The Integration of Material, Process and Product Design, Zabaras et al. (eds) © 1999 Balkema, Rotterdam, ISBN 90 5809 101 5

Configurational forces and conservation laws in couple stress elasticity

V.A. Lubarda & X. Markenscoff
Department of Applied Mechanics and Engineering Sciences, University of California, San Diego, Calif., USA

ABSTRACT: Interaction energies between the stress systems due to internal and external sources of stress in couple stress elasticity are determined in the spirit of Eshelby (1956). Expressions for the force on a singularity, and the energy-momentum tensor of elastic couple stress field are derived. Two conservation integrals of couple stress elasticity follow from Noether's theorem, which give rise to conservation laws $J_k = 0$ and $L_k = 0$. There is no M conservation integral in couple stress elasticity.

1 INTRODUCTION

The strain energy density in couple stress elasticity is a function of the strain and rotation gradient (e.g., Mindlin 1964, Koiter 1964). Consequently, the constitutive relations involve additional material parameters, some of which have the dimension of length. In the case of infinitesimal isotropic elasticity, there is one such parameter. It was found to be exceedingly small relative to geometric dimensions of the body in most conventional problems of stress analysis, since no observable effects of couple stresses were experimentally detected. The theory was thus intended to problems in which there was a relevant geometric length scale of the order of the material (microstructural) length. For example, Mindlin (1963) found that the stress concentration around a circular hole in an infinite medium is effected by the size of the hole, if this is of the order of the material length parameter. In the classical theory of elasticity, without couple stresses, the stress concentration factor is independent of the radius of the hole.

There has been a significant amount of research recently devoted to the analysis of plastic response of materials by using the couple stress, or more involved strain-gradient theories. The motivation was to account for experimentally observed size effects in problems, such as the indentation hardness in metals and ceramics, strengthening by hard particles, grain-size effect on the flow stress, shear band localization, etc. When plastic deformation occurs, the gradient effects can become important at much larger scales than in purely elastic problems, due to development of dislocation micro-structure at the scales of the order of sub-microns. A report on the progress in this research area is given in the recent review article by Fleck & Hutchinson (1996).

In view of this revived interest in non-local theories of material behavior, it appeared appealing to systematically extend some of the well-known results from classical elasticity developed by Eshelby (1956) to the framework of couple stress theory. Expressions for the interaction energy between the stress systems due to internal and external sources of stress in couple stress elasticity are derived, and the force on a singularity or a source of internal stress, together with the corresponding energy-momentum tensor of elastic couple stress field. Two conservation integrals of couple stress elasticity follow from Noether's theorem, which give rise to conservation laws $J_k = 0$ and $L_k = 0$. There is no M conservation integral in couple stress elasticity.

2 GOVERNING EQUATIONS OF COUPLE STRESS ELASTICITY

A surface element $\mathrm{d}S$ in couple stress theory can transmit a force vector $T_i\,\mathrm{d}S$, and a couple vector $M_i\,\mathrm{d}S$. T_i are the components of the force traction, and M_i are the components of the couple traction, relative to Cartesian coordinates x_i. The surface forces are in equilibrium with the unsymmetric Cauchy stress p_{ij}, and the surface couples are in equilibrium with the couple stress μ_{ij}, such that (e.g., Koiter 1964):

$$T_i = n_j p_{ji}, \quad M_i = n_j \mu_{ji}. \tag{1}$$

The unit normal to the surface element has the components n_j. Differential equations of the force and moment equilibrium are:

$$p_{ji,j} + f_i = 0, \quad e_{ijk}p_{jk} + \mu_{ji,j} = 0. \tag{2}$$

The body forces are f_i, and e_{ijk} stands for the skew-symmetric alternating tensor. The body couples have been neglected. It is convenient to decompose the Cauchy stress into its symmetric and anti-symmetric part, $p_{ij} = \sigma_{ij} + \tau_{ij}$. Likewise, the infinitesimal displacement gradient is decomposed into the strain and rotation tensor, $u_{i,j} = \epsilon_{ij} + \omega_{ij}$. The rotation vector ω_i, dual to the rotation tensor ω_{ij}, is

$$\omega_i = -\frac{1}{2} e_{ijk}\omega_{jk}, \quad \omega_{ij} = -e_{ijk}\omega_k. \tag{3}$$

A fundamental quantity in couple stress theory is the curvature tensor, defined as the gradient of the rotation vector. In view of the identity $\omega_{ij,k} = \epsilon_{ki,j} - \epsilon_{kj,i}$, the relationships hold:

$$\kappa_{ij} = \omega_{i,j} = -\frac{1}{2} e_{ikl}\omega_{kl,j} = -e_{ikl}\epsilon_{jk,l}. \tag{4}$$

Since ϵ_{ik} is symmetric and e_{ikl} skew-symmetric, the curvature tensor is deviatoric, $\kappa_{ii} = 0$. Thus, $\mu_{ji}\kappa_{ij} = m_{ji}\kappa_{ij}$, where $m_{ij} = \mu_{ij} - \frac{1}{3}\mu_{kk}\delta_{ij}$ is a deviatoric part of the couple stress μ_{ij}. A spherical part of the couple stress does not appear in any of the basic field equations of couple stress theory, and without any physical loss of generality it may be assumed to vanish. Consequently, from the moment equilibrium equation (2)

$$\tau_{ij} = -\frac{1}{2} e_{ijk}m_{lk,l}, \quad \tau_{ij}\omega_{ji} = -m_{ij,i}\omega_j. \tag{5}$$

The Saint-Venant compatibility equations and the three Bianchi conditions, written in terms of the curvature tensor, are:

$$e_{ikl}\kappa_{jk,l} = 0, \quad e_{ikl}\kappa_{jk,lj} = 0. \tag{6}$$

The elastic work done by the applied surface loading and the body forces is

$$\frac{1}{2}\int_S (T_i u_i + M_i\omega_i)\,\mathrm{d}S + \frac{1}{2}\int_V f_i u_i\,\mathrm{d}V = \int_V W\,\mathrm{d}V, \tag{7}$$

where W is the specific strain energy,

$$W = \frac{1}{2}\sigma_{ij}\epsilon_{ij} + \frac{1}{2} m_{ji}\kappa_{ij}. \tag{8}$$

For infinitesimal elasticity, W is a quadratic function of the strain and the curvature components,

$$W = \frac{1}{2} C_{ijkl}\epsilon_{ij}\epsilon_{kl} + \frac{1}{2}\mathcal{M}_{ijkl}\kappa_{ij}\kappa_{kl}. \tag{9}$$

Thus, the linear constitutive equations, $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ and $m_{ji} = \mathcal{M}_{ijkl}\kappa_{kl}$. If the material is isotropic, the stiffness tensors are the isotropic fourth-order tensors:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{10}$$

$$\mathcal{M}_{ijkl} = 2\mu l^2(\delta_{ik}\delta_{jl} + \eta\delta_{il}\delta_{jk}), \tag{11}$$

where λ and μ are the Lamé constants of classical elasticity, l is a material length parameter (characteristic material length scale), and η is a non-dimensional parameter of a similar character as Poisson's ratio ν. For positive definiteness of W, $-1 < \eta < 1$. The constitutive relations of linear isotropic couple stress elasticity are therefore:

$$\sigma_{ij} = 2\mu\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right), \tag{12}$$

$$m_{ji} = 2\mu l^2(\kappa_{ij} + \eta\kappa_{ji}). \tag{13}$$

3 RECIPROCAL PROPERTIES

Consider two equilibrium elastic fields, designated by the superscripts A and B. Clearly:

$$\sigma_{ji}^A\epsilon_{ij}^B = \sigma_{ji}^B\epsilon_{ij}^A, \quad m_{ji}^A\kappa_{ij}^B = m_{ji}^B\kappa_{ij}^A, \tag{14}$$

and

$$(p_{ji}^A - \tau_{ji}^A)u_{i,j}^B = (p_{ji}^B - \tau_{ji}^B)u_{i,j}^A. \tag{15}$$

The equilibrium equations (2) provide the relationships:

$$p_{ji}^A u_{i,j}^B - p_{ji}^B u_{i,j}^A = (p_{ji}^A u_i^B - p_{ji}^B u_i^A)_{,j} + f_i^A u_i^B - f_i^B u_i^A, \tag{16}$$

$$\tau_{ji}^A u_{i,j}^B - \tau_{ji}^B u_{i,j}^A = -(m_{ji}^A\omega_i^B - m_{ji}^B\omega_i^A)_{,j} \tag{17}$$

so that the substitution into Eq. (15) gives

$$(p_{ji}^A u_i^B + m_{ji}^A\omega_i^B - p_{ji}^B u_i^A - m_{ji}^B\omega_i^A)_{,j} = f_i^B u_i^A - f_i^A u_i^B. \tag{18}$$

Integrating over the volume of the body and using the Gauss divergence theorem, we thus obtain

$$\int_S (p_{ji}^A u_i^B + m_{ji}^A\omega_i^B)n_j\,\mathrm{d}S + \int_V f_i^A u_i^B\,\mathrm{d}V = \int_S (p_{ji}^B u_i^A + m_{ji}^B\omega_i^A)\,n_j\,\mathrm{d}S + \int_S f_i^B u_i^A\,\mathrm{d}V, \tag{19}$$

which is Betti's reciprocal theorem of couple stress elasticity.

If the absence of body forces, from Eq. (18) it follows that the vector

$$v_j(A,B) = p_{ji}^A u_i^B + m_{ji}^A \omega_i^B - p_{ji}^B u_i^A - m_{ji}^B \omega_i^A \quad (20)$$

has zero divergence. Thus,

$$\int_{S_1} v_j(A,B) n_j dS = \int_{S_2} v_j(A,B) n_j \, \mathrm{d}S, \quad (21)$$

for any two surfaces S_1 and S_2 which do not embrace any singularity of $v_j(A,B)$. In particular, if there are no singularities of $v_j(A,B)$ within a surface S,

$$\int_S v_j(A,B) n_j \, \mathrm{d}S = 0. \quad (22)$$

If the material is homogeneous ($C_{ijkl,\alpha} = 0$, $\mathcal{M}_{ijkl,\alpha} = 0$), in addition to Eq. (14) we have

$$\sigma_{ji,k}^A \epsilon_{ij}^B = \sigma_{ji}^B \epsilon_{ij,k}^A, \quad m_{ji,k}^A \kappa_{ij}^B = m_{ji}^B \kappa_{ij,k}^A. \quad (23)$$

Thus, for a homogeneous material, Eq. (21) also applies when $v_j(A,B)$ is replaced by

$$\begin{aligned} \vartheta_{jk}(A,B) = p_{ji,k}^A u_i^B + m_{ji,k}^A \omega_i^B \\ - p_{ji}^B u_{i,k}^A - m_{ji}^B \omega_{i,k}^A. \end{aligned} \quad (24)$$

4 ENERGY DUE TO INTERNAL STRESS SOURCES

Suppose that there are two systems of internal stresses in the body of volume V bounded by the surface S. The sources of internal stress system A lie entirely within the surface S_0, and those of the system B lie entirely outside of S_0. If E_A and E_B are the values of the elastic strain energy when A and B alone exists in the body, we may write the total strain energy as $E = E_A + E_B + E_{AB}$, when the sources coexist in the body. Here,

$$\begin{aligned} E_{AB} = \frac{1}{2} \int_V \big(\sigma_{ji}^A \epsilon_{ij}^B + m_{ji}^A \kappa_{ij}^B \\ + \sigma_{ji}^B \epsilon_{ij}^A + m_{ji}^B \kappa_{ij}^A \big) \, \mathrm{d}V \end{aligned} \quad (25)$$

is the interaction energy between A and B. Noting that $\epsilon_{ij}^B = \frac{1}{2}(u_{i,j}^B + u_{j,i}^B)$ and $\kappa_{ij}^B = \omega_{i,j}^B$ in the volume V_0 within the surface S_0, and $\epsilon_{ij}^A = \frac{1}{2}(u_{i,j}^A + u_{j,i}^A)$ and $\kappa_{ij}^A = \omega_{i,j}^A$ in the volume $V - V_0$ (but not conversely), and in view of the reciprocity properties from Section 3, Eq. (25) can be rewritten as

$$\begin{aligned} E_{AB} = \int_{V_0} \left(\sigma_{ji}^A \epsilon_{ij}^B + m_{ji}^A \kappa_{ij}^B \right) \mathrm{d}V \\ + \int_{V - V_0} \left(\sigma_{ji}^B \epsilon_{ij}^A + m_{ji}^B \kappa_{ij}^A \right) \mathrm{d}V. \end{aligned} \quad (26)$$

Furthermore, $\sigma_{ji}^A \epsilon_{ij}^B = \sigma_{ji}^A u_{i,j}^B = (p_{ji}^A - \tau_{ji}^A) u_{i,j}^B$, and since from equilibrium equations without body forces:

$$p_{ji}^A u_{i,j}^B = (p_{ji}^A u_i^B)_{,j}, \quad \tau_{ji}^A u_{i,j}^B = -m_{ji,j}^A \omega_i^B, \quad (27)$$

we obtain

$$\sigma_{ji}^A \epsilon_{ij}^B + m_{ji}^A \kappa_{ij}^B = (p_{ji}^A u_i^B + m_{ji}^A \omega_i^B)_{,j}. \quad (28)$$

An analogous equation holds when the superscripts A and B are interchanged in Eq. (28). Substituting these two equations into Eq. (26) and applying the Gauss divergence theorem then gives

$$\begin{aligned} E_{AB} = \int_{S_0} \big(p_{ji}^A u_i^B + m_{ji}^A \omega_i^B \\ - p_{ji}^B u_i^A - m_{ji}^B \omega_i^A \big) n_j \, \mathrm{d}S, \end{aligned} \quad (29)$$

where n_j is the outward normal to S_0. The surface integral over S vanishes, because no load is there applied. Thus, the interaction energy between two internal stress systems can be expressed by the integral over any surface S_0 that separates the two stress sources. Note that the bracketed terms of the integrand in Eq. (29) are equal to $v_j(A,B)$, defined by Eq. (20).

5 ENERGY DUE TO INTERNAL AND EXTERNAL STRESS SOURCES

Denote by A the stress system due to internal sources within the volume V, and by B the stress system due to external surface load T_i^B and M_i^B applied over S. The total potential energy of the system is $\Pi = \Pi_A + \Pi_B + \Pi_{AB}$, where:

$$\Pi_A = E_A = \frac{1}{2} \int_V (\sigma_{ji}^A \epsilon_{ij}^A + m_{ji}^A \kappa_{ij}^A) \, \mathrm{d}V, \quad (30)$$

$$\Pi_B = E_B - \int_S (T_i^B u_i^B + M_i^B \omega_i^B) \, \mathrm{d}S, \quad (31)$$

represent the potential energies of two systems when they act alone, and

$$\Pi_{AB} = E_{AB} - \int_S (T_i^B u_i^A + M_i^B \omega_i^A) \, \mathrm{d}S \quad (32)$$

is the interaction potential energy between the two systems. The interaction strain energy E_{AB} between stress systems due to internal and external stress sources is equal to zero. Indeed, in this case u_i^B and ω_i^B exist throughout the volume V, and from Eq. (25) and the reciprocity property we have

$$\begin{aligned} E_{AB} &= \int_V \left(\sigma_{ji}^A \epsilon_{ij}^B + m_{ji}^A \kappa_{ij}^B \right) \mathrm{d}V \\ &= \int_V (p_{ji}^A u_i^B + m_{ji}^A \omega_i^B)_{,j} \, \mathrm{d}V. \end{aligned} \quad (33)$$

Thus, upon application of the Gauss theorem, we obtain $E_{AB} = 0$, since $T_i^A = 0$ and $M_i^A = 0$ on S. The response of the body to external loading in couple stress theory is, therefore, the same whether the body is self-stressed or not, as in classical elasticity.

The interaction potential energy between external and internal stress systems is consequently

$$\Pi_{AB} = -\int_S (p_{ji}^B u_i^A + m_{ji}^B \omega_i^A) n_j \, dS. \tag{34}$$

The right-hand side of Eq. (34) can be rewritten in the same form as the right-hand side of Eq. (29), provided that S_0 is taken to be any surface that entirely encompasses the sources of internal stress A. This follows from the Gauss divergence theorem applied to the region $V - V_0$, noting that $T_i^A = M_i^A = 0$ on S, and that $v_j(A, B)$ is divergence free in the region between S and S_0.

In fact, if $(u_i^C, \omega_i^C, p_{ij}^C, m_{ij}^C)$ is any elastic field free of singularities within S_0 (thus $v_j(B, C)$ is divergence free in S_0), we can write

$$\begin{aligned}\Pi_{AB} = \int_{S_0} \big[&(p_{ji}^A - p_{ji}^C) u_i^B + (m_{ji}^A - m_{ji}^C)\omega_i^B \\ &- p_{ji}^B (u_i^A - u_i^C) - m_{ji}^B(\omega_i^A - \omega_i^C)\big] n_j \, dS,\end{aligned} \tag{35}$$

Since the field C is arbitrary field without singularities within S_0, $(A - C)$ field in Eq. (35) can be any field which has the same singularities within S_0 as does the field A. In applications it is convenient to take $(A - C)$ to be the elastic field of internal sources A, considered to be emerged in an infinite body. Thus, the interaction potential energy between the internal system A and external system B can be written as

$$\begin{aligned}\Pi_{AB} = \int_{S_0} \big(&p_{ji}^{\infty} u_i^B + m_{ji}^{\infty}\omega_i^B \\ &- p_{ji}^B u_i^{\infty} - m_{ji}^B \omega_i^{\infty}\big) n_j \, dS.\end{aligned} \tag{36}$$

This is a simple generalization of the corresponding Eshelby's result from classical elasticity.

6 THE FORCE ON AN ELASTIC SINGULARITY

The force on a singularity A due to external system B can be defined as the negative gradient of interaction energy with respect to the location of the singularity, i.e.,

$$J_k = -\frac{\partial \Pi_{AB}}{\partial x_k^A}. \tag{37}$$

To elaborate on this expression, it is convenient to use Eq. (36) for the interaction energy Π_{AB}. The stress field in an infinite body due to singularity at A evidently satisfies the property

$$p_{ij}^{\infty}(x_k, x_k^A + dx_k^A) = p_{ij}^{\infty}(x_k - dx_k^A, x_k^A), \tag{38}$$

since moving the singularity for dx_k^A toward the point of observation x_k, or approaching the point of observation toward the singularity by dx_k^A, equally effects the stress at the point of observation. Thus,

$$\frac{\partial p_{ij}^{\infty}}{\partial x_k^A} = -\frac{\partial p_{ij}^{\infty}}{\partial x_k}. \tag{39}$$

Analogous expressions apply to couple stress, displacement and rotation gradients. Substitution into Eq. (37), therefore, gives

$$\begin{aligned}J_k = \int_{S_0} \big(&p_{ji,k}^{\infty} u_i^B + m_{ji,k}^{\infty}\omega_i^B \\ &- p_{ji}^B u_{i,k}^{\infty} - m_{ji}^B \omega_{i,k}^{\infty}\big) n_j \, dS.\end{aligned} \tag{40}$$

Since $p_{ij}^A = p_{ij}^{\infty} + p_{ij}^I$ (and similarly for other fields), where the superscript I denotes the image field, free of singularities within S_0, Eq. (40) can be rewritten as

$$\begin{aligned}J_k = \int_{S_0} \big(&p_{ji,k}^{A} u_i^B + m_{ji,k}^{A}\omega_i^B \\ &- p_{ji}^B u_{i,k}^{A} - m_{ji}^B \omega_{i,k}^{A}\big) n_j \, dS.\end{aligned} \tag{41}$$

The terms within the brackets in the integrand are equal to $\vartheta_{jk}(A, B)$, defined by Eq. (24). Recall that $\vartheta_{jk}(I, B)$ is divergence free within S_0.

The right-hand side of Eq. (41) is symmetric with respect to superscripts A and B. Thus, the force on a singularity can also be expressed as

$$\begin{aligned}J_k = \int_{S_0} \big(&p_{ji,k}^{B} u_i^A + m_{ji,k}^{B}\omega_i^A \\ &- p_{ji}^A u_{i,k}^{B} - m_{ji}^A \omega_{i,k}^{B}\big) n_j \, dS.\end{aligned} \tag{42}$$

To see that the right-hand sides of Eqs. (41) and (42) are equal, we form their difference, which is

$$\begin{aligned}&\int_{S_0} \big[\vartheta_{jk}(A, B) - \vartheta_{jk}(B, A)\big] n_j \, dS \\ &= \int_{S_0} v_{j,k}(A, B) n_j \, dS = \int_{S_0} v_{j,j}(A, B) n_k \, dS.\end{aligned} \tag{43}$$

The last integral is equal to zero, because from Eq. (20) we obtain $v_{j,j}(A, B) = 0$, in view of the reciprocity properties and Eq. (27). In Eq. (43) we also assumed that u_i^A is single-valued on S_0.

7 THE ENERGY-MOMENTUM TENSOR OF A COUPLE STRESS FIELD

As in classical elasticity, it is possible to develop a general expression for the force on an elastic singularity, which embraces the foregoing results. To that goal, consider a body of volume V, loaded by the force traction T_k and couple traction M_k over its external surface S. The body contains a singularity which is a source of internal stress (e.g., a dislocation, an interstitial atom, or other lattice defect). The total potential energy is

$$\Pi = \int_V W \,\mathrm{d}V - \int_S \left(T_j u_j + M_j \omega_j\right) \mathrm{d}S, \tag{44}$$

where W is the elastic strain energy density, defined in linear couple stress theory by Eq. (9). If the singularity is moved a small distance ϵ in the direction n_k, the potential energy alters due to the change of the elastic strain energy and the load potential. The change of the elastic strain energy is

$$\delta_k \int_V W \,\mathrm{d}V = -\epsilon \int_S W n_k \,\mathrm{d}S + \int_S \left[T_j\left(\delta_k u_j + \epsilon u_{j,k}\right) + M_j\left(\delta_k \omega_j + \epsilon \omega_{j,k}\right)\right] \mathrm{d}S, \tag{45}$$

to within linear terms in ϵ. This can be established by the same consideration as in the original Eshelby's derivation of classical elasticity. The symbol δ_k indicates a variation associated with an infinitesimal displacement of the singularity in the direction n_k. The change of the load potential due to displacement of the singularity is

$$\delta_k \int_S \left(T_j u_j + M_j \omega_j\right) \mathrm{d}S = \int_S \left(T_j \delta_k u_j + M_j \delta_k \omega_j\right) \mathrm{d}S + O(\epsilon^2). \tag{46}$$

Thus, the total change of the potential energy becomes

$$\delta_k \Pi = -\epsilon \int_S \left(W n_k - T_j u_{j,k} - M_j \omega_{j,k}\right) \mathrm{d}S + O(\epsilon^2). \tag{47}$$

Since the force on the singularity can be defined as

$$J_k = -\lim_{\epsilon \to 0} \frac{\delta_k \Pi}{\epsilon}, \tag{48}$$

we obtain from Eq. (47)

$$J_k = \int_S \left(W n_k - T_j u_{j,k} - M_j \omega_{j,k}\right) \mathrm{d}S. \tag{49}$$

This can be rewritten as

$$J_k = \int_S P_{ik} n_i \,\mathrm{d}S, \tag{50}$$

where

$$P_{ik} = W \delta_{ik} - p_{ij} u_{j,k} - m_{ij} \kappa_{jk} \tag{51}$$

is the energy-momentum tensor of the couple stress elastic field. The generalization of the energy-momentum tensor for grade 2 and higher grade materials was discussed by Eshelby (1975). Xia & Hutchinson (1996) used the J integral representation of couple stress elasticity to study the crack tip fields in strain gradient plasticity. Previously, Jaric (1986) and Dai (1986) studied a path-independent J integral of micropolar elastic continua.

Pursuing Eshelby's (1956) analysis further, assume that in addition to a considered singularity A and external load B, there are other sources of internal stress, collectively denoted by C. Each of the elastic fields in the body, such as the displacement field u_i, can be represented as

$$u_i = u_i^{\infty} + u_i^{I} + u_i^{B} + u_i^{C}, \tag{52}$$

where the superscript I designates the image field of the singularity A, considered to be in an infinite medium. The total force on the singularity A is then

$$J_k = J_k^I + J_k^B + J_k^C, \tag{53}$$

where

$$J_k^X = \int_S \left(p_{ji,k}^{\infty} u_i^X + m_{ji,k}^{\infty} \omega_i^X - p_{ji}^X u_{i,k}^{\infty} - m_{ji}^X \kappa_{ik}^{\infty}\right) n_j \,\mathrm{d}S, \tag{54}$$

for each of the superscripts $X = I, B, C$. Indeed, a typical cross term in Eq. (49) is

$$(X,Y) = \int_S \left[\frac{1}{2}\left(\sigma_{ik}^X \epsilon_{kl}^Y + m_{ik}^X \kappa_{kl}^Y\right) n_l - \left(p_{jk}^X u_{k,i}^Y + m_{jk}^X \kappa_{ki}^Y\right) n_j\right] \mathrm{d}S. \tag{55}$$

This can be rewritten by using Eq. (28) as

$$(X,Y) = \int_S \left[\frac{1}{2}\left(p_{ik}^X u_k^Y + m_{ik}^X \omega_k^Y\right)_{,i} n_l - \left(p_{jk}^X u_{k,i}^Y + m_{jk}^X \kappa_{ki}^Y\right) n_j\right] \mathrm{d}S. \tag{56}$$

The property was employed that the surface integral of $\Phi_{,l} n_i$ is equal to that of $\Phi_{,i} n_l$ for any tensor field Φ defined within S. Thus, upon differentiation, it follows that

$$(X,Y) = \frac{1}{2} \int_S \vartheta_{ji}(X,Y) n_j \,\mathrm{d}S. \tag{57}$$

This is equal to zero because $\vartheta_{ji}(X,Y)$ is a divergence free field, unless one or other of the labels X and Y stands for ∞. The term (∞,∞) also vanishes because the products of the elastic fields in an infinite medium rapidly approach zero at large distances from the singularity.

8 CONSERVATION LAWS IN COUPLE STRESS ELASTICITY

Returning to Eq. (49), if S does not embrace a singularity, a conservation law $J_k = 0$ of couple stress elasticity is obtained. There is also a conservation law $L_k = 0$, where

$$L_k = \int_S \epsilon_{kij}\big(Wx_j n_i + T_i u_j + M_i\omega_j - T_l u_{l,i} x_j - M_l \omega_{l,i} x_j\big)\,\mathrm{d}S, \tag{58}$$

i.e., written more compactly by using the energy-momentum tensor of Eq. (51),

$$L_k = \int_S \epsilon_{kij}\big(P_{li}x_j + p_{li}u_j + m_{li}\omega_j\big)n_l\,\mathrm{d}S. \tag{59}$$

The two laws can be derived from a Noether's type theorem, and the invariance of the total strain energy with respect to translational and rotational type transformations (Lubarda & Markenscoff 1998). However, in contrast to classical elasticity, the strain energy in couple stress elasticity is not infinitesimally invariant under a self-similar scale change, because the material length parameter l in the expression for the strain energy remains unaltered under this transformation. Consequently, there is no M conservation law in couple stress elasticity. It should also be recalled from Knowles & Sternberg (1972) that the conservation integrals J_k and L_k of infinitesimal elasticity have their counterparts in the finite deformation elasticity, but the M integral does not.

9 CONCLUSIONS

Expressions for interaction energies between the stress systems due to internal and external sources of stress are derived in couple stress elasticity by extending the classical elasticity results of Eshelby (1956). The force on a singularity and the energy-momentum tensor of elastic couple stress field are obtained. Two conservation integrals of couple stress elasticity give rise to conservation laws $J_k = 0$ and $L_k = 0$. There is no M conservation integral in couple stress elasticity.

Acknowledgments - V.A.L. is grateful for the research support provided by the Alcoa Center, and to Dr. Owen Richmond for discussions and stimulated interest for non-local theories.

REFERENCES

Dai, T-A. 1986. Some path independent integrals for micropolar media. *Int. J. Solids Struc.* 22: 729-735.

Eshelby, J.D. 1956. The continuum theory of lattice defects. *Solid State Phys.* 3: 79-144.

Eshelby, J.D. 1975. The elastic energy-momentum tensor. *J. of Elasticity* 5: 321-335.

Fleck, N.A. & J.W. Hutchinson 1997. Strain gradient plasticity. *Advances in Appl. Mech.* 33: 295-361.

Jaric, J. P. 1986. The energy release rate in quasistatic crack propagation and J-integral. *Int. J. Solids Struct.* 22: 767-778.

Knowles, J.K. & E. Sternberg 1972. On a class of conservation laws in linearized and finite elastostatics. *Arch. Ration. Mech. Anal.* 44: 187-211.

Koiter, W.T. 1964. Couple stresses in the theory of elasticity, I and II. *Proc. Ned. Akad. Wet.* (B) 67(1): 17-44.

Lubarda, V.A. & X. Markenscoff 1998. Conservation integrals in couple stress elasticity. To be published.

Mindlin, R.D. 1963. Influence of couple-stresses on stress concentrations. *Exp. Mech.* 3: 1-7.

Mindlin, R.D. 1964. Micro-structure in linear elasticity. *Arch. Ration. Mech. Anal.* 16: 51-78.

Xia, Z.C. & J.W. Hutchinson 1996. Crack tip fields in strain gradient plasticity. *J. Mech. Phys. Solids* 44: 1621-1648.