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NON-ISOTHERMAL ELASTIC-PLASTIC DEFORMATION

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1. Introduction

The main objective of this work is to establish the rate (incremental) type constitutive law for non-isothermal elastic-plastic deformation of metals at finite strain. This is done be extanding the analysis presented in [2—5] which is concerned with the isothermal case. The result obtained here, i.e. the constitutive forms (4.15) and (4.16), appear to be established for the first time in this paper. They are not restricted to small strains and only restrictions imposed on them are isotropy requirements and time independence, as explained in the body of the paper.

2. Kinematics

Consider the body in its initial (stress free) configuraion $\mathcal{B}_0$ .Let it be deformed under the action of some external agency into the configuration $\mathcal{B}_t$ such that the motion (deformation) from $\mathcal{B}_0$ to $\mathcal{B}$ is given by a single valued mapping

$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t) \tag{2.1}$$

which carries the material particle from its initial position $\underset{\sim}{X}$ into its current position $\underset{\sim}{x}$ at time t (Fig. 2.1).

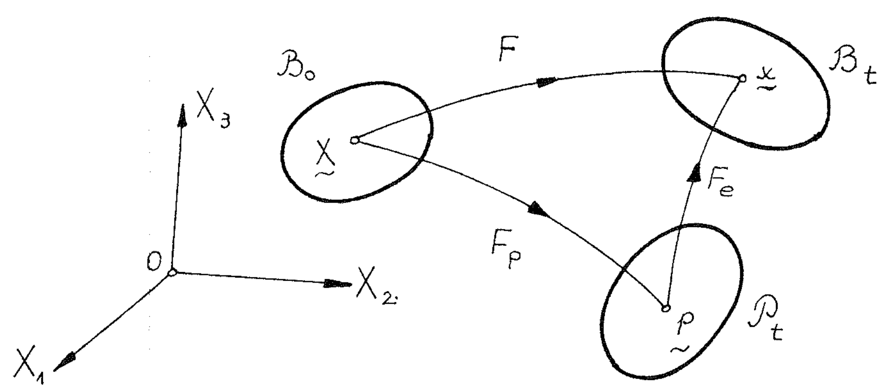

Fig. 2.1

Let the motion $\underset{\sim}{\chi}$ involve elastic-plastic deformation. Observations have shown such materials to be „simple" in the continuum sense, i.e. the deformation will be present in the constitutive equations only throught the deformation gradient matrix

$$F = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} \tag{2.2}$$

Following [1], we now introduce the intermidiate configuration $\mathcal{P}_t$ by destressing the whole body from its current configuration $\mathcal{B}_t$ and by reducing the temperature to the initial value. The configuration $\mathcal{P}_t$ then comprises pure plastic deformation, for thermal expansion and elastic strain components are both zero. One can then establish at each point of the deformed body the decomposition

$$F = F_e \, F_p \tag{2.3}$$

of the deformation gradient F into the pure (thermo-) elastic part F_e whoch corresponds to mapping from P_t to $\mathcal{B}_t$ and pure plastic part F_p which corresponds to mapping from $\mathcal{B}_0$ to $\mathcal{B}_t$. The decomposition (2.3) is not unique, for we can always have

$$F = F_e \, F_p = (F_e \, Q) \, (Q^T F_p) = \overline{F}_e \overline{F}_p \tag{2.4}$$

for an arbitrary orthogonal Q. However, for elastically isotropic bodies, the above choice of F_e is immaterial and we can choose that F_e which is the most convinient for us. We therefore chose F_e corresponding to destressing without rotation, i.e. $R_e = 1$ and

$$F_e = V_e \tag{2.5}$$

V_e being the symmetric left stretch tensor and R_e the orthogonal rotation tensor from the polar decomposition theorem $F_e = V_e R_e$. The decomposition (2.3) consequently becomes

$$F = V_e \, F_p. \tag{2.6}$$

Consider now the particle velocity in the state $\underset{\sim}{x}$

$$\underset{\sim}{v} = \frac{\partial \underset{\sim}{x}}{\partial t} \bigg/ \underset{\sim}{X} \tag{2.7}$$

The velocity gradient of the total deformation is then

$$L = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{X}} \frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{x}} = \dot{F} F^{-1} \tag{2.8}$$

where the superposed dot denotes material derivative, time differentiation at fixed $\underset{\sim}{X}$. Substitution from (2.6) then gives

$$L = L_e + V_e \, L_p \, V_e^{-1} \tag{2.9}$$

where $L_e = \dot{V}_e V_e^{-1}$ and $L_p = \dot{F}_p F_p^{-1}$ are the velocity gradients corresponding to the elastic and plastic part of deformation, respectively, Taking symmetric part of (2.9), we wbtain

$$D = D_e + (V_e \, L_p \, V_e^{-1})_s \tag{2.10}$$

where is the symmetrci part of $L = D + W$ called velocity strain, D_e is the symmetric part of L_e and subscript s denotes the symmetric part. Further $L_p = D_p + W_p$, D_p and W_p being the symmetric and antisymetric parts of L_p, and (2.10) becomes

$$D = D_e + (V_e D_p V_e^{-1})_s + (V_e W_p V_e^{-1})_s \tag{2.11}$$

Now, the first and third term on right hand side of (2.11) can be combined to give

$$D_e + (V_e W_p V_e^{-1})_s = \frac{1}{2} V_e^{-1} \overset{\nabla}{C}_e V_e^{-1} \tag{2.12}$$

where $C_e = V_e^T V_e = V_e^2$ is the right Cauchy-Green deformation tensor, and $(^\nabla)$ stands for the Jumann derivative with respect to the plastic spin W_p, i.e.

$$(^\nabla) = (\dot{\ }) - W_p (\ \ ) + W_p \tag{2.13}$$

Denoting (2.12) shortly by

$$\mathfrak{D}_e = \frac{1}{2} V_e^{-1} \overset{\nabla}{C}_e V_e^{-1} \tag{2.14}$$

we have from (2.11)

$$D = \mathfrak{D}_e + (V_e D_p V_e^{-1})_s \tag{2.15}$$

We shall further restrict ourselves to the case of isotropic hardening (plastic isotropy), when the principal directions of plastic stretching D_p coincide with the principal directions if stress. Since V_e has also the principal directions coincident with those of stress, matrices V_e and D_p in (2.15) are commutative and V_e and V_e^{-1} cancel each other. Hence, for the case of isotropic hardening (i.e. isotropic yield condition), (2.15) reduces to

$$D = \mathfrak{D}_e + D_p \tag{2.16}$$

which decomposed the stretching tensor (velocity strain) D into the elastic part $\mathfrak{D}_e$ and plastic part D_p. This relation, derived for the first time in the author's *PhD* thesis [2], presents the basis for the formulation of the rate-type elastic-plastic constitutive law.

3. Constitutive Laws

The elastic deformation F_e is governed by the classical finite elasticity law [9]

$$T = 2 \frac{\rho}{\rho_0} F_e \frac{\partial \Psi_e}{\partial C_e} F_e^T \tag{3.1}$$

where T is the Cauchy stress tensor, $\Psi_e = \Psi_e(C_e, \theta)$ is the Helmholtz free energy per unit initial volume, $C_e = F_e^T F_e$ is the right Cauchy — Green deformation tensor, θ is temperature, and ρ and ρ_0 are the densitiles in configurations $\mathcal{B}_t$ and $\mathcal{P}_t$, respectively. (We assume the incompressibility of plastic flow, hence the density in configuration $\mathcal{P}_t$ is the same as the intial density ρ_0 in the configuration $\mathcal{B}_0$. We also assume [8] that the elastic properties of material are not influenced by the prevoious plastic flow). But we have chosen $F_e = V_e$ and (3.1) becomes

$$\tau = 2 C_e \frac{\partial \Psi_e}{\partial C_e} \tag{3.2}$$

where $\tau = (\rho_0/\rho)\, T$ is the Kirchhoff stress and Ψ_e is an isotropic function of $C_e = V_e^2$. The law (3.2) is the constitutive law for the elastic part of deformation. It is seen to be in the finite form as one-to-one relation between the deformation C_e and the stress τ.

The structure of the constitutive law for the plastic part of deformation is quite different. Plasticity is a fluid type phenomenon which is governed by a rate (incremental, flow) type relation which involves the strain rate rather then strain in its structure. Restricting ourselves to the case of time independent plasticity, the law governing the plastic part of deformation in our theory, takes the form [1,6,7]

$$D_p = \frac{1}{\mathcal{J}} \left(\frac{\partial f}{\partial \tau} : \dot{\tau} + \frac{\partial f}{\partial \theta} \dot{\theta} \right) \frac{\partial f}{\partial \tau} \tag{3.3}$$

where $f = g(\tau) - c = 0$ is the yield function, c representing the hardening, „:" stands for the „trace", „." is the material derivative, $D_p = sym\,(F_p F_p^{-1})$ is the plastic straching tensor, and $\mathcal{J}$ is a slalar which contains information about the history of deformation and which for the time being is not of interest to us in this analysis.

Our objective is now to combine the elastic law (3.2), which is the finite (not rate-type) form, with the plastic law (3.3), which is in the rate-type form,) into a single relation — the constitutive law for non — isothermal elastic-plastic deformation.

4. Rate-Type Constitutive Law for Non-Isothermal Elastic-Plastic Deformation

The established kinematic relation (2.16) for the rate measures of the deformation will be the basis for the assembling of elastic and plastic constitutive laws into a single law. In fact, it's now just a matter of a proper mathematics to achieve the goal. Indeed, by taking the Jaumann derivative (2.13) of the relation (3.2), we have

$$\overset{\nabla}{\tau} = 2\, \overset{\nabla}{C}_e \frac{\partial \Psi_e}{\partial C_e} + 2\, C_e \left(\frac{\partial^2 \Psi_e}{\partial C_e^2} : C_e \right) + 2\, C_e \frac{\partial^2 \Psi_2}{\partial C_e\, \partial \theta}\, \dot{\theta} \tag{4.1}$$

or, in the component form (dropping the index e" for the moment)

$$\overset{\nabla}{\tau}_{ij} = \left[2\, \delta_{i\alpha} \left(\frac{\partial \Psi}{\partial C} \right)_{\beta j} + 2\, C_{ik} \left(\frac{\partial^2 \Psi}{\partial C^2} \right)_{kj\alpha\beta} \right] \overset{\nabla}{C}_{\alpha\beta} + 2 \left(C_{ik} \frac{\partial^2 \Psi}{\partial C_{kj}\, \partial \theta} \right) \dot{\theta} \tag{4.2}$$

Substituting $\overset{\nabla}{C}_e$ from (2.14) into (4.2), this becomes

$$\overset{\nabla}{\tau}_{ij} = 4 \left[V_{im} V_{n\beta} \left(\frac{\partial \Psi}{\partial C} \right)_{\beta j} + C_{ik} V_{\alpha m} V_{n\beta} \left(\frac{\partial^2 \Psi}{\partial C^2} \right)_{kj\alpha\beta} \right] \mathfrak{D}_{mn} + 2 \left(C_{ik} \frac{\partial^2 \Psi}{\partial C_{kj}\, \partial \theta} \right) \dot{\theta} \tag{4.3}$$

or, shortly

$$\overset{\nabla}{\tau}_{ij} = \tilde{\Pi}_{ijmn} \mathfrak{D}_{mn} + \Omega_{ij}\, \dot{\theta} \tag{4.4}$$

i.e, in dircet notation

$$\overset{\nabla}{\tau} = \tilde{\Pi}_e [\mathfrak{D}_e] + \Omega_e \dot{\theta} \tag{4.5}$$

Inverting (4.5) for $\mathfrak{D}_e$, we obtain

$$\mathfrak{D}_e = \tilde{\Lambda}_e [\overset{\nabla}{\tau}] + \tilde{\mathcal{M}}_e \dot{\theta} \tag{4.6}$$

which is the rate-type constitutive law for the elastic part of deformation.

With regard to the plastic part of deformation, we recall from [2—4] that in the case of isotropic hardening when f becomes an isotropic function of the stress, either the material or the Jaumann derivative with respect to any spin can be used in the structure of the law (3.3). Restricting, therefore, ourselves to the case of istotropic hardening, we rewrite the constitutive law (3.3) for the plastic part of deformation in the form

$$D_p = \frac{1}{\mathcal{J}} \left(\frac{\partial f}{\partial \tau} : \overset{\nabla}{\tau} + \frac{\partial f}{\partial \theta} \dot{\theta} \right) \frac{\partial f}{\partial \tau} \tag{4.7}$$

In the component form this reads

$$D_{ij}^p = \left(\frac{1}{\mathcal{J}} \frac{\partial f}{\partial \tau_{ij}} \frac{\partial f}{\partial \tau_{mn}} \right) \overset{\nabla}{\tau}_{mn} + \left(\frac{1}{\mathcal{J}} \frac{\partial f}{\partial \theta} \frac{\partial f}{\partial \tau_{ij}} \right) \dot{\theta} \tag{4.8}$$

i.e., with obvious notation,

$$D_{ij}^p = \Lambda_{ijmn}^p \overset{\nabla}{\tau}_{mn} + \mathcal{M}_{ij}^p \dot{\theta} \tag{4.9}$$

or

$$D_p = \Lambda_p [\overset{\nabla}{\tau}] + \mathcal{M}_p \dot{\theta} \tag{4.10}$$

This is the rate-type constitutive law for the plastic part of deformation.

To obtain the rate-type constitutive law for the total elastic-plastic deformation, we substitute (4.6) and (4.10) into the relation (2.16) to get

$$D = (\tilde{\Lambda}_e + \Lambda_p) [\overset{\nabla}{\tau}] + (\tilde{\mathcal{M}}_e + \mathcal{M}_p) \dot{\theta} \tag{4.11}$$

i.e.

$$D = \tilde{\Lambda} [\overset{\nabla}{\tau}] + \tilde{\mathcal{M}} \theta \tag{4.12}$$

This is the rate-type law for the elastic-plastic material in non-isothermal deformation. It gives the velocity strain D as a function of the stress rate $\overset{\nabla}{\tau}$, temperature rate $\dot{\theta}$ and the tensors (operators) $\tilde{\Lambda}$ and $\tilde{\mathcal{M}}$ which depend on the current state (i.e. stress, temperature and other quantities which define the state). Inverting (4.12) for $\overset{\nabla}{\tau}$, we obtain the other form of this rate-type constitutive law

$$\overset{\nabla}{\tau} = \tilde{\mathfrak{L}} [D] + \tilde{\mathcal{N}} \dot{\theta} \tag{4.13}$$

We, however, observe that in the laws (4.12) and (4.13), the Jaumann derivative is with respect to the plastic spin W_p. Although such a structure of the constitutive law is very prescient in revealing the nature of the kinematics of the elastic-plastic deformation process, in the application of the theory it would be awkward to work with this structure, since the spin W_p is not simply expressed in terms of the velocity field as is the total spin W. Fortunately, we can formulate the equivalent forms of the constitutive law in terms of the Jaumann derivative with respect to the total spin W, rather then plastic spin W_p. Following exactly the same mathematical procedure as given in [2—4], we can rigorously prove (for details, see [2,4]) that the elastic-plastic material also obeys the next two laws:

$$D = \Lambda [\overset{\circ}{\tau}] + \mathcal{M} \dot{\theta} \tag{4.14}$$

$$\overset{\circ}{\tau} = Z [D] + \mathcal{N} \dot{\theta} \tag{4.15}$$

where

$$(\overset{\circ}{\ }) = (\dot{\ }) - W(\ ) + (\ ) W \tag{4.16}$$

is the Jaumann derivative with respect to total spin W.

The expressions (4.14) and (4.15) are the final forms of the constitutive law for the material under the conditions of non-isothermal elastic-plastic deformation. They are in this form derived for the first time in this paper and present the extension of the previous author result obtained for the isothermal deformation, presented in [2]. The constitutive laws (4.14) or (4.15) are not restricted to small strains, and the only major restrictions imposed on them are the isotropy requirements and time independence. Elimination of these restrictions, i.e. inclusion of anisotropy and time dependent (rheological, viscous) effects in a satisfactory manner, would be, of course, worthy goals for future investigations.

Finally, it should be mentioned that in the case of isothermal deformation, the laws (4.14) and (4.15) reduce to

$$D = \Lambda [\overset{\circ}{\tau}] \tag{4.17}$$

$$\overset{\circ}{\tau} = \mathcal{L} [D] \tag{4.18}$$

as already established in [2—4].

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LA DEFORMATION ELASTO-PLASTIQUE NON-ISOTHERMALE

R é s u m é

Le but de ce travail est l'établissement de la loi constitutive du type incrémental pour les matériaux élasto-plastiques aux conditions de la déformation non-isotermale. C'est effectué en étendant l'analyse qui est presentée dans [2—5] au cas isothermal. Le résultat obtenu, les lois constitutives (4.15) et (4.16), sont formulées ici pour la première fois. Elles ne sont pas restreintes au cas des petites déformations et leur seule restriction vient de la supposition d' isotropie de la déformation et de l' absence des effets visqueux (rhéologiques).

NEIZOTERMALNA ELASTO-PLASTIČNA DEFORMACIJA

R e z i m e

Cilj ovog rada je da se formira izvodni (inkrementalni) tip konstitutivnog zakona za neizotermalnu elasto-plastičnu deformaciju metala. Ovo je postignuto proširujući analizu izvršenu u [2—5] za izotermalan slučaj. Dobijeni rezultat, tj. konstitutivni zakoni (4.15) i (4.16), pojavljuju se izvedeni po prvi put u ovom radu. Oni nisu ograničeni na slučaj malih deformacija i jedina njihova veća ograničenja proizilaze iz pretpostavki izotropnosti deformacije i odsustva od vremena zavisnih (viskoznih, reoloških) efekata.

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