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On one basic half-plane elasticity solution*

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Abstract

The stress distribution in a half-plane loaded by uniform pressure over half of its straight edge is given in elasticity books, such as Timoshenko and Goodier [1], and is obtained by specializing the solution of an appropriate wedge problem, or by integration using the Flamant concentrated force solution. An alternative stress distribution is derived in this paper and compared with the existing well-known solution.

1 Introduction

Consider a half-plane loaded by a uniformly distributed pressure p over half of its straight edge, as shown in Fig. 1. By using polar coordinates (r, θ) the stress components can be expressed in terms of Airy stress function $\Phi = \Phi(r, \theta)$ by the well-known formulas

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2},$$
$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}, \tag{1}$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right).$$

*(Dedicated to the memory of Professor Batrić Vulićević)

Since there is no characteristic length in the considered problem, stress components are proportional to applied pressure p and independent of the radial coordinate r. This implies from (1) that Φ is proportional to pr^2, i.e. $\Phi = pr^2 f(\theta)$. Substituting $pr^2 f(\theta)$ into the governing biharmonic equation for Φ and solving gives

$$\Phi = pr^2 \left(a \cos 2\theta + b \sin 2\theta + c\theta + d\right), \tag{2}$$

where a, b, c and d are integration constants. The stress components in (1) can be expressed as

$$\begin{aligned} \sigma_r &= -p\left(2a \cos 2\theta + 2b \sin 2\theta - 2c\theta - 2d\right), \\ \sigma_\theta &= -p\left(-2a \cos 2\theta - 2b \sin 2\theta - 2c\theta - 2d\right), \\ \tau_{r\theta} &= p\left(2a \sin 2\theta - 2b \cos 2\theta - c\right). \end{aligned} \tag{3}$$

The constants a, b, c and d are obtained from the appropriate boundary conditions.

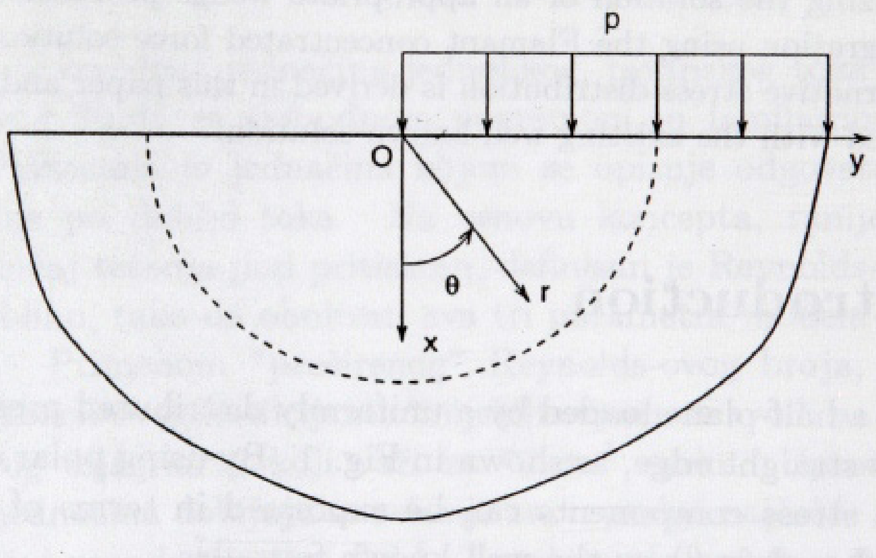

Fig. 1. - A half-plane loaded by uniform pressure p over half of its straight edge.

From the conditions along the straight edge ($\theta = \pm\pi/2$) it follows that $2b = c = -1/2\pi$, and $2a = 2d + 1/2$,with the constant d remaining unspecified. Similar situations with an unspecified integration constant also arise in other elasticity problems, as discussed in [2]. The stress components in (3) accordingly become

$$\sigma_r = -p\left[\left(2d + \frac{1}{2}\right)\cos 2\theta - \frac{1}{2\pi}\sin 2\theta + \frac{1}{\pi}\theta - 2d\right],$$

$$\sigma_\theta = -p\left[-\left(2d + \frac{1}{2}\right)\cos 2\theta + \frac{1}{2\pi}\sin 2\theta + \frac{1}{\pi}\theta - 2d\right], \tag{4}$$

$$\tau_{r\theta} = p\left[\left(2d + \frac{1}{2}\right)\sin 2\theta + \frac{1}{2\pi}\cos 2\theta + \frac{1}{2\pi}\right].$$

The longitudinal normal stress along the loaded part of the straight edge σ_r $(\theta = \pi/2)$ is $4pd$, while along the unloaded part it is $(4d+1)\,p$. Indeterminacy of the constant d can be discussed by considering an arbitrary semicircle around the point O as the closing boundary of the semicircular region. This is indicated by a dashed line in Fig. 1. Whatever the selected value of the constant d is, expressions (4) represent the stress distribution in the region provided that the tractions σ_r and $\tau_{r\theta}$ are applied over the semicircular boundary in accordance with (4) and the selected value of d. Note that two separate sets of tractions that correspond to different values of d and act over the semicircular boundary differ by a self-equilibrating set of tractions.

Letting $d = -1/4$ gives a solution which coincides with that obtained for an appropriate wedge problem (Timoshenko and Goodier [1], p.141, Eq. (d)), in which the wedge angle is π. The corresponding stress distribution is

$$\sigma_r = -\frac{p}{2\pi}\left(-\sin 2\theta + 2\theta + \pi\right),$$

$$\sigma_\theta = -\frac{p}{2\pi}\left(\sin 2\theta + 2\theta + \pi\right), \tag{5}$$

$$\tau_{r\theta} = \frac{p}{2\pi}\left(\cos 2\theta + 1\right).$$

Indeed, by considering an infinite wedge of angle 2α loaded by uniform pressure p over the face $\theta = \alpha$, the zero shear stress condition over the faces $\theta = \pm\alpha$ $(\alpha \neq \pi/2)$ requires that $a = 0$, hence $d = -1/4$. However, if $\alpha = \pi/2$ the product $a \sin 2\alpha$ (required to be zero by shear stress on the wedge faces) is zero even if $a \neq 0$, hence the more general expressions (4). Note that the stress distribution in (5) can also be obtained by integration using the Flamant concentrated force solution. According to (4), the radial stress (σ_r) immediately below the unloaded part ($\theta = -\pi/2$) of the straight edge is zero, while compression of magnitude p exists below the loaded part ($\theta = \pi/2$) of the straight edge.

It is instructive to rewrite the stress distribution in (4) using the Cartesian stress components, which gives

$$\sigma_x = -\frac{p}{2\pi}\left(\sin 2\theta + 2\theta + \pi\right),$$

$$\sigma_y = -\frac{p}{2\pi}\left(-\sin 2\theta + 2\theta - \pi\right) + 4pd, \tag{6}$$

$$\tau_{xy} = \frac{p}{2\pi}\left(\cos 2\theta + 1\right).$$

Hence, two solutions corresponding to two different values of the constant d differ only by a uniform stress of magnitude $\sigma_y = 4p\,(d_1 - d_2)$.

2 An alternative stress distribution

An alternative specification of the constant d in the stress distribution of (4) appears to be physically appealing. To obtain this value of d, we consider the problem in Fig. 1 as the superposition of two problems. That of the symmetric loading by uniform pressure of magnitude $p/2$ all along the straight edge, and of the antisymmetric pressure-tension loading of magnitude $p/2$. The stress distribution corresponding to the first problem is given by

$$\sigma_r = -p\left[\left(2d + \frac{1}{2}\right)\cos 2\theta - 2d\right],$$

$$\sigma_\theta = -p\left[-\left(2d + \frac{1}{2}\right)\cos 2\theta - 2d\right], \tag{7}$$

$$\tau_{r\theta} = p\left(2d + \frac{1}{2}\right)\sin 2\theta.$$

This is the symmetric part of the stress distribution in (4), and has the simple Cartesian representation $\sigma_x = -p/2$, $\sigma_y = (8d+1)\,p/2$, and $\tau_{xy} = 0$. The antisymmetric stress distribution corresponding to the second problem is given by

$$\sigma_r = -p\left(-\frac{1}{2\pi}\sin 2\theta + \frac{2}{\pi}\theta\right),$$

$$\sigma_\theta = -p\left(\frac{1}{2\pi}\sin 2\theta + \frac{1}{\pi}\theta\right), \tag{8}$$

$$\tau_{r\theta} = \frac{p}{2\pi}\left(\cos 2\theta + 1\right),$$

which is the antisymmetric part of the stress distribution in (4). The antisymmetric stress distribution in (8) is completely specified. However, the symmetric stress distribution in (7) is specified only to within a constant d. In order to determine the constant d, we calculate the radial strain corresponding to (7). Along the straight edge this strain is given by

$$\epsilon_r\left(\theta = \pm\frac{\pi}{2}\right) = \frac{p}{E^*}\left(\frac{1+\nu^*}{2} + 4d\right), \tag{9}$$

where ν^* is equal to Poisson's ratio ν in the case of plane stress, and to $\nu/(1-\nu)$ in the case of plane strain; E^* is equal to Young's modulus E in the case of plane stress, and to $E/(1-\nu^2)$ in the case of plane strain. The right-hand side of (9) also represents the horizontal (ϵ_y) strain everywhere in the half-plane. Assuming that the points of the half plane only move vertically, the strain in the horizontal direction is zero, and from (9) we obtain

$$d = -\frac{1+\nu^*}{8}. \tag{10}$$

Substituting (10) into (7) gives the stress distribution

$$\sigma_r = -p\left(\frac{1-\nu^*}{4}\cos 2\theta + \frac{1+\nu^*}{4}\right),$$

$$\sigma_\theta = -p\left(-\frac{1-\nu^*}{4}\cos 2\theta + \frac{1+\nu^*}{4}\right), \tag{11}$$

$$\tau_{r\theta} = p\frac{1-\nu^*}{4}\sin 2\theta,$$

which has the simple Cartesian representation $\sigma_x = -p/2$, $\sigma_y = -\nu^* p/2$ and $\tau_{xy} = 0$. This result is also given (for plane strain) by Eringen [3], p. 211, Eq. (6.7.4).. It is interesting to note that integrating the Flamant concentrated force solution gives the stress distribution $\sigma_x = \sigma_y = -p/2$ and $\tau_{xy} = 0$. The difference between these two stress distributions is a consequence of the zero horizontal displacement imposed in the first solution.

Superimposing (8) and (11) gives the stress distribution in the half-plane corresponding to the problem in Fig. 1 as

$$\sigma_r = -\frac{p}{2\pi}\left(\pi\frac{1-\nu^*}{2}\cos 2\theta - \sin 2\theta + 2\theta + \pi\frac{1+\nu^*}{2}\right),$$

$$\sigma_\theta = -\frac{p}{2\pi}\left(-\pi\frac{1-\nu^*}{2}\cos 2\theta + \sin 2\theta + 2\theta + \pi\frac{1+\nu^*}{2}\right), \qquad (12)$$

$$\tau_{r\theta} = \frac{p}{2\pi}\left(\pi\frac{1-\nu^*}{2}\sin 2\theta + \cos 2\theta + 1\right).$$

The radial stresses immediately below the straight edge are $\sigma_r\,(\theta = \pi/2) = -\,(1+\nu^*)\,p/2$ and $\sigma_r\,(\theta = -\pi/2) = (1-\nu^*)\,p/2$. As an example, if the Poisson ratio is $\nu = 1/3$, this gives $-2p/3$ and $p/3$ in the case of plane stress, and $-3p/4$ and $p/4$ in the case of plane strain. These values parallel the previously obtained (ν independent) values of $-p$ and 0,associated with the wedge type solution given in (5).

3 Discussion

The two stress distributions defined by (5) and (12) differ by a uniform stress $\sigma_y = (1-\nu^*)\,p/2$. Indeed, the Airy stress function given by (2) can be expressed as

$$\Phi = pr^2 d\,(\cos 2\theta + 1) + pr^2\frac{1}{4\pi}\,(\pi\cos 2\theta - \sin 2\theta - 2\theta)\,. \qquad (13)$$

Only the first part of (12), i.e. $\Phi = pr^2 d\,(\cos 2\theta + 1) = 2pdx^2$, depends on the constant d, and represents a stress function associated with the uniform stress $\sigma_y = \partial^2\Phi/\partial x^2 = 4pd$. It is observed in (4) that for $d = 0$ the radial stress immediately below the loaded section of the straight edge for the problem in Fig. 1 vanishes, while a tensile radial stress of magnitude p exists below the unloaded section of the straight edge. Superimposing on the stress distribution corresponding to $d = 0$ the uniform stress $\sigma_y = -p$, gives the wedge-type solution (5), while superimposing the uniform stress $\sigma_y = -(1+\nu^*)\,p/2$ leads to the alternative solution given by (12). It is noted that in the case of plane strain the two solutions coincide for an incompressible material (i.e. $\nu = 1/2$). It should also be pointed out that the radial strains immediately below the unloaded and loaded sections of the straight edge are $(4d+1)$ and $(4d+\nu^*)$, respectively, multiplied by $(1+\nu)\,p/E\,(1+\nu^*)$. If $d = -1/4$ the radial strain below the unloaded part of the straight edge is equal to zero. However, if d is given by (10) the radial strain below the loaded part of straight edge is equal in magnitude but opposite to the radial strain below the loaded part of straight edge. Thus, in this case the average radial strain for any line segment along the straight edge symmetric about the origin is equal to zero.

References

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O jednom baznom rešenju teorije elastičnosti za poluravan

Raspored napona na poluravni opterećenoj ravnomernim pritiskom duž polovine njene ravne ivice je dat u udžbenicima teorije elastičnosti, kao na primer u Timošenko i Gudijer [1], a dobijan je specificiranjem rešenja pogodnog ugaonog problema, ili integracijom pomoću Flamantovog rešenja za slučaj koncentrisane sile. U ovom radu izveden je alternativni raspored napona, koji je uporedjen sa postojećim dobro poznatim rešenjima.