (\sigma_a \otimes \sigma_b) : \mathcal{L}, \tag{6.34}$$

using the parameter $\widehat{H} = H + \sigma_a : \mathcal{L} : \sigma_b$. Finally, the rate of the cumulative damage parameter, expressed in terms of the strain rate, is given by

$$\dot{\rho} = \frac{1}{\widehat{H}}(\sigma_b : \mathcal{L} : \dot{\epsilon}). \tag{6.35}$$

7. CONCLUSION

Some of the basic features of the rate-type analysis of damage processes in brittle elastic solids and presented in this paper, assuming infinitesimal and path-dependant deformation response. Stress and strain based formulations are both given. Appropriate structures of the evolution equations for damage fluxes are established, and constitutive expressions for the damage stress and strain rates derived. Explicit representations of the tangent stiffness and compliance tensors are derived for arbitrary forms of the damage potentials and damage functions. By virtue of its incremental nature, the presented formulation is capable of accommodating path-dependant degradation of elastic properties, and induced elastic anisotropy. Simple

and computationally attractive results are obtained for some physically appealing choices of the damage functions, analogous to those used in other damage and pressure-dependent plasticity models. In summary, this paper presents a basic formulation and discusses some essential features of the rate-type damage elasticity. Modifications needed to model different tensile and compressive responses of brittle materials are presented in a separate paper [13]. Further research is also directed toward incorporation of the residual (plastic) component of deformation and finite strain and rotation effects, [26].

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