SHORT COMMUNICATION

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LUBARDA, V. A.

On the Rate-Type Finite Elasticity Constitutive Law

1. Introduction

The constitutive law for the elastic deformation is normally given as an "one to one" relationship between the measures of stress and deformation [1, 2]. However, by direct differentiation this finite-type elasticity law can be expressed in the rate-type form which is more characteristic for the history dependent behaviour of deforming materials (such as occurs, for example, in metal plasticity [3]). In this paper we derive some useful relations between different rate measures of strain and then establish two alternative expressions for the rate-type constitutive law valid for finite elastic deformation of isotropic materials. These expressions can be of interest in the rubber elasticity and have also their significance in the analysis of finite elastic-plastic deformation, when one combines elastic and plastic part of deformation in an appropriate manner [4].

2. Kinematic preliminaries

Let $\mathscr{B}_0$ be the original (virgin) configuration of the body, and $\mathscr{B}_t$ its deformed configuration at time t. Let the motion (deformation) from $\mathscr{B}_0$ to $\mathscr{B}_t$ be given by a one to one mapping

$$x = \chi(X, t)\,, \tag{2.1}$$

where x is the current position (at time t) of the material point which was originally at place X. The function (2.1) is assumed to be of class C_2. The infinitesimal vector $\mathrm{d}X$ in the configuration $\mathscr{B}_0$ becomes $\mathrm{d}x$ in the configuration $\mathscr{B}_t$, such that

$$\mathrm{d}x = F\,\mathrm{d}X\,, \qquad F = \partial x/\partial X \tag{2.2}$$

where F is the deformation gradient matrix, which is the basic kinematic quantity in the nonlinear continuum mechanics. The right Cauchy-Green deformation tensor

$$C = F^{\mathrm{T}}F \tag{2.3}$$

("T" stands for the transpose) is then defined such that the square length of the deformed element $\mathrm{d}x$ is

$$(\mathrm{d}s)^2 = \mathrm{d}X^{\mathrm{T}} \cdot C \cdot \mathrm{d}X\,. \tag{2.4}$$

Similarly, the left Cauchy-Green deformation tensor

$$B = FF^{\mathrm{T}} \tag{2.5}$$

is defined such that the square length of the undeformed element $\mathrm{d}X$ is

$$(\mathrm{d}S)^2 = \mathrm{d}x^{\mathrm{T}} \cdot B^{-1} \cdot \mathrm{d}x\,, \tag{2.6}$$

where "-1" stands for the inverse. Since F is a nonsingular matrix, the polar decomposition theorem gives

$$F = RU = VR\,, \tag{2.7}$$

R being the orthogonal matrix, and U and V symmetric positive-definite matrices. The decomposition (2.7) physically means that the deformation F can be obtained by pure stretch U followed by rotation R, or by rotation R followed by stretch V. R is called the rotation tensor, and U and V are the right and left stretch tensors. Substitution of (2.7) into (2.3) and (2.5) clearly gives the relationships

$$C = U^2\,, \qquad B = V^2\,. \tag{2.8), (2.9}$$

Consider now the velocity field in the current configuration

$$v = v(x, t)\,. \tag{2.10}$$

The difference in the velocity between two points distance $\mathrm{d}x$ away is

$$\mathrm{d}v = L\,\mathrm{d}x\,, \qquad L = \partial v/\partial x\,, \tag{2.11}$$

where L is the velocity gradient matrix. This matrix can be splitted into symmetric and antisymmetric part

$$L = D + W\,, \tag{2.12}$$

where

$$D = \tfrac{1}{2}\,(L + L^{\mathrm{T}}) \tag{2.13}$$

is the velocity strain tensor which gives the rate of current straining, while

$$W = \tfrac{1}{2}\,(L - L^{\mathrm{T}}) \tag{2.14}$$

is the spin tensor, which gives the rate of current rotation of material element. For example, it can be shown that the rate o change of the square length $(\mathrm{d}s)^2$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,[(\mathrm{d}s)^2] = 2\,\mathrm{d}x^{\mathrm{T}} \cdot D \cdot \mathrm{d}x\,. \tag{2.15}$$

Further, it is easy to show that the following relationships hold:

$$L = \dot{F}F^{-1}\,, \qquad \dot{C} = 2F^{\mathrm{T}}DF\,, \tag{2.16), (2.17}$$

$$\dot{B} = LB + BL^{\mathrm{T}}\,. \tag{2.18}$$

In (2.16)–(2.18) the dot "·" stands for the material derivative $\mathrm{d}/\mathrm{d}t$. By using the Jaumann corrotational derivative [5]

$$\overset{\circ}{(\;)} = \dot{(\;)} - W(\;) + (\;)\,W \tag{2.19}$$

(2.18) can be rewritten as

$$\overset{\circ}{B} = DB + BD\,. \tag{2.20}$$

Substituting decomposition (2.7) into (2.16), we get

$$L = \dot{V}V^{-1} + V\Omega V^{-1}\,, \tag{2.21}$$

where Ω stands for the spin

$$\Omega = \dot{R}R^{-1}\,. \tag{2.22}$$

Finally, if we define the Jaumann derivative with respect to spin Ω

$$\overset{\nabla}{(\;)} = \dot{(\;)} - \Omega(\;) + (\;)\,\Omega\,, \tag{2.23}$$

the velocity strain D and the spin tensor W can be represented in the form:

$$D = \tfrac{1}{2}\,V^{-1}\overset{\nabla}{B}V^{-1}\,, \tag{2.24}$$

$$W = \tfrac{1}{2}\,V^{-1}[(V\dot{V} - \dot{V}V) + (\Omega B + B\Omega)]\,V^{-1}\,. \tag{2.25}$$

The relationship (2.24) between the velocity strain (D) and the selected Jaumann derivative of the left Cauchy-Green deformation tensor ($\overset{\nabla}{B}$) appears to be first given here, althought a similar relationship was derived and utilised in [4] to properly separate elastic from plastic part of the total velocity strain.

3. Finite elasticity constitutive law

The structure of the finite elasticity constitutive law is well established [1, 2] and is given by

$$T = 2\,\frac{\varrho}{\varrho_0}\,F\,\frac{\partial W}{\partial C}\,F^{\mathrm{T}}\,, \tag{3.1}$$

where T is the Cauchy stress tensor, $W = W(C)$ is the strain energy per unit initial volume, C is the right Cauchy-Green deformation tensor, while ϱ and ϱ_0 are the densities in the current and initial configurations. Restricting further attention to isotropic bodies, the strain energy must be an isotropic function of C, i.e.

$$W(C) = W(QCQ^{\mathrm{T}}) \tag{3.2}$$

for an arbitrary orthogonal tensor Q ($QQ^\mathsf{T} = I$). Selecting $Q = R$ (the rotation tensor from the polar decomposition theorem (2.7)), we have

$$W(C) = W(B) \qquad \text{and} \qquad \frac{\partial W}{\partial C} = R^\mathsf{T} \frac{\partial W}{\partial B} R\,, \qquad (3.3),\ (3.4)$$

which substituted into (3.1) gives

$$\tau = 2V \frac{\partial W}{\partial B} V\,, \qquad (3.5)$$

$\tau = \frac{\varrho_0}{\varrho} T$ being the Kirchoff stress tensor. But for the isotropic materials the principal directions of the stress tensor τ and the stretch tensor V are parallel, consequently the matric product on the right hand side of (3.5) is commutative and hence

$$\tau = 2B \frac{\partial W}{\partial B}\,, \qquad (3.6)$$

where W is an isotropic function of B. Relationship (3.6) is the constitutive law for finite elastic deformation of isotropic material. It gives a one to one relation between the total deformation B and corresponding stress τ, ragardless of the deformation history. This is so only for the elastic deformation of material. It is often said that the elastic body remembers only the initial (virgin) configuration (because it returns to it when the forces are removed), but not the intermediate configurations through which it passes toward the final state. That is why the elasticity law is a one to one (function) relationship, and not the functional relationship (as in viscoelasticity [6]), or rate-type relationship (as in plasticity [7]).

4. Rate-type finite elasticity law

Althought the nature of the elastic deformation is such that it allows one to one relationship between the stress and deformation, one can express the elastic constitutive law also in the rate-type form. Indeed, by applying on (3.6) the Jaumann derivative (2.19), we get

$$\mathring{\tau} = 2\mathring{B} \frac{\partial W}{\partial B} + 2B \frac{\partial^2 W}{\partial B^2} : \mathring{B}\,, \qquad (4.1)$$

where ":" stands for the inner product or trace. In the index notation, (4.1) reads

$$\mathring{\tau}_{ij} = 2B_{i\alpha} \frac{\partial W}{\partial B_{\alpha j}} + 2B_{i\alpha} \left(\frac{\partial^2 W}{\partial B^2}\right)_{\alpha jmn} \mathring{B}_{mn}\,, \qquad (4.2)$$

which can be rewritten as

$$\mathring{\tau}_{ij} = 2\left[\delta_{im} \frac{\partial W}{\partial B_{nj}} + B_{i\alpha} \left(\frac{\partial^2 W}{\partial B^2}\right)_{\alpha jmn}\right] \mathring{B}_{mn}\,. \qquad (4.3)$$

This establishes the structure

$$\mathring{\tau}_{ij} = A_{ijmn} \mathring{B}_{mn}\,, \qquad (4.4)$$

with the obvious representation of the fourth order tensor A. Substituting (2.20) into (4.4) now gives

$$\mathring{\tau}_{ij} = (A_{ijpn} B_{qn} + A_{ijmq} B_{mp})\, D_{pq}\,, \qquad (4.5)$$

or shortly

$$\mathring{\tau}_{ij} = \Lambda_{ijpq} D_{pq}\,, \qquad (4.6)$$

where the fourth order tensor Λ, from (4.3) and (4.5), is clearly

$$\Lambda_{ijpq} = 2\left[\delta_{ip} B_{qn} \left(\frac{\partial W}{\partial B}\right)_{nj} + B_{i\alpha} B_{qn} \left(\frac{\partial^2 W}{\partial B^2}\right)_{\alpha jpn} + B_{ip} \left(\frac{\partial W}{\partial B}\right)_{qj} + B_{i\alpha} B_{mp} \left(\frac{\partial^2 W}{\partial B^2}\right)_{\alpha jmq}\right]. \qquad (4.7)$$

Expression (4.6) with (4.7) presents the rate-type constitutiv law for finite elastic deformation of an isotropic body. It give the Jaumann derivative of stress ($\mathring{\tau}$) as a function of the velocity strain (D) and the tensor (operator) Λ which depends on th current state (i.e. deformation B or stress τ).

An alternative expression for the elastic rate-type constitutiv law can be obtained by applying on (3.6) the Jaumann deriva tive (2.23)

$$\overset{\nabla}{\tau} = 2\overset{\nabla}{B} \frac{\partial W}{\partial B} + 2B \frac{\partial^2 W}{\partial B^2} : \overset{\nabla}{B}\,, \qquad (4.8$$

i.e.

$$\overset{\nabla}{\tau}_{ij} = A_{ijmn} \overset{\nabla}{B}_{mn}\,, \qquad (4.9$$

with the previously defined tensor A_{ijmn} (see (4.3) and (4.4)) But, from (2.24) we have

$$\overset{\nabla}{B} = 2VDV \qquad (4.10$$

which substituted into (4.9) gives

$$\overset{\nabla}{\tau}_{ij} = (2A_{ijmn} V_{mp} V_{qn})\, D_{pq}\,. \qquad (4.11$$

Shortly (4.11) can be written as

$$\overset{\nabla}{\tau}_{ij} = \mathcal{L}_{ijpq} D_{pq} \qquad (4.12$$

where, from (4.3), (4.9) and (4.11), the fourth order tensor $\mathcal{L}$ ha the representation

$$\mathcal{L}_{ijpq} = 4\left[V_{ip} V_{qn} \left(\frac{\partial W}{\partial B}\right)_{nj} + B_{i\alpha} V_{mp} V_{qn} \left(\frac{\partial^2 W}{\partial B^2}\right)_{\alpha jmn}\right]. \qquad (4.13$$

Expression (4.12) with (4.13) presents another (alternative form of the rate-type finite elasticity law for the isotropi material.

5. Discussion

The main difference between the constitutive laws (4.6) wit (4.7) and (4.12) with (4.13) is the presence of different Jauman derivatives in their structure, which leads to different represen tations of the corresponding operators Λ and $\mathcal{L}$. As is seen fro (4.13), the operator Λ has simpler structure than the opera tor $\mathcal{L}$ given by (4.7). However, the constitutive law (4.6) contain the Jaumann derivative with respect to spin W which is directl expressible in terms of the velocity components, what is no the case with the spin Ω which appears in the Jaumann deriva tive of the law (4.12). Therefore, in the integration (step by step procedure of solving finite elasticity problems, the law (4.6 should be more suitable to apply.

References

1 RIVLIN, R. S., Some Topics in Finite Elasticity, in Structural Mechanics (Ed GOODIER and HOFF), Pergamon Press, New York, 1960.
2 TRUESDELL, C.; NOLL, W., The Nonlinear Field Theories of Mechanics, i Handbuch der Physik, (Ed. S. FLÜGGE), Vol. III/3, Springer-Verlag, Berli 1965.
3 HILL, R., Mathematical Theory of Plasticity, Oxford Press, 1950.
4 LUBARDA, V. A.; LEE, E. H., A Correct Definition of Elastic and Plastic Defo mation and its Computational Significance, J. Appl. Mech. **48** (1981).
5 PRAGER, W., Introduction to Mechanics of Continua, Dover, New York, 196
6 CHRISTENSEN, R. M., Theory of Viscoelasticity, Academic Press, 1971.
7 HILL, R., Aspects of Invariance in Solid Mechanics, Advances in Appl. Mec **18**, 1978.

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Address: Dr. ing. VLADO LUBARDA, docent, 81000 Tito grad, University of Titograd, Mechanical Eng. Dept. Yugoslavia