NON-ISOHERMAL ELASTIC-PLASTIC DEFORMATION

V. Lubarda

(Received — December 4, 1980)
(Revised March 7, 1981)

1. Introduction

The main objective of this work is to establish the rate (incremental) type constitutive law for non-isothermal elastic-plastic deformation of metals at finite strain. This is done by extending the analysis presented in [2—5] which is concerned with the isothermal case. The result obtained here, i.e. the constitutive forms (4.15) and (4.16), appear to be established for the first time in this paper. They are not restricted to small strains and only restrictions imposed on them are isotropy requirements and time independence, as explained in the body of the paper.

2. Kinematics

Consider the body in its initial (stress free) configuration $B_0$. Let it be deformed under the action of some external agency into the configuration $B_t$ such that the motion (deformation) from $B_0$ to $B$ is given by a single valued mapping

$$\hat{x} = \hat{x}(X, t)$$

which carries the material particle from its initial position $X$ into its current position $\hat{x}$ at time $t$ (Fig. 2.1).

Fig. 2.1
Let the motion \( \dot{X} \) involve elastic-plastic deformation. Observations have shown such materials to be "simple" in the continuum sense, i.e., the deformation will be present in the constitutive equations only through the deformation gradient matrix

\[
F = \frac{\delta X}{\delta \dot{X}}
\]  

(2.2)

Following [1], we now introduce the intermediate configuration \( \dot{\mathcal{R}}_t \) by destressing the whole body from its current configuration \( \mathcal{R}_t \) and by reducing the temperature to the initial value. The configuration \( \dot{\mathcal{R}}_t \) then comprises pure plastic deformation, for thermal expansion and elastic strain components are both zero. One can then establish at each point of the deformed body the decomposition

\[
F = F_e F_p
\]  

(2.3)

of the deformation gradient \( F \) into the pure (thermo-) elastic part \( F_e \) which corresponds to mapping from \( \mathcal{R}_t \) to \( \mathcal{R}_e \) and pure plastic part \( F_p \) which corresponds to mapping from \( \mathcal{R}_e \) to \( \mathcal{R}_t \). The decomposition (2.3) is not unique, for we can always have

\[
F = F_e F_p = (F_e Q) (Q^T F_p) = F_e F_p
\]  

(2.4)

for an arbitrary orthogonal \( Q \). However, for elastically isotropic bodies, the above choice of \( F_e \) is immaterial and we can choose that \( F_e \) which is the most convenient for us. We therefore chose \( F_e \) corresponding to destressing without rotation, i.e., \( \mathcal{R}_e = 1 \) and

\[
F_e = V_e
\]  

(2.5)

\( V_e \) being the symmetric left stretch tensor and \( R_e \) the orthogonal rotation tensor from the polar decomposition theorem \( F_e = V_e R_e \). The decomposition (2.3) consequently becomes

\[
F = V_e F_p.
\]  

(2.6)

Consider now the particle velocity in the state \( \dot{X} \)

\[
\dot{\mathcal{Z}} = \frac{\delta \dot{X}}{\delta \dot{X}}
\]  

(2.7)

The velocity gradient of the total deformation is then

\[
L = \frac{\delta \dot{\mathcal{Z}}}{\delta \dot{X}} = \frac{\delta \dot{\mathcal{Z}}}{\delta \dot{X}} \frac{\delta \dot{X}}{\delta \dot{X}} = \dot{V} F^{-1}
\]  

(2.8)

where the superposed dot denotes material derivative, time differentiation at fixed \( \dot{X} \). Substitution from (2.6) then gives

\[
L = \dot{V}_e + V_e L_p V_e^{-1}
\]  

(2.9)

where \( \dot{V}_e = \dot{V}_e V_e^{-1} \) and \( L_p = \dot{F}_p F_p^{-1} \) are the velocity gradients corresponding to the elastic and plastic part of deformation, respectively. Taking symmetric part of (2.9), we obtain

\[
D = D_e + (V_e L_p V_e^{-1}) s
\]  

(2.10)
where is the symmetric part of \( L = D + W \) called velocity strain, \( D_e \) is the symmetric part of \( L_p \) and subscript \( s \) denotes the symmetric part. Further \( L_p = D_p + W_p \), \( D_p \) and \( W_p \) being the symmetric and antisymmetric parts of \( L_p \), and (2.10) becomes

\[
D = D_e + (V_e D_p V_e^{-1})_e + (V_e W_p V_e^{-1})_s
\]  
(2.11)

Now, the first and third term on right hand side of (2.11) can be combined to give

\[
D_e + (V_e W_p V_e^{-1})_e = \frac{1}{2} V_e^{-1} C_e V_e^{-1}
\]  
(2.12)

where \( C_e = V_e^T V_e = V_e^2 \) is the right Cauchy-Green deformation tensor, and \((\nabla)\) stands for the Jumann derivative with respect to the plastic spin \( W_p \), i.e.

\[
(\nabla) = (\cdot) - W_p (\cdot) + W_p
\]  
(2.13)

Denoting (2.12) shortly by

\[
\mathcal{D}_e = \frac{1}{2} V_e^{-1} C_e V_e^{-1}
\]  
(2.14)

we have from (2.11)

\[
D = \mathcal{D}_e + (V_e D_p V_e^{-1})_s
\]  
(2.15)

We shall further restrict ourselves to the case of isotropic hardening (plastic isotropy), when the principal directions of plastic stretching \( D_p \) coincide with the principal directions of stress. Since \( V_e \) has also the principal directions coincident with those of stress, matrices \( V_e \) and \( D_p \) in (2.15) are commutative and \( V_e \) and \( V_e^{-1} \) cancel each other. Hence, for the case of isotropic hardening (i.e. isotropic yield condition), (2.15) reduces to

\[
D = \mathcal{D}_e + D_p
\]  
(2.16)

which decomposed the stretching tensor (velocity strain) \( D \) into the elastic part \( \mathcal{D}_e \) and plastic part \( D_p \). This relation, derived for the first time in the author's PhD thesis [2], presents the basis for the formulation of the rate-type elastic-plastic constitutive law.

3. Constitutive Laws

The elastic deformation \( F_e \) is governed by the classical finite elasticity law [9]

\[
T = 2 \frac{\rho}{\rho_0} F_e \frac{\partial \Psi_e}{\partial C_e} F_e^T
\]  
(3.1)

where \( T \) is the Cauchy stress tensor, \( \Psi_e = \Psi_e(C_e, \theta) \) is the Helmholtz free energy per unit initial volume, \( C_e = F_e F_e^T \) is the right Cauchy-Green deformation tensor, \( \theta \) is temperature, and \( \rho \) and \( \rho_0 \) are the densities in configurations \( \mathcal{F}_e \) and \( \mathcal{F}_0 \), respectively. (We assume the incompressibility of plastic flow, hence the density in configuration \( \mathcal{F}_e \) is the same as the initial density \( \rho_0 \) in the configuration \( \mathcal{F}_0 \). We also assume [8] that the elastic properties of material are not influenced by the previous plastic flow). But we have chosen \( F_e = V_e \) and (3.1) becomes

\[
\tau = 2 C_e \frac{\partial \Psi_e}{\partial C_e}
\]  
(3.2)
where $\tau = (\rho_0/\rho)T$ is the Kirchhoff stress and $\Psi_e$ is an isotropic function of $C_e = V_e^2$. The law (3.2) is the constitutive law for the elastic part of deformation. It is seen to be in the finite form as one-to-one relation between the deformation $C_e$ and the stress $\tau$.

The structure of the constitutive law for the plastic part of deformation is quite different. Plasticity is a fluid type phenomenon which is governed by a rate (incremental, flow) type relation which involves the strain rate rather than strain in its structure. Restricting ourselves to the case of time independent plasticity, the law governing the plastic part of deformation in our theory, takes the form

$$D_p = \frac{1}{f} \left( \frac{\partial f}{\partial \tau}; \frac{\partial f}{\partial \theta} \right)$$

(3.3)

where $f = g(\tau) - c = 0$ is the yield function, $c$ representing the hardening, "" stands for the ""trace"", "" is the material derivative, $D_p = zy m (\varepsilon_p \varepsilon^p)$ is the plastic straching tensor, and $f$ is a scalar which contains information about the history of deformation and which for the time being is not of interest to us in this analysis.

Our objective is now to combine the elastic law (3.2), which is the finite (not rate-type) form, with the plastic law (3.3), which is in the rate-type form, into a single relation — the constitutive law for non — isothermal elastic-plastic deformation.

4. Rate-Type Constitutive Law for Non-Isothermal Elastic-Plastic Deformation

The established kinematic relation (2.16) for the rate measures of the deformation will be the basis for the assembling of elastic and plastic constitutive laws into a single law. In fact, it's now just a matter of a proper mathematics to achieve the goal. Indeed, by taking the Jaumann derivative (2.15) of the relation (3.2), we have

$$\bar{\tau} = 2 \bar{\tau}_e \frac{\partial \Psi_e}{\partial C_e} + 2 C_e \left( \frac{\partial^2 \Psi_e}{\partial C_e^2} \right) \bar{C}_e + 2 C_e \frac{\partial^2 \Psi_e}{\partial C_e \partial \theta} \dot{\theta}$$

(4.1)

or, in the component form (dropping the index $e$ for the moment)

$$\bar{\tau}_{ij} = \left[ 2 \delta_{kl} \left( \frac{\partial \Psi}{\partial C_{ij}} \right) + 2 C_{ikl} \left( \frac{\partial^2 \Psi}{\partial C_{ij} \partial C_{kl}} \right) \right] \bar{C}_{ij} + 2 \left( C_{ikl} \frac{\partial^2 \Psi}{\partial C_{ij} \partial \theta} \right) \dot{\theta}$$

(4.2)

Substituting $\bar{C}_e$ from (2.14) into (4.2), this becomes

$$\bar{\tau}_{ij} = 4 \left[ V_{im} V_{n\beta} \left( \frac{\partial \Psi}{\partial C_{ij}} \right) + C_{ikl} V_{am} V_{n\beta} \left( \frac{\partial^2 \Psi}{\partial C_{ij} \partial C_{kl}} \right) \right] \bar{D}_{mn} + 2 \left( C_{ikl} \frac{\partial^2 \Psi}{\partial C_{ij} \partial \theta} \right) \dot{\theta}$$

(4.3)

or, shortly

$$\bar{\tau}_{ij} = \Pi_{ijmn} \bar{D}_{mn} + \Omega_{ij} \dot{\theta}$$

(4.4)
Non-Isothermal Elastic-Plastic Deformation

i.e., in direct notation

\[ \tau = \Pi_{t} [\mathcal{D}_{e}] + \Omega_{e} \dot{\theta} \]  \hspace{1cm} (4.5)

Inverting (4.5) for \( \mathcal{D}_{e} \), we obtain

\[ \mathcal{D}_{e} = \tilde{\Lambda}_{e} [\tau] + \tilde{\mathcal{M}}_{e} \dot{\theta} \]  \hspace{1cm} (4.6)

which is the rate-type constitutive law for the elastic part of deformation.

With regard to the plastic part of deformation, we recall from \([2-4]\) that in the case of isotropic hardening when \( f \) becomes an isotropic function of the stress, either the material or the Jaumann derivative with respect to any spin can be used in the structure of the law \((3.3)\). Restricting, therefore, ourselves to the case of isotropic hardening, we rewrite the constitutive law \((3.3)\) for the plastic part of deformation in the form

\[ D_{p} = \frac{1}{f} \left( \frac{\partial f}{\partial \tau} \tau + \frac{\partial f}{\partial \theta} \dot{\theta} \right) \frac{\partial f}{\partial \tau} \]  \hspace{1cm} (4.7)

In the component form this reads

\[ D_{ij}^{p} = \left( \frac{1}{f} \frac{\partial f}{\partial \tau_{ij}} \tau_{mn} + \frac{1}{f} \frac{\partial f}{\partial \theta} \frac{\partial f}{\partial \tau_{ij}} \right) \dot{\theta} \]  \hspace{1cm} (4.8)

i.e., with obvious notation,

\[ D_{ij}^{p} = \Lambda_{ijmn}^{p} \tau_{mn} + \mathcal{M}_{ij}^{p} \dot{\theta} \]  \hspace{1cm} (4.9)

or

\[ D_{p} = \Lambda_{p} [\tau] + \mathcal{M}_{p} \dot{\theta} \]  \hspace{1cm} (4.10)

This is the rate-type constitutive law for the plastic part of deformation.

To obtain the rate-type constitutive law for the total elastic-plastic deformation, we substitute \((4.6)\) and \((4.10)\) into the relation \((2.16)\) to get

\[ D = (\tilde{\Lambda}_{e} + \Lambda_{p}) [\tau] + (\tilde{\mathcal{M}}_{e} + \mathcal{M}_{p}) \dot{\theta} \]  \hspace{1cm} (4.11)

i.e.

\[ D = \tilde{\Lambda} [\tau] + \tilde{\mathcal{M}} \dot{\theta} \]  \hspace{1cm} (4.12)

This is the rate-type law for the elastic-plastic material in non-isothermal deformation. It gives the velocity strain \( D \) as a function of the stress rate \( \tau \), temperature rate \( \dot{\theta} \) and the tensors (operators) \( \tilde{\Lambda} \) and \( \tilde{\mathcal{M}} \) which depend on the current state (i.e. stress, temperature and other quantities which define the state). Inverting \((4.12)\) for \( \tau \), we obtain the other form of this rate-type constitutive law

\[ \tau = \tilde{\tau} [D] + \tilde{\mathcal{M}} \dot{\theta} \]  \hspace{1cm} (4.13)
We, however, observe that in the laws (4.12) and (4.13), the Jaumann derivative is with respect to the plastic spin \( W_p \). Although such a structure of the constitutive law is very present in revealing the nature of the kinematics of the elastic-plastic deformation process, in the application of the theory it would be awkward to work with this structure, since the spin \( W_p \) is not simply expressed in terms of the velocity field as is the total spin \( W \). Fortunately, we can formulate the equivalent forms of the constitutive law in terms of the Jaumann derivative with respect to the total spin \( W \), rather than plastic spin \( W_p \). Following exactly the same mathematical procedure as given in [2–4], we can rigorously prove (for details, see [2,4]) that the elastic-plastic material also obeys the next two laws:

\[
D = \Lambda \left[ \dot{\tau} \right] + \dot{\mathcal{M}} \dot{\Omega}
\]

\[
\dot{\tau} = Z \{ D \} + \mathcal{N} \dot{\Omega}
\]

where

\[
(\cdot) = (\cdot) - W(\cdot) + (\cdot) W
\]

is the Jaumann derivative with respect to total spin \( W \).

The expressions (4.14) and (4.15) are the final forms of the constitutive law for the material under the conditions of non-isothermal elastic-plastic deformation. They are in this form derived for the first time in this paper and present the extension of the previous author result obtained for the isothermal deformation, presented in [2]. The constitutive laws (4.14) or (4.15) are not restricted to small strains, and the only major restrictions imposed on them are the isotropy requirements and time independence. Elimination of these restrictions, i.e. inclusion of anisotropy and time dependent (rheological, viscous) effects in a satisfactory manner, would be, of course, worthy goals for future investigations.

Finally, it should be mentioned that in the case of isothermal deformation, the laws (4.14) and (4.15) reduce to

\[
D = \Lambda \left[ \tau \right]
\]

\[
\dot{\tau} = \Omega \{ D \}
\]

as already established in [2–4].

Acknowledgement. This work was carried out under the sponsorship of the SIZ — Crna Gora through a contract with University of Tirograd for the project "Stability Problems in Continuum Mechanics". This support is appreciated by the author.

REFERENCES


LA DEFORMATION ELASTO-PLASTIQUE NON-ISOTHERMAL

Résumé

Le but de ce travail est l'établissement de la loi constitutive du type incrémental pour les matériaux élasto-plastiques aux conditions de la déformation non-isothermale. C’est effectué en étendant l'analyse qui est présentée dans [2–5] au cas isothermale. Le résultat obtenu, les lois constitutives (4.15) et (4.16), sont formulées ici pour la première fois. Elles ne sont pas restreintes au cas des petites déformations et leur seule restriction vient de la supposition d’isotropie de la déformation et de l’absence des effets visqueux (rhéologiques).

NEIZOTERMALNA ELASTO-PLASTIČNA DEFORMACIJA

Résume


Vlado Lubarda, 81000 Titograd
Mašinski fakultet, Yugoslavia