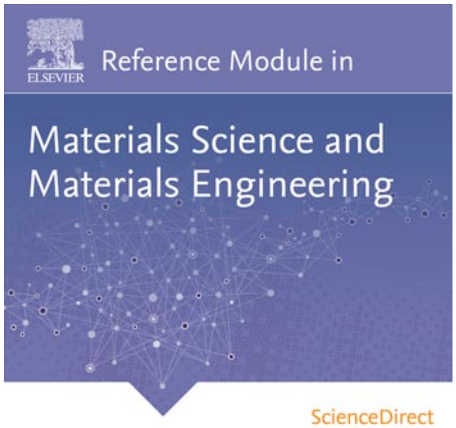

Lubarda V.A., Mechanics of Materials: Plasticity. In: Saleem Hashmi (editor-in-chief), Reference Module in Materials Science and Materials Engineering. Oxford: Elsevier; 2016. pp. 1-24.

ISBN: 978-0-12-803581-8

Mechanics of Materials: Plasticity ☆

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1 Yield Surface

Materials capable of plastic deformation usually have an elastic range of purely elastic response. This range is a closed domain in either stress or strain space whose boundary is called the yield surface. The shape of the yield surface depends on the entire deformation path from the reference state. The yield surfaces for actual materials are mainly smooth, but may have or develop pointed pyramidal or conical vertices. Physical theories of plasticity Hill (1967) imply the formation of a corner or vertex at the

☆*Change History*: March 2015. V.A. Lubarda has changed the title to Mechanics of Materials: Plasticity. Section 11: 'Strain Gradient Plasticity' has been added, The Section References has been enriched by additionally cited references in Section 11, and 3 more references in the rest of the manuscript.

 doi:10.1016/B978-0-12-803581-8.02841-1

loading point on the yield surface. Experimental evidence, on the other hand, suggests that, while relatively high curvature at the loading point is often observed, sharp corners are seldom seen Hecker (1976). Experiments also indicate that yield surfaces for metals are convex in Cauchy stress space, if elastic response within the yield surface is linear and unaffected by plastic flow.

1.1 Yield Surface in Strain Space

The yield surface in strain space is defined by $g(\mathrm{E},\mathcal{H})=0$, where E is the strain tensor, and $\mathcal{H}$ the pattern of internal rearrangements due to plastic flow, i.e., the set of appropriate internal variables including the path history by which they are achieved Rice (1971). The shape of the yield surface specified by function g is different for different choices of E. If elastic response within the yield surface is Green-elastic, associated with the strain energy $\psi=\psi(\mathbf{E},\mathcal{H})$ per unit reference volume, the corresponding stress is $\mathbf{T}=\partial\psi/\partial\mathbf{E}$. From the strain state on the current yield surface, an increment of strain dE directed inside the yield surface constitutes an elastic unloading. The associated incremental elastic response is governed by

$$\dot{\mathbf{T}}=\boldsymbol{\Lambda}:\dot{\mathbf{E}},\quad \Lambda=\frac{\partial^2\psi}{\partial\mathbf{E}\otimes\partial\mathbf{E}} \qquad [1]$$

where $\Lambda=\Lambda(\mathbf{E},\mathcal{H})$ is a tensor of elastic moduli of the material at the considered state of strain and internal structure. An increment of strain directed outside the current yield surface constitutes plastic loading. The resulting increment of stress consists of elastic and plastic parts, such that

$$\dot{\mathbf{T}}=\dot{\mathbf{T}}^{\mathrm{e}}+\dot{\mathbf{T}}^{\mathrm{p}}=\boldsymbol{\Lambda}:\dot{\mathbf{E}}+\dot{\mathbf{T}}^{\mathrm{p}} \qquad [2]$$

The elastic part of the stress rate $\dot{\mathbf{T}}^{\mathrm{e}}$ gives a stress decrement $\mathrm{d}^{\mathrm{e}}\mathbf{T}$ associated with elastic removal of the strain increment d**E**. The plastic part of the stress rate $\dot{\mathbf{T}}^{\mathrm{p}}$ gives a residual stress decrement $\mathrm{d}^{\mathrm{p}}\mathbf{T}$ in a considered infinitesimal strain cycle. A transition between elastic unloading and plastic loading is a neutral loading, in which an infinitesimal strain increment is tangential to the yield surface and represents pure elastic deformation. Thus,

$$\frac{\partial g}{\partial\mathbf{E}}:\dot{\mathbf{E}}\begin{cases} >0, & \text{for plastic loading}\\ =0, & \text{for neutral loading}\\ <0, & \text{for elasticun loading}\end{cases} \qquad [3]$$

The gradient $\partial g/\partial\mathbf{E}$ is codirectional with the outward normal to a locally smooth yield surface $g=0$ at the state of strain **E**. For incrementally linear response, all infinitesimal increments d**E** with equal projections on the normal $\partial g/\partial\mathbf{E}$, produce equal plastic increments of stress $\mathrm{d}^{\mathrm{p}}\mathbf{T}$, since the components of dE obtained by projection on the plane tangential to the yield surface represent elastic deformation only.

1.2 Yield Surface in Stress Space

The yield surface in stress space is defined by $f(\mathrm{T},\mathcal{H})=0$. The stress T is a work conjugate to strain **E**, in the sense that $\mathbf{T}:\dot{\mathbf{E}}$ represents the rate of work per unit initial volume Hill (1978). The function f is related to g by

$$f[\mathbf{T}(\mathbf{E},\mathcal{H}),\mathcal{H}]=g(\mathbf{E},\mathcal{H})=0 \qquad [4]$$

provided that physically identical conditions of yield are imposed in both spaces. If elastic response within the yield surface is Green-elastic, associated with the complementary strain energy $\phi=\phi(\mathbf{T},\mathcal{H})$ per unit reference volume, the corresponding strain is $\mathbf{E}=\partial\phi/\partial\mathbf{T}$. For material in the hardening range relative to measures **E** and **T**, an increment of stress d**T** from the stress state on the yield surface, directed inside the yield surface constitutes an elastic unloading. The associated incremental elastic response is governed by

$$\dot{\mathbf{E}}=\mathbf{M}:\dot{\mathbf{T}},\quad \mathbf{M}=\frac{\partial^2\phi}{\partial\mathbf{T}\otimes\partial\mathbf{T}} \qquad [5]$$

The tensor $\mathbf{M}=\mathbf{M}(\mathbf{T},\mathcal{H})$ is a tensor of elastic compliances of the material at the considered state of stress and internal structure. An increment of stress directed outside the current yield surface constitutes plastic loading in the hardening range of material response. The resulting increment of stain consists of elastic and plastic parts, such that

$$\dot{\mathbf{E}}=\dot{\mathbf{E}}^{\mathrm{e}}+\dot{\mathbf{E}}^{\mathrm{p}}=\mathbf{M}:\dot{\mathbf{T}}+\dot{\mathbf{E}}^{\mathrm{p}} \qquad [6]$$

During plastic loading of hardening material, the yield surface locally expands, the stress state remaining on the yield surface. The elastic part of the strain rate $\dot{\mathbf{E}}^{\mathrm{e}}$ gives an elastic increment of strain $\mathrm{d}^{\mathrm{e}}\mathbf{E}$ which is recovered upon elastic unloading of the stress increment d**T**. The plastic part of the strain rate $\dot{\mathbf{E}}^e$ gives a residual increment of strain $d^p\mathbf{E}$ which is left upon removal of the stress increment d**T**. A transition between elastic unloading and plastic loading is a neutral loading, in which an infinitesimal stress increment is tangential to the yield surface and produces only elastic deformation. Thus, in the hardening range

$$\frac{\partial f}{\partial \mathbf{T}} : \dot{\mathbf{T}} \begin{cases} > 0, & \text{for plastic loading} \\ = 0, & \text{for neutral loading} \\ < 0, & \text{for elastic unloading} \end{cases} \qquad [7]$$

The gradient $\partial f/\partial \mathbf{T}$ is codirectional with the outward normal to a locally smooth yield surface $f=0$ at the state of stress **T**. For incrementally linear response, all infinitesimal increments d**T** with equal projections on $\partial f/\partial \mathbf{T}$ produce equal plastic increments of deformation $d^p\mathbf{E}$, since the components of d**T** obtained by projection on the plane tangential to the yield surface give rise to elastic deformation only.

In a softening range of material response eqn [6] still holds, although the elastic and plastic parts of the strain rate have purely formal significance, since in the softening range it is not physically possible to perform an infinitesimal cycle of stress starting from a stress point on the yield surface. The hardening is, however, a relative term: material that is in the hardening range relative to one pair of stress and strain measures, may be in the softening range relative to another pair.

2 Plasticity Postulates, Normality and Convexity of the Yield Surface

Several postulates in the form of constitutive inequalities have been proposed for certain types of materials undergoing plastic deformation. The two most well-known are by Drucker (1960) and Ilyushin (1961).

2.1 Ilyushin's Postulate

According to Ilyushin's postulate the network in an isothermal cycle of strain must be positive

$$\oint_E \mathbf{T} : d\mathrm{E} > 0 \qquad [8]$$

if a cycle at some stage involves plastic deformation. The integral in [8] over an elastic strain cycle is equal to zero. Since a cycle of strain that includes plastic deformation in general does not return the material to its state at the beginning of the cycle, the inequality [8] is not a law of thermodynamics. For example, it does not apply to materials which dissipate energy by friction. For materials obeying Ilyushin's postulate it can be shown that (Hill and Rice, 1973; Havner, 1992)

$$d^p\mathbf{T} : d\mathbf{E} < 0 \qquad [9]$$

Since during plastic loading the strain increment d**E** is directed outward from the yield surface, and since the same $d^p\mathbf{T}$ is associated with a fan of infinitely many d**E** around the normal $\partial g/\partial \mathbf{E}$, all having the same projection on that normal, the inequality [9] requires that $d^p\mathbf{T}$ is codirectional with the inward normal to a locally smooth yield surface in strain space,

$$d^p\mathrm{T} = -d\gamma \frac{\partial g}{\partial \mathrm{E}} \qquad [10]$$

The scalar multiplier $d\gamma > 0$ is called a loading index. At a vertex of the yield surface, $d^p\mathbf{T}$ must lie within the cone of limiting inward normals.

The inequality [9] and the normality rule [10] hold for all pairs of conjugate stress and strain measures, irrespective of the nature of elastic changes caused by plastic deformation, or possible elastic nonlinearities within the yield surface. Also, [10] applies regardless of whether the material is in a hardening or softening range.

If elastic response within the yield surface is nonlinear, Ilyushin's postulate does not imply that the yield surface is necessarily convex. For a linearly elastic response, however, it follows that

$$(\mathbf{E}^0 - \mathbf{E}) : d^p\mathbf{T} > 0 \qquad [11]$$

provided that there is no change of elastic stiffness caused by plastic deformation ($d\mathbf{\Lambda}=0$), or the change is such that $d\mathbf{\Lambda}$ is negative semi-definite. The strain $\mathbf{E}^0$ is an arbitrary strain state within the yield surface. Since $d^p\mathbf{T}$ is codirectional with the inward normal to a locally smooth yield surface in strain space, [11] implies that the yield surface is convex. The convexity of the yield surface is not an invariant property, because $d\mathbf{\Lambda}$ can be negative definite for some measures (**E**, **T**), but not for others.

Plastic stress and strain rates are related by $\dot{\mathbf{T}}^{\mathrm{p}} = -\boldsymbol{\Lambda} : \dot{\mathbf{E}}^{\mathrm{p}}$, so that, to first order,

$$\mathrm{d}^{\mathrm{p}}\mathbf{T} = -\boldsymbol{\Lambda} : \mathrm{d}^{\mathrm{p}}\mathbf{E} \qquad [12]$$

Since for any elastic strain increment $\delta\mathbf{E}$, emanating from a point on the yield surface in strain space and directed inside of it,

$$\mathrm{d}^{\mathrm{p}}\mathbf{T} : \delta\mathbf{E} > 0 \qquad [13]$$

substitution of [12] into [13] gives

$$\mathrm{d}^{\mathrm{p}}\mathbf{E} : \delta\mathbf{T} < 0 \qquad [14]$$

Here, $\delta\mathbf{T} = \boldsymbol{\Lambda} : \delta\mathbf{E}$ is the stress increment from the point on the yield surface in stress space, directed inside of it (elastic unloading increment associated with elastic strain increment $\delta\mathbf{E}$). Inequality [14] holds for any $\delta\mathbf{T}$ directed inside the yield surface. Consequently, $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ must be codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{T}$ space,

$$\mathrm{d}^{\mathrm{p}}\mathbf{E} = \mathrm{d}\gamma \frac{\partial f}{\partial \mathbf{T}}, \quad \mathrm{d}\gamma > 0 \qquad [15]$$

At a vertex of the yield surface, $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ must lie within the cone of limiting outward normals. Inequality [14] and the normality rule [15] hold for all pairs of conjugate stress and strain measures.

If material is in a hardening range relative to $\mathbf{E}$ and $\mathbf{T}$, the stress increment $\mathrm{d}\mathbf{T}$ producing plastic deformation $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ is directed outside the yield surface, satisfying

$$\mathrm{d}^{\mathrm{p}}\mathbf{E} : \mathrm{d}\mathbf{T} > 0 \qquad [16]$$

If material is in the softening range, the stress increment $\mathrm{d}\mathbf{T}$ producing plastic deformation $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ is directed inside the yield surface, satisfying the reversed inequality in [16]. The normality rule [15] applies to both hardening and softening. Inequality [16] is not measure invariant, since material may be in the hardening range relative to one pair of conjugate stress and strain measures, and in the softening range relative to another pair.

The normals to the yield surfaces in stress and strain space are related by

$$\frac{\partial g}{\partial \mathbf{E}} = \boldsymbol{\Lambda} : \frac{\partial f}{\partial \mathbf{T}} \qquad [17]$$

This follows directly from eqn [4] by partial differentiation.

2.2 Drucker's Postulate

A non-invariant dual to [8] is

$$\oint_T \mathbf{E} : \mathrm{d}\mathbf{T} < 0 \qquad [18]$$

requiring that the net complementary work (relative to measures $\mathbf{E}$ and $\mathbf{T}$) in an isothermal cycle of stress must be negative, if the cycle at some stage involves plastic deformation. Inequality [18] is non-invariant because the value of the integral in [18] depends on the selected measures $\mathbf{E}$ and $\mathbf{T}$, and the reference state with respect to which they are defined. This is because $\mathbf{T}$ is introduced as a conjugate stress to $\mathbf{E}$ such that, for the same geometry change, $\mathbf{T}$:d$\mathbf{E}$ (and not $\mathbf{E}$:d$\mathbf{T}$) is measure invariant. If inequality [18] applies to conjugate pair ($\mathbf{E}$, $\mathbf{T}$), it follows that in the hardening range [16] holds, and $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ is codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{T}$ space, eqn [15]. At a vertex of the yield surface, $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ must lie within the cone of limiting outward normals. In the softening range,

$$\mathrm{d}^{\mathrm{p}}\mathbf{E} : \mathrm{d}\mathbf{T} < 0 \qquad [19]$$

Since $\mathrm{d}\mathbf{T}$ is now directed inside the current yield surface, [19] also requires that $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ is codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{T}$ space, with the same generalization at a vertex as in the case of hardening behavior.

If elastic response is nonlinear, the yield surface in stress space is not necessarily convex. A concavity of the yield surface in the presence of nonlinear elasticity for a particular material model has been demonstrated by Palmer *et al.*, (1967). For linear elastic response, however,

$$(\mathbf{T} - \mathbf{T}^0) : \mathrm{d}^{\mathrm{p}}\mathbf{E} > 0 \qquad [20]$$

provided that there is no change of elastic stiffness caused by plastic deformation ($d\mathbf{M}=0$), or that the change is such that $d\mathbf{M}$ is positive semi-definite. The stress state $\mathbf{T}^0$ is an arbitrary stress state within the yield surface. Since $d^p\mathbf{E}$ is codirectional with the outward normal to a locally smooth yield surface in strain $\mathbf{T}$ space, [20] implies that the yield surface in a considered stress space is convex. Inequality [20] is often referred to as the principle of maximum plastic work (Hill, 1950; Johnson and Mellor, 1973; Lubliner, 1990). If inequality is assumed at the outset, it by itself assures both normality and convexity.

3 Constitutive Equations of Elastoplasticity

3.1 Strain Space Formulation

The stress rate is a sum of elastic and plastic parts, such that

$$\dot{\mathbf{T}} = \dot{\mathbf{T}}^e + \dot{\mathbf{T}}^p = \mathbf{\Lambda} : \dot{\mathbf{E}} - \dot{\gamma}\frac{\partial g}{\partial \mathbf{E}} \quad [21]$$

For incrementally linear and continuous response between loading and unloading, the loading index is

$$\dot{\gamma} = \frac{1}{h}\left(\frac{\partial g}{\partial \mathbf{E}} : \dot{\mathbf{E}}\right), \qquad \frac{\partial g}{\partial \mathbf{E}} : \dot{\mathbf{E}} > 0 \quad [22]$$

where $h>0$ is a scalar function of the plastic state on the yield surface in strain space, determined from the consistency condition $\dot{g}=0$. Consequently, the constitutive equation for elastoplastic loading is

$$\dot{\mathbf{T}} = \left[\mathbf{\Lambda} - \frac{1}{h}\left(\frac{\partial g}{\partial \mathbf{E}} \otimes \frac{\partial g}{\partial \mathbf{E}}\right)\right] : \dot{\mathbf{E}} \quad [23]$$

The fourth-order tensor within the square brackets is the elastoplastic stiffness tensor associated with the considered measure and reference state. Within the framework based on Green-elasticity and normality rule, the elastoplastic stiffness tensor possesses reciprocal or self-adjoint symmetry (with respect to first and second pair of indices), in addition to symmetries in the first and last two indices associated with the symmetry of stress and strain tensors.

The inverted form of [23] is

$$\dot{\mathbf{E}} = \left[\mathbf{M} + \frac{1}{H}\left(\mathbf{M} : \frac{\partial g}{\partial \mathbf{E}}\right) \otimes \left(\frac{\partial g}{\partial \mathbf{E}} : \mathbf{M}\right)\right] : \dot{\mathbf{T}} \quad [24]$$

where

$$H = h - \frac{\partial g}{\partial \mathbf{E}} : \mathbf{M} : \frac{\partial g}{\partial \mathbf{E}} \quad [25]$$

3.2 Stress Space Formulation

The strain rate is a sum of elastic and plastic parts, such that

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p = \mathbf{M} : \dot{\mathbf{T}} + \dot{\gamma}\frac{\partial f}{\partial \mathbf{T}} \quad [26]$$

The loading index is obtained from the consistency condition $\dot{f}=0$,

$$\dot{\gamma} = \frac{1}{H}\left(\frac{\partial f}{\partial \mathbf{T}} : \dot{\mathbf{T}}\right) \quad [27]$$

where H is a scalar function of the plastic state on the yield surface in stress space. Thus,

$$\dot{\mathbf{E}} = \left[\mathbf{M} + \frac{1}{H}\left(\frac{\partial f}{\partial \mathbf{T}} \otimes \frac{\partial f}{\partial \mathbf{T}}\right)\right] : \dot{\mathbf{T}} \quad [28]$$

The fourth-order tensor within the square brackets is the elastoplastic compliance tensor associated with the considered measure and reference state.

The scalar parameter H can be positive, negative or equal to zero. Three types of response can be identified within this constitutive framework. These are Hill (1978)

$$\begin{aligned} &H>0, \quad \frac{\partial f}{\partial \mathbf{T}}:\dot{\mathbf{T}}>0 \quad \text{hardening} \\ &H<0, \quad \frac{\partial f}{\partial \mathbf{T}}:\dot{\mathbf{T}}<0 \quad \text{softening} \\ &H=0, \quad \frac{\partial f}{\partial \mathbf{T}}:\dot{\mathbf{T}}=0 \quad \text{ideally plastic} \end{aligned} \qquad [29]$$

Starting from the current yield surface in stress space, the yield point moves outward in the case of hardening, inward in the case of softening, and tangentially to the yield surface in the case of ideally plastic response. In the case of softening, $\dot{\mathbf{E}}$ is not uniquely determined by prescribed stress rate $\dot{\mathbf{T}}$, since either eqn [28] applies, or the elastic unloading expression $\dot{\mathbf{E}}=\mathbf{M}:\dot{\mathbf{T}}$. In the case of ideally plastic response, the plastic part of the strain rate is indeterminate to the extent of an arbitrary positive multiple, since $\dot{\gamma}$ in eqn [27] is indeterminate.

3.3 Yield Surface with a Vertex

Physical theories of plasticity imply the formation of a corner or vertex at the loading point on the yield surface. Suppose that the yield surface in stress space has a pyramidal vertex formed by n intersecting segments $f_{<i>}=0$, then near the vertex

$$\prod_{i=1}^{n} f_{<i>}(\mathbf{T},\mathcal{H})=0, \quad n\geq 2 \qquad [30]$$

It follows that

$$\dot{\mathbf{E}}=\left[\mathbf{M}+\sum_{i=1}^{n}\sum_{j=1}^{n}H_{<ij>}^{-1}\left(\frac{\partial f_{<i>}}{\partial \mathbf{T}}\otimes\frac{\partial f_{<j>}}{\partial \mathbf{T}}\right)\right]:\dot{\mathbf{T}} \qquad [31]$$

This is an extension of the constitutive structure [28] for the smooth yield surface to the yield surface with a vertex. Elements of the matrix inverse to plastic moduli matrix $H_{<ij>}$ are denoted by $H_{<ij>}^{-1}$. The references Koiter (1953), Hill (1978), Asaro (1983), and Asaro and Lubarda (2006) can be consulted for further analysis.

4 Constitutive Models of Plastic Deformation

4.1 Isotropic Hardening

Experimental determination of the yield surface shape is commonly done with respect to Cauchy stress σ. Suppose that this is given by $f(\boldsymbol{\sigma}, k)=0$, where f is an isotropic function of $\boldsymbol{\sigma}$ and $k=k(\vartheta)$ is a scalar which defines the size of the yield surface. This depends on the history parameter, such as the effective plastic strain

$$\vartheta=\int_0^t (2\,\mathbf{D}^{\mathrm{p}}:\mathbf{D}^{\mathrm{p}})^{1/2}\,\mathrm{d}t \qquad [32]$$

The hardening model in which the yield surface expands during plastic deformation preserving its shape is known as the isotropic hardening model. Since f is an isotropic function of stress, the material is assumed to be isotropic. For non-porous metals the onset of plastic deformation and plastic yielding is unaffected by moderate superimposed pressure. The yield condition can consequently be written as an isotropic function of the deviatoric part of the Cauchy stress, i.e., its second and third invariant, $f(J_2, J_3, k)=0$. The well-known examples are the Tresca maximum shear stress criterion, or the von Mises yield criterion. In the latter case,

$$f=J_2-k^2(\vartheta)=0, \quad J_2=\frac{1}{2}\boldsymbol{\sigma}':\boldsymbol{\sigma}' \qquad [33]$$

The corresponding plasticity theory is referred to as the J_2 flow theory of plasticity. The yield stress in simple shear is k. If Y is the yield stress in uniaxial tension, $k=Y/\sqrt{3}$. The consistency condition gives

$$\dot{\gamma}=\frac{1}{4k^2h_{\mathrm{t}}^{\mathrm{p}}}\left(\boldsymbol{\sigma}':\overset{\circ}{\underline{\boldsymbol{\tau}}}\right), \quad H=4k^2h_{\mathrm{t}}^{\mathrm{p}} \qquad [34]$$

where the plastic tangent modulus in shear test is $h_t^p = dk/d\vartheta$. The stress rate

$$\overset{o}{\underline{\tau}} = \overset{o}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\ \text{tr}\ \mathbf{D}, \quad \overset{o}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{W} \qquad [35]$$

represents the Jaumann rate of the Kirchhoff stress $\tau = (\det \mathbf{F})\sigma$, when the current state is taken as the reference ($\det \mathbf{F} = 0$). The deformation gradient is $\mathbf{F}$, and the material spin is $\mathbf{W}$. The total rate of deformation is therefore

$$\mathbf{D} = \left(\mathbf{M} + \frac{1}{2h_t^p}\frac{\boldsymbol{\sigma}'\otimes\boldsymbol{\sigma}'}{\boldsymbol{\sigma}':\boldsymbol{\sigma}'}\right) : \overset{o}{\underline{\tau}} \qquad [36]$$

The elastic compliance tensor for infinitesimal elasticity is

$$\mathbf{M} = \frac{1}{2\mu}\left(\mathbf{I} - \frac{\lambda}{2\mu + 3\lambda}\boldsymbol{\delta}\otimes\boldsymbol{\delta}\right) \qquad [37]$$

The Lamé elastic constants are λ and μ. The second- and fourth-order unit tensors are designated by δ and $\mathbf{I}$. The plastic deformation is in this case isochoric ($\text{tr}\ \mathbf{D}^p = 0$), and principal directions of $\mathbf{D}^p$ are parallel to those of $\boldsymbol{\sigma}(\mathbf{D}^p \sim \boldsymbol{\sigma}')$. The inverted form of [36] is

$$\overset{o}{\underline{\tau}} = \left(\Lambda - \frac{2\mu}{1 + h_t^p/\mu}\frac{\boldsymbol{\sigma}'\otimes\boldsymbol{\sigma}'}{\boldsymbol{\sigma}':\boldsymbol{\sigma}'}\right) : \mathbf{D} \qquad [38]$$

where

$$\boldsymbol{\Lambda} = \lambda\boldsymbol{\delta}\otimes\boldsymbol{\delta} + 2\mu\mathbf{I} \qquad [39]$$

is the elastic stiffness tensor. Constitutive structures [36] and [38] have been extensively used in analytical and numerical studies of large plastic deformation problems (Neale, 1981; Needleman, 1982). Infinitesimal strain formulation, derivation of classical Prandtl-Reuss equations for elastic-ideally plastic, and Levy–Mises equations for rigid-ideally plastic material models can be found in standard texts or review papers (Hill, 1950; Naghdi, 1960).

4.2 Kinematic Hardening

To account for the Bauschinger effect and anisotropy of hardening, a simple model of kinematic hardening was introduced by Prager (1956). According to this model, the initial yield surface does not change its size and shape during plastic deformation, but translates in the stress space according to some prescribed rule. Thus, $f(\boldsymbol{\sigma} - \boldsymbol{\alpha}, k) = 0$, where $\boldsymbol{\alpha}$ represents the current center of the yield surface (back stress), and f is an isotropic function of the stress difference $\boldsymbol{\sigma} - \boldsymbol{\alpha}$. The size of the yield surface is specified by the constant k. The evolution of the back stress is governed by

$$\overset{o}{\boldsymbol{\alpha}} = c(\boldsymbol{\alpha})\mathbf{D}^p + \mathbf{C}(\boldsymbol{\alpha})(2\mathbf{D}^p : \mathbf{D}^p)^{1/2} \qquad [40]$$

where c and $\mathbf{C}$ are appropriate scalar and tensor functions of $\boldsymbol{\alpha}$. This representation is in accord with assumed time independence of plastic deformation, which requires eqn [40] to be homogeneous relation of degree one.

If $\mathbf{C} = 0$ and c is taken to be constant, the model corresponds to Prager's linear kinematic hardening. The plastic tangent modulus h^p in shear test is in this case constant and related to c by $c = 2h_t^p$. The resulting constitutive structure is

$$\mathbf{D} = \left[\mathbf{M} + \frac{1}{2h_t^p}\frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha})\otimes(\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}):(\boldsymbol{\sigma}' - \boldsymbol{\alpha})}\right] : \overset{o}{\underline{\tau}} \qquad [41]$$

with the inverse

$$\overset{o}{\underline{\tau}} = \left[\boldsymbol{\Lambda} - \frac{2\mu}{1 + h_t^p/\mu}\frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha})\otimes(\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}):(\boldsymbol{\sigma}' - \boldsymbol{\alpha})}\right] : \mathbf{D} \qquad [42]$$

If $\mathbf{C}$ in eqn [40] is taken to be proportional to $\boldsymbol{\alpha}$ (i.e., $\mathbf{C} = -c_0\boldsymbol{\alpha}$, $c_0 = \text{const.}$), a nonlinear kinematic hardening model of Armstrong and Frederick (1966) is obtained. Details can be found in Khan and Huang (1995). Ziegler (1959) used as an evolution equation for the back stress

$$\overset{o}{\boldsymbol{\alpha}} = \dot{\beta}(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \qquad [43]$$

The proportionality factor $\dot{\beta}$ is determined from the consistency condition in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$.

4.3 Combined Isotropic-Kinematic Hardening

In this hardening model the yield surface expands and translates during plastic deformation, so that

$$f(\boldsymbol{\sigma} - \boldsymbol{\alpha}, k) = 0, \quad k = k(\vartheta) \qquad [44]$$

The function $k(\vartheta)$, with ϑ defined by eqn [32], specifies expansion of the yield surface, while evolution eqn [40] specifies its translation.

4.4 Multisurface Models

Motivated by the need to better model nonlinearities in stress–strain loops, cyclic hardening or softening, cyclic creep, and stress relaxation, more involved hardening models were suggested. Mroz (1967) introduced a multi-yield surface model in which there is a field of hardening moduli, one for each yield surface. Initially the yield surfaces are assumed to be concentric. When the stress point reaches the inner-most yield surface, the plastic deformation develops according to linear hardening model with a prescribed plastic tangent modulus, until the active yield surface reaches the adjacent yield surface. Subsequent plastic deformation develops according to linear hardening model with another specified value of the plastic tangent modulus, until the next yield surface is reached, etc. Dafalias and Popov (1975) and Krieg (1975) suggested a hardening model which uses the yield (loading) surface and the limit (bounding) surface. A smooth transition from elastic to plastic regions on loading is assured by introducing a continuous variation of the plastic tangent modulus between the two surfaces.

5 Pressure-Dependent Plasticity

For porous metals, concrete and geomaterials like soils and rocks, plastic deformation has its origin in pressure-dependent microscopic processes and the yield condition for these materials, in addition to deviatoric components, depends on hydrostatic component of stress, i.e., its first invariant $I_1 = \mathrm{tr}\boldsymbol{\sigma}$.

5.1 Drucker–Prager Yield Condition for Geomaterials

Drucker and Pruger (1952) suggested that yielding of soil occurs when the shear stress on octahedral planes overcomes cohesive and frictional resistance to sliding on those planes. The yield condition is consequently

$$f = J_2^{1/2} + \frac{1}{3}\alpha I_1 - k = 0 \qquad [45]$$

where α is a frictional parameter. This geometrically represents a cone in the principal stress space with its axis parallel to hydrostatic axis. The radius of the circle in the deviatoric plane is $\sqrt{2}k$, where k is the yield stress in simple shear. The angle of the cone is $\tan^{-1}(\sqrt{2}\alpha/3)$. The yield stresses in uniaxial tension and compression are according to eqn (45),

$$Y^+ = \frac{\sqrt{3}k}{1 + \alpha/\sqrt{3}}, \quad Y^- = \frac{\sqrt{3}k}{1 - \alpha/\sqrt{3}} \qquad [46]$$

For the yield condition to be physically meaningful, the restriction holds $\alpha < \sqrt{3}$. If the compressive states of stress are considered positive (as commonly done in geomechanics), the minus sign appears in front of the second term in eqn [45].

When Drucker–Prager cone is applied to porous rocks, it overestimates the yield stress at higher pressures and inadequately predicts inelastic volume changes. To circumvent this, DiMaggio and Sandler (1971) introduced an ellipsoidal cap to close the cone at certain level of pressure. Other shapes of the cap were also used. Details can be found in Chen and Han (1988).

5.2 Gurson Yield Condition for Porous Metals

Based on a rigid-perfectly plastic analysis of spherically symmetric deformation around a spherical cavity, Gurson (1977) suggested a yield condition for porous metals in the form

$$f = J_2 + \frac{2}{3}\upsilon Y_0^2 \cosh\left(\frac{I_1}{2Y_0}\right) - (1 + \upsilon^2)\frac{Y_0^2}{3} = 0 \qquad [47]$$

where u is the porosity (void/volume fraction), and $Y_0 = \text{const.}$ is the tensile yield stress of the matrix material. Generalizations to include hardening matrix material were also made. The change in porosity during plastic deformation is given by the evolution equation

$$\dot{\upsilon} = (1-\upsilon)\mathrm{tr}\ \mathbf{D}^{\mathrm{p}} \quad [48]$$

Other evolution equations, which take into account nucleation and growth of voids, have been considered. To improve its predictions and agreement with experimental data, Tvergaard (1982) introduced two additional material parameters in the structure of the Gurson yield criterion. Mear and Hutchinson (1985) incorporated the effects of anisotropic (kinematic) hardening by replacing J_2 of $\boldsymbol{\sigma}'$ in eqn [47] with J_2 of $\boldsymbol{\sigma}' - \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the back stress.

5.3 Constitutive Equations of Pressure-Dependent Plasticity

The two considered pressure-dependent yield conditions are of the type

$$f(J_2, I_1, \mathcal{H}) = 0 \quad [49]$$

For materials obeying Ilyushin's postulate, the plastic part of the rate of deformation tensor is normal to the yield surface, so that

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial J_2}\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\boldsymbol{\delta} \quad [50]$$

The loading index is

$$\dot{\gamma} = \frac{1}{H}\left(\frac{\partial f}{\partial J_2}\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\boldsymbol{\delta}\right) : \underline{\overset{\circ}{\tau}} \quad [51]$$

where H is an appropriate hardening modulus. Therefore,

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{H}\left[\left(\frac{\partial f}{\partial J_2}\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\boldsymbol{\delta}\right) \otimes \left(\frac{\partial f}{\partial J_2}\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\boldsymbol{\delta}\right)\right] : \underline{\overset{\circ}{\tau}} \quad [52]$$

The volumetric part of the plastic rate of deformation is

$$\mathrm{tr}\ \mathbf{D}^{\mathrm{p}} = \frac{3}{H}\frac{\partial f}{\partial I_1}\left(\frac{\partial f}{\partial J_2}\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\boldsymbol{\delta}\right) : \underline{\overset{\circ}{\tau}} \quad [53]$$

For the Drucker–Prager yield condition,

$$\frac{\partial f}{\partial J_2} = \frac{1}{2}J_2^{-1/2}, \quad \frac{\partial f}{\partial I_1} = \frac{1}{3}\alpha \quad [54]$$

and

$$H = \frac{\mathrm{d}k}{\mathrm{d}\vartheta}, \quad \vartheta = \int_0^t (2\mathbf{D}^{\mathrm{p}'} : \mathbf{D}^{\mathrm{p}'})^{1/2}\mathrm{d}t \quad [55]$$

For the Gurson yield condition,

$$\frac{\partial f}{\partial J_2} = 1, \quad \frac{\partial f}{\partial I_1} = \frac{1}{3}\upsilon Y_0 \sinh\left(\frac{I_1}{2Y_0}\right) \quad [56]$$

and

$$H = \frac{2}{3}\upsilon(1-\upsilon)Y_0^3 \sinh\left(\frac{I_1}{2Y_0}\right)\left[\upsilon - \cosh\left(\frac{I_1}{2Y_0}\right)\right] \quad [57]$$

6 Non-Associative Plasticity

Constitutive equations in which plastic part of the rate of strain is normal to locally smooth yield surface $f=0$ in stress space,

$$\dot{\mathbf{E}}^{\mathrm{p}} = \dot{\gamma}\frac{\partial f}{\partial \mathbf{T}} \quad [58]$$

are often referred to as associative flow rules. A sufficient condition for this constitutive structure to hold is that material obeys the Ilyushin's postulate. However, many pressure-dependent dilatant materials with internal frictional effects are not well described by associative flow rules. For example, associative flow rules largely overestimate inelastic volume changes in geomaterials like rocks and soils (Rudnicki and Rice, 1975), and in certain high-strength steels exhibiting the strength-differential effect by which the yield strength is higher in compression than in tension (Spitzig *et al.*, 1975). For such materials, plastic part of the rate of strain is taken to be normal to plastic potential surface $\pi = 0$, which is distinct from the yield surface. The resulting constitutive structure,

$$\dot{\mathbf{E}}^{\mathrm{p}} = \dot{\gamma} \frac{\partial \pi}{\partial \mathbf{T}} \tag{59}$$

is known as non-associative flow rule Nemat-Nasser (1983). The consistency condition $\dot{f} = 0$ gives

$$\dot{\gamma} = \frac{1}{H} \frac{\partial f}{\partial \mathbf{T}} : \dot{\mathbf{T}} \tag{60}$$

so that

$$\dot{\mathbf{E}}^{\mathrm{p}} = \frac{1}{H} \left(\frac{\partial \pi}{\partial \mathbf{T}} \otimes \frac{\partial f}{\partial \mathbf{T}} \right) : \dot{\mathbf{T}} \tag{61}$$

Since $\pi \neq f$, the plastic compliance tensor in eqn [61] does not possess a reciprocal symmetry.

Consider inelastic behavior of geomaterials whose yield is governed by the Drucker–Prager yield condition of eqn [45]. A non-associative flow rule can be used with the plastic potential

$$\pi = J_2^{1/2} + \frac{1}{3}\beta I_1 - k = 0 \tag{62}$$

The material parameter β is in general different from the frictional parameter α of eqn [45]. The rate of plastic deformation is

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma} \frac{\partial \pi}{\partial \boldsymbol{\sigma}} = \dot{\gamma} \left(\frac{1}{2} J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \beta \boldsymbol{\delta} \right) \tag{63}$$

The consistency condition $\dot{f} = 0$ gives the loading index

$$\dot{\gamma} = \frac{1}{H} \left(\frac{1}{2} J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \alpha \boldsymbol{\delta} \right) : \underset{\sim}{\overset{\circ}{\tau}}, \quad H = \frac{\mathrm{d}k}{\mathrm{d}\vartheta} \tag{64}$$

Consequently,

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{H} \left[\left(\frac{1}{2} J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \beta \boldsymbol{\delta} \right) \otimes \left(\frac{1}{2} J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \alpha \boldsymbol{\delta} \right) \right] : \underset{\sim}{\overset{\circ}{\tau}} \tag{65}$$

The deviatoric and spherical parts are

$$\mathbf{D}^{\mathrm{p}'} = \frac{1}{2H} \frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \left(\boldsymbol{\sigma}' : \frac{\underset{\sim}{\overset{\circ}{\tau}}}{2J_2^{1/2}} + \frac{1}{3} \alpha \mathrm{tr}\, \underset{\sim}{\overset{\circ}{\tau}} \right) \tag{66}$$

$$\mathrm{tr}\, \mathbf{D}^{\mathrm{p}} = \frac{\beta}{H} \left(\boldsymbol{\sigma}' : \frac{\underset{\sim}{\overset{\circ}{\tau}}}{2J_2^{1/2}} + \frac{1}{3} \alpha \, \mathrm{tr}\, \underset{\sim}{\overset{\circ}{\tau}} \right) \tag{67}$$

The parameter β can be expressed as

$$\beta = \frac{\mathrm{tr}\, \mathbf{D}^{\mathrm{p}}}{(2\mathbf{D}^{\mathrm{p}'} : \mathbf{D}^{\mathrm{p}'})^{1/2}} \tag{68}$$

which shows that β is the ratio of the volumetric and shear part of the plastic strain rate, often called the dilatancy factor (Rudnicki and Rice, 1975). Frictional parameter and inelastic dilatancy of material actually change with progression of inelastic deformation. An analysis which accounts for their variation is presented by Nemat-Nasser and Shokooh (1980). Constitutive formulation of elastoplastic theory with evolving elastic properties is studied by Lubarda and Krajcinovic (1995) and others; see also Lubarda (2002).

6.1 Yield Vertex Model for Fissured Rocks

In a brittle rock, modeled to contain a collection of randomly oriented fissures, inelastic deformation results from frictional sliding on the fissure surfaces. Inelastic dilatancy under overall compressive loads is a consequence of opening the fissures at asperities and local tensile fractures at some angle at the edges of fissures. Individual yield surface may be associated with each fissure. The macroscopic yield surface is the envelope of individual yield surfaces for fissures of all orientations, similarly to slip models of metal plasticity (Rudnicki and Rice, 1975; Rice, 1976). Continued stressing in the same direction will cause continuing sliding on (already activated) favorably oriented fissures, and will initiate sliding for a progressively greater number of orientations. After certain amount of inelastic deformation, the macroscopic yield envelope develops a vertex at the loading point. The stress increment normal to the original stress direction will initiate or continue sliding of fissure surfaces for some fissure orientations. In isotropic hardening idealization with smooth yield surface, however, a stress increment tangential to the yield surface will cause only elastic deformation, overestimating the stiffness of response. In order to take into account the effect of the yield vertex in an approximate way, a second plastic modulus Hi is introduced, which governs the response to part of the stress increment directed tangentially to what is taken to be the smooth yield surface through the same stress point. Since no vertex formation is associated with hydrostatic stress increments, tangential stress increments are taken to be deviatoric, and eqn [66] is replaced with

$$\mathbf{D}^{\mathrm{p}'} = \frac{1}{2H}\frac{\boldsymbol{\sigma}'}{J_2^{1/2}}\left(\boldsymbol{\sigma}' : \frac{\overset{\circ}{\underline{\boldsymbol{\tau}}}}{2J_2^{1/2}} + \frac{1}{3}\alpha \,\mathrm{tr}\, \overset{\circ}{\underline{\boldsymbol{\tau}}}\right) + \frac{1}{2H_1}\left(\overset{\circ}{\underline{\boldsymbol{\tau}}}' - \frac{\boldsymbol{\sigma}' : \overset{\circ}{\underline{\boldsymbol{\tau}}}}{2J_2}\boldsymbol{\sigma}'\right) \qquad [69]$$

The dilation induced by the small tangential stress increment is assumed to be negligible, so that eqn [67] applies for tr $\mathbf{D}^{\mathrm{p}}$. The constitutive structure of eqn [69] is intended to model the response at a yield surface vertex for small deviations from proportional (straight ahead) loading $\overset{\circ}{\underline{\boldsymbol{\tau}}} \sim \boldsymbol{\sigma}'$. For the full range of directions of stress increment, the relationship between the rates of stress and plastic deformation is not expected to be necessarily linear, although it is homogeneous in these rates in the absence of time-dependent creep effects.

7 Deformation Theory of Plasticity

Simple plasticity theory has been suggested for proportional loading and small deformation by Hencky (1924) and Ilyushin (1963). A large deformation version of the theory can be formulated by using the logarithmic strain and its conjugate stress. Since stress proportionally increase, elastoplastic response is described macroscopically by constitutive structure of nonlinear elasticity, where strain is a function of stress. The strain tensor is decomposed into elastic and plastic part, $\mathbf{E} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}}$, elastic part is expressed in terms of stress by generalized Hooke's law, and plastic part is assumed to be

$$\mathbf{E}^{\mathrm{p}} = \varphi \mathbf{T}' \qquad [70]$$

where φ is an appropriate scalar function. Suppose that a nonlinear relationship $\bar{\tau} = \bar{\tau}(\bar{\gamma}^{\mathrm{p}})$ is available from the elastoplastic shear test. Define the plastic secant and tangent moduli by $h_{\mathrm{s}}^{\mathrm{p}} = \bar{\tau}/\bar{\gamma}^{\mathrm{p}}$, $h_{\mathrm{t}}^{\mathrm{p}} = \mathrm{d}\bar{\tau}/\mathrm{d}\bar{\gamma}^{\mathrm{p}}$, and let

$$\bar{\tau} = \left(\frac{1}{2}\mathbf{T}' : \mathbf{T}'\right)^{1/2}, \quad \bar{\gamma}^{\mathrm{p}} = (2\mathbf{E}^{\mathrm{p}} : \mathbf{E}^{\mathrm{p}})^{1/2} \qquad [71]$$

The scalar function ϕ is then $\phi = 1/2h_{\mathrm{s}}^{\mathrm{p}}$. Although deformation theory of plasticity is total strain theory, it is useful to cast it in the rate-type form, particularly when the considered boundary value problem needs to be solved in an incremental manner. The resulting expression for the plastic part of the total rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\overset{\circ}{\boldsymbol{\tau}}' + \left(\frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} - \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'} \qquad [72]$$

where $\overset{\circ}{\boldsymbol{\tau}}$ is the Jaumann derivative of the Kirchhoff stress.

7.1 Application of Deformation Theory Beyond Proportional Loading

Deformation theory agrees with flow theory of plasticity only under proportional loading, since then specification of the final state of stress also specifies the stress history. For general (non-proportional) loading, more accurate and physically appropriate is the flow theory of plasticity, particularly with an accurate modeling of the yield surface and hardening behavior. Budiansky (1959), however, indicated that deformation theory can be successfully used for certain nearly proportional loading paths, as well. The stress rate $\overset{\circ}{\boldsymbol{\tau}}'$ in eqn [72] does not then have to be codirectional with $\boldsymbol{\tau}'$, and the plastic part of the rate of deformation depends on both components of the stress rate $\overset{\circ}{\boldsymbol{\tau}}'$ one in the direction of $\boldsymbol{\tau}'$and the other normal to it. In contrast, according to flow theory with the von Mises smooth yield surface, the component of the stress rate $\overset{\circ}{\boldsymbol{\tau}}'$ normal to $\boldsymbol{\tau}'$ (thus tangential to the yield surface) does not affect the plastic part of the rate of deformation. Since the structure of the deformation theory of plasticity under proportional loading does not use any notion of the yield surface, eqn [72] can be used to approximately describe the response when the yield surface develops a vertex. Rewriting eqn [72] in the form

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\left[\overset{\circ}{\boldsymbol{\tau}}' - \frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\right] + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'} \qquad [73]$$

the first term on the right-hand side gives the response to component of the stress increment normal to $\boldsymbol{\tau}'$. The associated plastic modulus is $h_{\mathrm{s}}^{\mathrm{p}}$. The plastic modulus associated with the component of the stress increment in the direction of $\boldsymbol{\tau}'$ is $h_{\mathrm{t}}^{\mathrm{p}}$. A corner theory that predicts continuous variation of the stiffness and allows increasingly non-proportional increments of stress was formulated by Christoffersen and Hutchinson (1979). When applied to the analysis of necking in thin sheets under biaxial stretching, the results were in better agreement with experiments than those obtained from the theory with smooth yield characterization. Similar observations were long known in the field of elastoplastic buckling. Deformation theory predicts buckling loads better than flow theory with a smooth yield surface Hutchinson (1974).

8 Thermoplasticity

Non-isothermal plasticity is here considered assuming that temperature is not too high, so that creep deformation can be neglected. The analysis may also be adequate for certain applications under high stresses of short duration, where temperature increase is more pronounced but viscous (creep) strains have no time to develop (Prager, 1958; Kachanov, 1971). Thus, infinitesimal changes of stress and temperature applied to the material at a given state produce a unique infinitesimal change of strain that is independent of the speed with which these changes are made. Rate-dependent plasticity models will be presented in Section 9.

The formulation of thermoplastic analysis under described conditions can proceed by introducing a non-isothermal yield condition in either stress or strain space. For example, the yield condition is stress space is $f(\mathbf{T},\theta,\mathcal{H}) = 0$. The response within the yield surface is thermoelastic. If the Gibbs energy relative to selected stress and strain measures is $\phi = \phi(\mathbf{T},\theta,\mathcal{H})$ per unit reference volume, the strain is $\mathbf{E}=\partial\phi/\partial\mathbf{T}$.

Let the stress state $\mathbf{T}$ be on the current yield surface. The rates of stress and temperature associated with thermoplastic loading satisfy the consistency condition $\dot{f} = 0$, which gives

$$\frac{\partial f}{\partial \mathbf{T}} : \dot{\mathbf{T}} + \frac{\partial f}{\partial \theta}\dot{\theta} - \dot{\gamma}H = 0 \qquad [74]$$

The hardening parameter is $H = H(\mathbf{T},\theta,\mathcal{H})$, and the loading index is $\dot{\gamma} > 0$. Three types of response are possible,

$$\begin{aligned}
&H>0, \quad \frac{\partial f}{\partial \mathbf{T}}\dot{\mathbf{T}} : + \frac{\partial f}{\partial \theta}\dot{\theta} > 0 \quad \text{thermoplastic hardening}\\
&H<0, \quad \frac{\partial f}{\partial \mathbf{T}}\dot{\mathbf{T}} : + \frac{\partial f}{\partial \theta}\dot{\theta} < 0 \quad \text{thermoplastic softening}\\
&H=0, \quad \frac{\partial f}{\partial \mathbf{T}} : \dot{\mathbf{T}} + \frac{\partial f}{\partial \theta}\dot{\theta} = 0 \quad \text{ideally thermoplastic}
\end{aligned} \qquad [75]$$

This parallels the isothermal classification of eqn [29].

Since rate-independence is assumed, the constitutive relationship has to be homogeneous of degree one in rates of stress, strain and temperature. For thermoplastic part of the rate of strain this is satisfied by the normality structure

$$\dot{\mathbf{E}}^{\mathrm{p}} = \dot{\gamma}\frac{\partial f}{\partial \mathbf{T}} \qquad [76]$$

which, in view of eqn [74], becomes

$$\dot{\mathbf{E}}^{\mathrm{p}} = \frac{1}{H}\left(\frac{\partial f}{\partial \mathbf{T}} : \dot{\mathbf{T}} + \frac{\partial f}{\partial \theta}\dot{\theta}\right)\frac{\partial f}{\partial \mathbf{T}} \qquad [77]$$

The strain rate is a sum of thermoelastic and thermoplastic parts. The thermoelastic part is

$$\dot{\mathbf{E}}^{\mathrm{e}} = \frac{\partial^2 \phi}{\partial \mathbf{T} \otimes \partial \mathbf{T}} : \dot{\mathbf{T}} + \frac{\partial^2 \phi}{\partial \mathbf{T} \partial \theta} \dot{\theta} \tag{78}$$

For example, if

$$\phi = \frac{1}{4\mu} \left(\mathrm{tr}\, \mathbf{T}^2 - \frac{\lambda}{3\lambda + 2\mu} \mathrm{tr}^2 \mathbf{T} \right) + \alpha(\theta) \mathrm{tr}\, \mathbf{T} + \beta(\theta, \mathcal{H}) \tag{79}$$

there follows

$$\dot{\mathbf{E}}^{\mathrm{e}} = \frac{1}{2\mu} \left(\mathbf{I} - \frac{\lambda}{2\mu + 3\lambda} \boldsymbol{\delta} \otimes \boldsymbol{\delta} \right) : \dot{\mathbf{T}} + \alpha'(\theta) \dot{\theta} \boldsymbol{\delta} \tag{80}$$

where λ and β are the Lame type elastic constants corresponding to selected measures, α and β are appropriate functions of indicated arguments, and $\alpha' = \mathrm{d}\alpha/\mathrm{d}\theta$.

Suppose that non-isothermal yield condition in the Cauchy stress space is temperature-dependent von Mises condition

$$f = \frac{1}{2} \boldsymbol{\sigma}' : \boldsymbol{\sigma}' - [\varphi(\theta) k(\vartheta)]^2 = 0 \tag{81}$$

The thermoplastic part of the deformation rate is then

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2\varphi h_{\mathrm{t}}^{\mathrm{p}}} \left(\frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} : \underset{\sim}{\overset{\circ}{\tau}} - \boldsymbol{\sigma}' \frac{\varphi'}{\varphi} \dot{\theta} \right) \tag{82}$$

where $h_{\mathrm{t}}^{\mathrm{p}} = \mathrm{d}k/\mathrm{d}\vartheta$, and $\varphi' = \mathrm{d}\varphi/\mathrm{d}\theta$. Combining with eqn [80], the total rate of deformation is

$$\mathbf{D} = \left[\frac{1}{2\mu} \left(\mathbf{I} - \frac{\lambda}{2\mu + 3\lambda} \boldsymbol{\delta} \otimes \boldsymbol{\delta} \right) + \frac{1}{2\varphi h_{\mathrm{t}}^{\mathrm{p}}} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} \right] : \underset{\sim}{\overset{\circ}{\tau}} + \left[\alpha'(\theta) \boldsymbol{\delta} - \frac{\varphi'}{2\varphi^2 h_{\mathrm{t}}^{\mathrm{p}}} \boldsymbol{\sigma}' \right] \dot{\theta} \tag{83}$$

The inverse equation is

$$\underset{\sim}{\overset{\circ}{\tau}} = \left(\lambda \boldsymbol{\delta} \otimes \boldsymbol{\delta} + 2\mu \mathbf{I} - \frac{2\mu}{1 + \varphi h_{\mathrm{t}}^{\mathrm{p}}/\mu} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} \right) : \mathbf{D} - \left[(3\lambda + 2\mu) \alpha' \boldsymbol{\delta} - \frac{1}{1 + \varphi h_{\mathrm{t}}^{\mathrm{p}}/\mu} \frac{\varphi'}{\varphi} \boldsymbol{\sigma}' \right] \dot{\theta} \tag{84}$$

Infinitesimal strain formulation for rigid-thermoplastic material was given by Prager (1958). See also Lee (1969), and Naghdi (1990). Experimental investigation of non-isothermal yield surfaces was reported by Phillips (1982).

In the case of thermoplasticity with linear kinematic hardening ($c = 2h_{\mathrm{t}}^{\mathrm{p}}$), and the temperature-dependent yield surface

$$f = \frac{1}{2} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) - [\varphi(\theta) k]^2 = 0, \quad k = \mathrm{const.} \tag{85}$$

there follows

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} \left[\frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} : \underset{\sim}{\overset{\circ}{\tau}} - \frac{\varphi'}{\varphi} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \dot{\theta} \right] \tag{86}$$

9 Rate-Dependent Plasticity

This section is devoted to inelastic constitutive equations for metals in the strain rate sensitive range of material response, where time effects play an important role. There is an indication from the dislocation dynamics point of view (Johnston and Gilman, 1959) that plasticity caused by crystallographic slip in metals is inherently time-dependent. Once it is assumed that the rate of shearing on a given slip system depends on local stresses only through the resolved shear stress in slip direction, the plastic part of the rate of strain is derivable from a scalar flow potential Rice (1971) as

$$\dot{\mathbf{E}}^{\mathrm{p}} = \frac{\partial \Omega(\mathbf{T}, \theta, \mathcal{H})}{\partial \mathbf{T}} \tag{87}$$

The history of deformation is represented by the pattern of internal rearrangements $\mathcal{H}$, and the absolute temperature is θ. Geometrically, the plastic part of the strain rate is normal to surfaces of constant flow potential in stress space. There is no yield surface in the model and plastic deformation commences from the onset of loading. Time-independent behavior can be recovered, under certain idealizations – neglecting creep and rate effects, as an appropriate limit Rice (1970).

9.1 Power-Law and Johnson–Cook Models

The power-law representation of the flow potential in the Cauchy stress space is

$$\Omega = \frac{2\dot{\gamma}^0}{m+1}\left(\frac{J_2^{1/2}}{k}\right)^m J_2^{1/2}, \quad J_2 = \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}' \qquad [88]$$

where $k = \mathrm{k}(\theta, \mathcal{H})$ is the reference shear stress, $\dot{\gamma}^0$ is the reference shear strain rate to be selected for each material, and m is the material parameter (of the order of 100 for metals at room temperature and strain rates below $10^4\ \mathrm{s}^{-1}$; Nemat-Nasser, 1992). The corresponding plastic part of the rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}^0 \left(\frac{J_2^{1/2}}{k}\right)^m \frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \qquad [89]$$

The equivalent plastic strain is usually used as the only history parameter $\mathcal{H}$, and the reference shear stress depends on ϑ and θ according to

$$k = k^0\left(1 + \frac{\vartheta}{\vartheta^0}\right)^{\alpha} \exp\left(-\beta\frac{\theta - \theta_0}{\theta_{\mathrm{m}} - \theta_0}\right) \qquad [90]$$

Here, k^0 and ϑ^0 are the normalizing stress and strain, θ_0 and θ_{m} are the room and melting temperatures, and α and β are the material parameters. From the onset of loading the deformation rate consists of elastic and plastic constituents, although for large m the plastic contribution may be small if J_2 is less than k.

Another representation of the flow potential, constructed according to Johnson and Cook (1983) model, is

$$\Omega = \frac{2\dot{\gamma}^0}{a} k \exp\left[a\left(\frac{J_2^{1/2}}{k} - 1\right)\right] \qquad [91]$$

The reference shear stress is

$$k = k^0\left[1 + b\left(\frac{\vartheta}{\vartheta^0}\right)^c\right]\left[1 - \left(\frac{\theta - \theta_0}{\theta_{\mathrm{m}} - \theta_0}\right)^d\right] \qquad [92]$$

where a, b, c, d are the material parameters. The corresponding plastic part of the rate of deformation is in this case

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}^0 \, exp\left[a\left(\frac{J_2^{1/2}}{k} - 1\right)\right]\frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \qquad [93]$$

9.2 Viscoplasticity Models

For high strain rate applications in dynamic plasticity (Cristescu, 1967; Clifton, 1983), the flow potential can be taken as

$$\Omega = \frac{1}{\zeta}\left[J_2^{1/2} - k_{\mathrm{s}}(\vartheta)\right]^2 \qquad [94]$$

where ζ is the viscosity coefficient, and $k_{\mathrm{s}}(\vartheta)$ represents the shear stress–plastic strain relationship from the (quasi) static shear test. The positive difference $J_2^{1/2} - k_{\mathrm{s}}(\vartheta)$between the measure of the current dynamic stress state and corresponding static stress state (at the given level of equivalent plastic strain ϑ) is known as the overstress measure Malvern (1951). The plastic part of the rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{\zeta}\left[J_2^{1/2} - k_{\mathrm{s}}(\vartheta)\right]\frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \qquad [95]$$

The inverted form of eqn (95) is

$$\boldsymbol{\sigma}' = \zeta \mathbf{D}^{\mathrm{p}} + 2k_{\mathrm{s}}(\vartheta)\frac{\mathbf{D}^{\mathrm{p}}}{(2\mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}})^{1/2}} \qquad [96]$$

which shows that the rate-dependence in the model comes from the first term on the right-hand side. In quasi-static tests, viscosity ζ is taken to be equal to zero, and eqn [96] reduces to time-independent von-Mises isotropic hardening plasticity. In this case, flow potential Ω is constant within the elastic range bounded by the yield surface $J_2^{1/2} = k_{\mathrm{s}}(\vartheta)$.

More general representation for Ω is possible by using the Perzyna (1966) viscoplastic model. For example, one can take

$$\Omega = \frac{C}{m+1}[f(\boldsymbol{\sigma}) - k_{\mathrm{s}}(\vartheta)]^{m+1} \qquad [97]$$

which yields

$$\mathbf{D}^{\mathrm{p}} = C[f(\boldsymbol{\sigma}) - k_{\mathrm{s}}(\vartheta)]^{m}\frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad [98]$$

If $f = J_2^{1/2}$, $C = 2/\zeta$, and $k_{\mathrm{s}}(\vartheta) = k^0 = \text{const.}$, eqn [98] gives

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{\zeta}\left(J_2^{1/2} - k^0\right)^m \frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \qquad [99]$$

which is a nonlinear Bingham model. If $k_{\mathrm{s}}(\vartheta) = 0$, $f = J_2^{1/2}$, and $C = 2\dot{\gamma}^0/k^m$, eqn (98) reproduces the power-law J_2 creep given by eqn (89).

10 Phenomenological Plasticity Based on the Multiplicative Decomposition

In this section we introduce a multiplicative decomposition of the total deformation gradient into elastic and plastic parts to provide an additional framework for dealing with finite elastic and plastic deformation. We apply the decomposition in the specific context of a strain rate dependent, J_2-flow theory of plasticity. Consider the current elastoplastically deformed configuration of the material sample. Let $\mathbf{F}$ be the deformation gradient that maps an infinitesimal material element $d\mathbf{X}$ from initial configuration to $d\mathbf{x}$ in current configuration, such that $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$. Introduce an intermediate configuration by elastically destressing the current configuration to zero stress. Such configuration differs from the initial configuration by a residual (plastic) deformation, and from the current configuration by a reversible (elastic) deformation. If $d\mathbf{x}^{\mathrm{p}}$ is the material element in the intermediate configuration, corresponding to $d\mathbf{x}$ in the current configuration, then $d\mathbf{x} = \mathrm{F}^{\mathrm{e}} \cdot d\mathbf{x}^{\mathrm{p}}$, where $\mathbf{F}^{\mathrm{e}}$ represents a deformation gradient associated with elastic loading from the intermediate to current configuration. If the deformation gradient of plastic transformation is $\mathbf{F}^{\mathrm{p}}$, such that $d\mathbf{x}^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}} \cdot d\mathbf{X}$, the multiplicative decomposition of the total deformation gradient into its elastic and plastic parts holds

$$\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}} \qquad [100]$$

The decomposition was introduced in the phenomenological rate-independent theory of plasticity by Lee (1969). The historical background and the role of the multiplicative decomposition in the constitutive description of other types of inelastic deformation can be found in Lubarda (2004). In the case when elastic destressing to zero stress is not physically achievable due to possible onset of reverse inelastic deformation before the state of zero stress is reached, the intermediate configuration can be conceptually introduced by virtual destressing to zero stress, locking all inelastic structural changes that would take place during the actual destressing. The deformation gradients $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ are not uniquely defined because the intermediate unstressed configuration is not unique. Arbitrary local material rotations can be superposed to the intermediate configuration, preserving it unstressed. In applications, however, the decomposition (100) can be made unique by additional specifications, dictated by the nature of the considered material model. For example, for elastically isotropic materials the elastic stress response depends only on the elastic stretch $\mathbf{V}^{\mathrm{e}}$, and not on the rotation $\mathbf{R}^{\mathrm{e}}$ from the polar decomposition $\mathbf{F}^{\mathrm{e}} = \mathbf{V}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}}$. Consequently, the intermediate configuration can be specified uniquely by requiring that elastic unloading takes place without rotation ($\mathbf{F}^{\mathrm{e}} = \mathbf{V}^{\mathrm{e}}$). An alternative choice will be pursued in the constitutive derivation presented here. See also Lubarda (2002) and Nemat-Nasser (2009).

The velocity gradient in the current configuration at time t is defined by

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \qquad [101]$$

The superposed dot designates the material time derivative. By introducing the multiplicative decomposition of deformation gradient [100], the velocity gradient becomes

$$\mathbf{L} = \dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1} + \mathbf{F}^{e} \cdot \left(\dot{\mathbf{F}}^{p} \cdot \mathbf{F}^{p-1}\right) \cdot \mathbf{F}^{e-1} \quad [102]$$

The rate of deformation D and the spin W are, respectively, the symmetric and antisymmetric part of L,

$$\mathbf{D} = \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{s} + \left[\mathbf{F}^{e} \cdot \left(\dot{\mathbf{F}}^{p} \cdot \mathbf{F}^{p-1}\right) \cdot \mathbf{F}^{e-1}\right]_{s} \quad [103]$$

$$\mathbf{W} = \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{a} + \left[\mathbf{F}^{e} \cdot \left(\dot{\mathbf{F}}^{p} \cdot \mathbf{F}^{p-1}\right) \cdot \mathbf{F}^{e-1}\right]_{a} \quad [104]$$

Since $\mathbf{F}^{e}$ is specified up to an arbitrary rotation, and since the stress response of elastically isotropic materials does not depend on the rotation, we shall choose the unloading program such that

$$\left[\mathbf{F}^{e} \cdot \left(\dot{\mathbf{F}}^{p} \cdot \mathbf{F}^{p-1}\right) \cdot \mathbf{F}^{e-1}\right]_{a} = \mathbf{0} \quad [105]$$

With this choice, therefore, the rate of deformation and the spin tensors are

$$\mathbf{D} = \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{s} + \mathbf{F}^{e} \cdot \left(\dot{\mathbf{F}}^{p} \cdot \mathbf{F}^{p-1}\right) \cdot \mathbf{F}^{e-1} \quad [106]$$

$$\mathbf{W} = \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{a} \quad [107]$$

10.1 Elastic and Plastic Constitutive Contributions

It is assumed that the material is elastically isotropic in its initial undeformed state, and that plastic deformation does not affect its elastic properties. The elastic response is then given by

$$\boldsymbol{\tau} = \mathbf{F}^{e} \cdot \frac{\partial \Psi^{e}(\mathbf{E}^{e})}{\partial \mathbf{E}^{e}} \cdot \mathbf{F}^{eT} \quad [108]$$

The elastic strain energy per unit unstressed volume, Ψ^{e}, is an isotropic function of the Lagrangian strain $\mathbf{E}^{e} = (\mathbf{F}^{eT} \cdot \mathbf{F}^{e} - \mathbf{I})/2$. Plastic deformation is assumed to be incompressible ($\det \mathbf{F}^{e} = \det \mathbf{F}$), so that $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress. By differentiating eqn [108], we obtain

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)^{T} = \hat{\mathcal{L}} : \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{s} \quad [109]$$

The rectangular components of $\hat{\mathcal{L}}$ are

$$\hat{\mathcal{L}}_{ijkl} = F^{e}_{im} F^{e}_{jn} \frac{\partial^{2} \Psi^{e}}{\partial E^{e}_{mn} \partial E^{e}_{pq}} F^{e}_{kp} F^{e}_{lq} \quad [110]$$

Equation [109] can be equivalently written as

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{a} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{a} = \hat{\mathcal{L}} : \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{s} \quad [111]$$

The modified elastic moduli tensor $\mathcal{L}$ has the components

$$\mathcal{L}_{ijkl} = \hat{\mathcal{L}}_{ijkl} + \frac{1}{2}\left(\tau_{ik}\delta_{jl} + \tau_{jk}\delta_{il} + \tau_{il}\delta_{jk} + \tau_{jl}\delta_{ik}\right) \quad [112]$$

In view of eqn [107], we can rewrite eqn [111] as

$$\overset{\circ}{\boldsymbol{\tau}} = \mathcal{L} : \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{s} \quad [113]$$

where

$$\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W} \quad [114]$$

is the Jaumann rate of the Kirchhoff stress with respect to total spin. By inversion, eqn [3] gives the elastic rate of deformation as

$$\mathbf{D}^{e} = \left(\dot{\mathbf{F}}^{e} \cdot \mathbf{F}^{e-1}\right)_{s} = \mathcal{L}^{-1} : \overset{o}{\boldsymbol{\tau}} \qquad [115]$$

Physically, the strain increment $\mathbf{D}^{e}$ dt is a reversible part the total strain increment $\mathbf{D}$ dt, which is recovered upon loading-unloading cycle of the stress increment $\overset{o}{\boldsymbol{\tau}}dt$. The remaining part of the total rate of deformation,

$$\mathbf{D}^{p} = \mathbf{D} - \mathbf{D}^{e} \qquad [116]$$

is the plastic part, which gives a residual strain increment left upon the considered infinitesimal cycle of stress. When the material obeys Ilyushin's work postulate, the so defined plastic rate of deformation $\mathbf{D}^{p}$ is codirectional with the outward normal to a locally smooth yield surface in the Cauchy stress space, i.e.,

$$\mathbf{D}^{p} || \frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad [117]$$

10.2 Rate Dependent J_2 Flow Theory

Classical J_2 flow theory uses the yield surface as generated earlier as a flow potential. Thus the current yield criteria $\overline{\boldsymbol{\sigma}} = \kappa$ defines a series of yield surfaces in stress space, where k serves the role of a scaling parameter. Here we rephrase the yield criterion in terms of the effective stress, $\overline{\boldsymbol{\sigma}} = (3/2\boldsymbol{\sigma}'_{ij}\boldsymbol{\sigma}'_{ij})^{1/2}$; k is then the uniaxial yield stress. J_2 flow theory assumes that $\mathbf{D}^{p} || \boldsymbol{\sigma}'$. This amounts to taking

$$\mathbf{D}^{p} || \frac{\partial \overline{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}'} \qquad [118]$$

or

$$\mathbf{D}^{p}_{ij} || \frac{\partial \overline{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}'_{ij}} = \frac{3}{2} \frac{\boldsymbol{\sigma}'_{ij}}{\overline{\boldsymbol{\sigma}}} \qquad [119]$$

Thus we can write

$$\mathbf{D}^{p} = \dot{\overline{\varepsilon}}^{p} \frac{3}{2} \frac{\boldsymbol{\sigma}'}{\overline{\boldsymbol{\sigma}}} \qquad [120]$$

where $\dot{\overline{\varepsilon}}^{p}$ is an effective plastic strain rate whose specification requires an additional model statement. By incorporating [120] we can write from eqn [113]

$$\overset{o}{\boldsymbol{\tau}} = \mathcal{L} : \mathbf{D}^{e} = \mathcal{L} : (\mathbf{D} - \mathbf{D}^{p}) = \mathcal{L} : \left(\mathbf{D} - \dot{\overline{\varepsilon}}^{p} \frac{3}{2} \frac{\boldsymbol{\sigma}'}{\overline{\boldsymbol{\sigma}}}\right) \qquad [121]$$

We adopt a simple power-law expression of the form

$$\dot{\overline{\varepsilon}}^{p} = \dot{\varepsilon}_{0} \left(\frac{\overline{\boldsymbol{\sigma}}}{g}\right)^{1/m} \qquad [122]$$

where $\dot{\varepsilon}_{0}$ is a reference strain rate and 1/m represents a strain rate sensitivity coefficient. For common metals, $50 < 1/\mathrm{m} < 200$. For values of $1/\mathrm{m} \sim 100$, or larger, the materials will display a very nearly rate-independent response in the sense that $\overline{\boldsymbol{\sigma}}$ will track g at nearly any value of strain rate.

Strain hardening is described as an evolution of the hardness function g. This is often taken to be

$$g(\overline{\varepsilon}^{p}) = \boldsymbol{\sigma}_{0} \left(1 + \frac{\overline{\varepsilon}^{p}}{\varepsilon_{y}}\right)^{n} \qquad [123]$$

where

$$\dot{\overline{\varepsilon}}^{p} = \left(\frac{2}{3}\mathbf{D}^{p} : \mathbf{D}^{p}\right)^{1/2}, \quad \overline{\varepsilon}^{p} = \int_{0}^{t} \left(\frac{2}{3}\mathbf{D}^{p} : \mathbf{D}^{p}\right)^{1/2} \mathrm{d}t \qquad [124]$$

are the effective plastic strain rate and effective plastic strain, respectively. The remaining parameters are the material parameters; the initial yield stress is σ_{0}, and n is the hardening exponent.

11 Strain Gradient Plasticity

In classical plasticity there is no material length scale in the framework of the constitutive theory, so that this theory cannot predict the size effects experimentally observed in plastic deformation problems at micron or smaller scales, as in the bending and torsion testing of very thin beams and wires, inelastic response of nanograined materials, dispersion strengthening by small particles, measurements of indentation hardness at the micron and submicron scales, micro-electromechanical systems and thin film applications, etc. (Fleck *et al.*, 1994; Nix and Gao, 1998; Stoulken and Evans, 1998). In general, the observed trend is that smaller is stronger. This size-dependent strengthening has been attributed to the effects of strain gradients on plastic deformation. The theory which includes these effects has been put forward by Aifantis (1984), Muhlhaus and Aifantis (1991), Fleck and Hutchinson (1993, 1997, 2001), and Gao *et al.* (1999), with subsequent developments by many investigators, including, inter alia, Huang *et al.* (2004), Gurtin and Anand (2009), Fleck and Willis (2009), Gudmundson (2004), Hutchinson (2012), and Fleck *et al.* (2014).

From the dislocations point of view, the gradients of plastic strain can be associated with the storage of geometrically necessary dislocations, while the work hardening of material under uniform strain is associated with random trapping and storage of dislocations referred to as statistically stored dislocations (Fleck *et al.*, 1994; Nix and Gao, 1998). In this section we present a simple formulation of the phenomenological strain gradient plasticity which includes only one material (constitutive) length scale, in the absence of which the theory reduces to the classical J_2 plasticity theory. To a large extent, the presented formulation is based on the infinitesimal strain formulation by Hutchinson (2012). No explicit referral is made to specific dislocation mechanisms and interactions among individual dislocations, which is on the agenda of the discrete dislocation dynamics and dislocation based plasticity at even smaller scales, for example, Devincre and Kubin (1997), Tadmor *et al.* (1999), Needleman (2000), Zbib *et al.* (2002), and Bittencourt *et al.* (2003).

11.1 Gradient-Enhanced Effective Plastic Strain

The rectangular components of the infinitesimal strain are denoted by ε_{ij}, which are related to the displacement components u_i by $\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$, where $(\ )_{,i}$ designates the derivative with respect to the spatial coordinate x_i. It is assumed that the elastoplastic rate of strain is the sum of elastic and plastic contributions, such that $\dot{\varepsilon}_{ij}=\dot{\varepsilon}^{\mathrm{e}}_{ij}+\dot{\varepsilon}^{\mathrm{p}}_{ij}$. The rates are defined with respect to any monotonically increasing time-like parameter t. The elastic part of the strain rate depends on the rate of the Cauchy stress (σ_{ij}) according to the generalized Hooke's law

$$\dot{\varepsilon}^{\mathrm{e}}_{ij}=\frac{1}{2\mu}\dot{\boldsymbol{\sigma}}'_{ij}+\frac{1}{9\kappa}\dot{\boldsymbol{\sigma}}_{kk}\delta_{ij} \tag{125}$$

where μ and κ are the elastic shear and bulk moduli, respectively. The plastic part of the strain rate is assumed to be codirectional with the deviatoric part of the Cauchy stress (σ'_{ij}), as in the classical J_2 flow theory of plasticity,

$$\dot{\varepsilon}^{\mathrm{p}}_{ij}=\dot{e}_{\mathrm{p}}m_{ij},\quad m_{ij}=\frac{3}{2}\frac{\boldsymbol{\sigma}'_{ij}}{\boldsymbol{\sigma}_{\mathrm{eq}}} \tag{126}$$

The equivalent (von Mises) stress σ_{eq} is

$$\boldsymbol{\sigma}_{eq}=\left(\frac{3}{2}\boldsymbol{\sigma}'_{ij}\boldsymbol{\sigma}'_{ij}\right)^{1/2},\quad \boldsymbol{\sigma}'_{ij}=\boldsymbol{\sigma}_{ij}-\frac{1}{3}\boldsymbol{\sigma}_{kk}\delta_{ij} \tag{127}$$

while the loading index $\dot{e}_{\mathrm{p}}$ satisfies

$$\dot{e}_{\mathrm{p}}=\left(\frac{2}{3}\dot{\varepsilon}^{\mathrm{p}}_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij}\right)^{1/2}>0 \tag{128}$$

Its path-dependent integral over the history of deformation gives the effective plastic strain e_{p}. The spatial gradient of e_{p} will be used as a cumulative measure of plastic strain gradients, i.e.,

$$e_{\mathrm{p}}=\int\dot{e}_{\mathrm{p}}\,\mathrm{d}t,\quad e_{\mathrm{p},k}=\int\dot{e}_{\mathrm{p},k}\mathrm{d}t \tag{129}$$

In the strain gradient plasticity, a gradient-enhanced effective plastic strain can be the defined by Hutchinson (2012)

$$E_{\mathrm{p}}=\left(e_{\mathrm{p}}^2+l^2e_{\mathrm{p},k}e_{\mathrm{p},k}\right)^{1/2} \tag{130}$$

where l is the material length scale of the specific problem at hand, introduced in [130] by the dimensional consideration. With the so defined measure of the cumulative plastic strain, it will be assumed that the specific plastic work (per unit volume) is

$$w_{\mathrm{p}} = \int_0^{E_{\mathrm{p}}} \boldsymbol{\sigma}_0(\varepsilon_{\mathrm{p}})\mathrm{d}\varepsilon_{\mathrm{p}} \qquad [131]$$

where $\sigma_0 = \sigma_0(\varepsilon_{\mathrm{p}})$ is obtained from the stress–plastic strain curve in one-dimensional simple tension test. The assumed form [131] implies that the plastic work required to deform the material element to the strain level represented by E_p, in the presence of strain gradients, is equal to that at the strain level $e_p = E_p$, in the absence of plastic gradients Hutchinson (2012). More involved representation of the expression for w_{p} could be introduced, such as one proposed by Fleck *et al.* (2014). The elastic portion of the work, associated with elastic strain $\varepsilon_{ij}^{\mathrm{e}}$, is

$$w_{\mathrm{e}} = \mu\varepsilon_{ij}^{\mathrm{e}'}\varepsilon_{ij}^{\mathrm{e}'} + \frac{1}{2}\kappa\varepsilon_{kk}^{\mathrm{e}2}, \quad \varepsilon_{ij}^{\mathrm{e}'} = \varepsilon_{ij}^{\mathrm{e}} - \frac{1}{3}\varepsilon_{kk}^{\mathrm{e}}\delta_{ij} \qquad [132]$$

such that the total work done per unit volume is $w = w_{\mathrm{e}} + w_{\mathrm{p}}$. If the plastic part of the total strain $\varepsilon_{ij}^{\mathrm{p}}$ is obtained from $\dot{\varepsilon}_{ij}^{\mathrm{p}}$ by the integration along a specified history of deformation, the elastic part follows from $\varepsilon_{ij}^{\mathrm{e}} = \varepsilon_{ij} - \varepsilon_{ij}^{\mathrm{p}}$. This is related to the Cauchy stress by Hooke's law $\boldsymbol{\sigma}_{ij} = 2\mu\varepsilon_{ij}^{\mathrm{e}'} + \kappa\varepsilon_{kk}^{\mathrm{e}}\delta_{ij}$.

11.2 Rate of Work

The rate of work is the sum of elastic and plastic parts, $\dot{w} = \dot{w}_{\mathrm{e}} + \dot{w}_{\mathrm{p}}$. The elastic part is obtained from [132] as $\dot{w}_{\mathrm{e}} = \boldsymbol{\sigma}_{ij}\dot{\varepsilon}_{ij}^{\mathrm{e}}$. The expression for the plastic part is derived by differentiation of [131], which gives

$$\dot{w}_{\mathrm{p}} = \boldsymbol{\sigma}_0(E_{\mathrm{p}})\dot{E}_{\mathrm{p}}, \quad \dot{E}_{\mathrm{p}} = \frac{e_{\mathrm{p}}}{E_{\mathrm{p}}}\dot{e}_{\mathrm{p}} + l^2\frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}}\dot{e}_{\mathrm{p},k} \qquad [133]$$

The rate $\dot{w}_{\mathrm{p}}$ can be either positive or negative, depending on the sign of $\dot{E}_{\mathrm{p}}$. Furthermore, it can be expressed as

$$\dot{w}_{\mathrm{p}} = \bar{q}\dot{e}_{\mathrm{p}} + \bar{\boldsymbol{\tau}}_k\dot{e}_{\mathrm{p},k} \qquad [134]$$

where the quantities $\bar{q}$ and $\bar{\boldsymbol{\tau}}_k$ are the work conjugates to the plastic strain and strain gradient measures e_{p} and $e_{\mathrm{p},k}$. By comparing [133] and [145], these are

$$\bar{q} = \boldsymbol{\sigma}_0(E_{\mathrm{p}})\frac{e_{\mathrm{p}}}{E_{\mathrm{p}}}, \quad \bar{\boldsymbol{\tau}}_k = l^2\boldsymbol{\sigma}_0(E_{\mathrm{p}})\frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}} \qquad [135]$$

Thus, the total rate of work can be expressed as

$$\dot{w} = \boldsymbol{\sigma}_{ij}\dot{\varepsilon}_{ij}^{\mathrm{e}} + \bar{q}\dot{e}_{\mathrm{p}} + \bar{\boldsymbol{\tau}}_k\dot{e}_{\mathrm{p},k} \qquad [136]$$

11.3 Principle of Virtual Work

In absence of body forces, the principal of virtual work for the strain gradient plasticity reads

$$\int_V (\boldsymbol{\sigma}_{ij}\delta\varepsilon_{ij}^{\mathrm{e}} + q_{ij}\delta\varepsilon_{ij}^{\mathrm{p}} + \boldsymbol{\tau}_{ijk}\delta\varepsilon_{ij,k}^{\mathrm{p}})\mathrm{d}V = \int_S (T_i\delta u_i + t_{ij}\delta\varepsilon_{ij}^{\mathrm{p}})\mathrm{d}S \qquad [137]$$

where the (deviatoric) microstress q_{ij} is the work conjugate to plastic strain $\varepsilon_{ij}^{\mathrm{p}}$, and the moment stress τ_{ijk} is the work conjugate to plastic strain gradient $\varepsilon_{ij,k}^{\mathrm{p}}$. The virtual elastic strain increment is $\delta\varepsilon_{ij}^{\mathrm{e}} = \delta\varepsilon_{ij} - \delta\varepsilon_{ij}^{\mathrm{p}}$. The Gauss divergence theorem applied to [137] yields the equations of equilibrium

$$\boldsymbol{\sigma}_{ij,j} = 0, \quad \boldsymbol{\tau}_{ijk,k} + \boldsymbol{\sigma}_{ij}' - q_{ij} = 0 \qquad [138]$$

and the relations between the traction vector T_i and the Cauchy stress tensor σ_{ij}, and between the (deviatoric) moment traction tensor t_{ij} are the moment stress tensor τ_{ijk},

$$T_i = \boldsymbol{\sigma}_{ij}n_j, \quad t_{ij} = \boldsymbol{\tau}_{ijk}n_k \qquad [139]$$

The components of the outward unit vector, orthogonal to the considered surface element, are denoted by n_i. The rate of internal work per unit volume is

$$\dot{w} = \boldsymbol{\sigma}_{ij}\dot{\varepsilon}^{\mathrm{e}}_{ij} + q_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij} + \boldsymbol{\tau}_{ijk}\dot{\varepsilon}^{\mathrm{p}}_{ij,k} \quad [140]$$

The principle of virtual work can also be expressed in terms of δe_{p} and $\delta e_{\mathrm{p},k}$. Since $\delta\varepsilon^{\mathrm{p}}_{ij} = \delta e_{\mathrm{p}} m_{ij}$ and $\sigma_{\mathrm{eq}} = \sigma'_{ij} m_{ij}$, the virtual work principle [137] can be recast as

$$\int_V (\boldsymbol{\sigma}_{ij}\delta\varepsilon^{\mathrm{e}}_{ij} + \bar{q}\delta e_{\mathrm{p}} + \bar{\boldsymbol{\tau}}_k \delta e_{\mathrm{p},k})\mathrm{d}V = \int_S (T_i \delta u_i + \bar{t}\delta e_{\mathrm{p}})\mathrm{d}S \quad [141]$$

where

$$\bar{t} = t_{ij}m_{ij}, \quad \bar{\boldsymbol{\tau}}_k = \boldsymbol{\tau}_{ijk}m_{ij}, \quad \bar{q} = q_{ij}m_{ij} + \boldsymbol{\tau}_{ijk}m_{ij,k} \quad [142]$$

provided that the equilibrium conditions hold

$$\boldsymbol{\sigma}_{ij,j} = 0, \quad \bar{\boldsymbol{\tau}}_{k,k} + \boldsymbol{\sigma}_{\mathrm{eq}} - \bar{q} = 0 \quad [143]$$

together with

$$T_i = \boldsymbol{\sigma}_{ij}n_j, \quad \bar{t} = \bar{\boldsymbol{\tau}}_k n_k \quad [144]$$

11.4 Helmholtz Free Energy

By considering $\dot{e}_{\mathrm{p}}$ and $\dot{e}_{\mathrm{p},k}$ to be the fluxes whose conjugate thermodynamic forces (affinities) are denoted by f and g_k, the rate of internal energy dissipation due to inelastic deformation processes is

$$\mathcal{D} = f\dot{e}_{\mathrm{p}} + g_k\dot{e}_{\mathrm{p},k} > 0 \quad [145]$$

The rate of Helmholtz free energy under isothermal conditions can then be expressed as $\dot{\psi} = \dot{w} - \mathcal{D}$, i.e.,

$$\dot{\psi} = \dot{w} - f\dot{e}_{\mathrm{p}} - g_k\dot{e}_{\mathrm{p},k} \quad [146]$$

By substituting the rate of work expression [136] into [146] the rate of the free energy becomes

$$\dot{\psi} = \boldsymbol{\sigma}_{ij}\dot{\varepsilon}^{\mathrm{e}}_{ij} + \dot{\psi}_{\mathrm{p}}, \quad \dot{\psi}_{\mathrm{p}} = (\bar{q} - f)\dot{e}_{\mathrm{p}} + (\bar{\boldsymbol{\tau}}_k - g_k)\dot{e}_{\mathrm{p},k} \quad [147]$$

In view of [135], the rate of the plastic part of the free energy becomes

$$\dot{\psi}_{\mathrm{p}} = \left[\boldsymbol{\sigma}_0(E_{\mathrm{p}})\frac{e_{\mathrm{p}}}{E_{\mathrm{p}}} - f\right]\dot{e}_{\mathrm{p}} + \left[l^2\boldsymbol{\sigma}_0(E_{\mathrm{p}})\frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}} - g_k\right]\dot{e}_{\mathrm{p},k} \quad [148]$$

Physically, the free energy in an elastoplastically deformed material consists of the elastic strain energy associated with the overall elastic strain $\varepsilon^{\mathrm{e}}_{ij}$, and the locked-in stored energy around statistically stored and geometrically necessary dislocations. Consequently, it is assumed that the free energy can be expressed as

$$\psi = \psi_{\mathrm{e}}(\varepsilon^{\mathrm{e}}_{ij}) + \psi_{\mathrm{p}}(e_{\mathrm{p}}, E_{\mathrm{p}}) \quad [149]$$

Its rate is then

$$\dot{\psi} = \frac{\partial\psi_{\mathrm{e}}}{\partial\varepsilon^{\mathrm{e}}_{ij}}\dot{\varepsilon}^{\mathrm{e}}_{ij} + \frac{\partial\psi_{\mathrm{p}}}{\partial e_{\mathrm{p}}}\dot{e}_{\mathrm{p}} + \frac{\partial\psi_{\mathrm{p}}}{\partial E_{\mathrm{p}}}\dot{E}_{\mathrm{p}} \quad [150]$$

In view of the expression for $\dot{E}_{\mathrm{p}}$ from [133], the above becomes

$$\dot{\psi} = \boldsymbol{\sigma}_{ij}\dot{\varepsilon}^{\mathrm{e}}_{ij} + \left(\frac{\partial\psi_{\mathrm{p}}}{\partial E_{\mathrm{p}}}\frac{e_{\mathrm{p}}}{E_{\mathrm{p}}} + \frac{\partial\psi_{\mathrm{p}}}{\partial e_{\mathrm{p}}}\right)\dot{e}_{\mathrm{p}} + \left(l^2\frac{\partial\psi_{\mathrm{p}}}{\partial E_{\mathrm{p}}}\frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}}\right)\dot{e}_{\mathrm{p},k} \quad [151]$$

By comparing [148] and [151], there follows

$$f = -\frac{\partial \psi_p}{\partial e_p}, \quad g_k = 0, \quad \frac{\partial \psi_p}{\partial E_p} = \boldsymbol{\sigma}_0(E_p) \qquad [152]$$

These expressions suggest that the plastic part of the free energy can be taken as

$$\psi_p = \int_0^{E_p} \boldsymbol{\sigma}_0(\varepsilon_p)\mathrm{d}\varepsilon_p - \int_0^{e_p} \eta(\varepsilon_p)\boldsymbol{\sigma}_0(\varepsilon_p)\mathrm{d}\varepsilon_p, \quad 0.9 \le \eta(\varepsilon_p) \le 1 \qquad [153]$$

The coefficient $\eta(e_p)$ is specified to be in the indicated range, so that [153] reproduces the locked-in strain energy around dislocations in the case of classical plasticity (without strain gradient effects), which is estimated to be about 10% of the increment of plastic work at the beginning of plastic deformation, decreasing to zero with further progression of plastic deformation (Taylor and Quinney, 1934). If it is assumed that all plastic work in the model of classical plasticity is dissipated ($\eta = 1$), [153] reduces to the expression originally proposed by Hutchinson (2012). In that case, the entire free energy due to plastic deformation is associated with the existence of plastic strain gradients.

With the free energy contribution ψ_p specified by [153], the affinity f conjugate to the effective plastic strain e_p follows from [152] as

$$f = \eta(e_p)\boldsymbol{\sigma}_0(e_p) \qquad [154]$$

11.5 Yield Condition and Plastic Loading Conditions

In the considered framework of the strain gradient plasticity, the yield condition is of the von Mises (J_2) type in the deviatoric Cauchy stress space, $(3/2)\boldsymbol{\sigma}'_{ij}\boldsymbol{\sigma}'_{ij} = \boldsymbol{\sigma}_Y^2$, where σ_Y specifies the radius of the current yield surface. The loading/unloading conditions are

$$\boldsymbol{\sigma}'_{ij}\dot{\varepsilon}_{ij} \begin{cases} \le 0, & \text{elastic unloading } (\dot{e}_p = 0) \\ > 0, & \text{plastic loading } (\dot{e}_p > 0) \end{cases} \qquad [155]$$

The consistency condition during plastic loading is $(3/2)\boldsymbol{\sigma}'_{ij}\dot{\boldsymbol{\sigma}}'_{ij} = \boldsymbol{\sigma}_Y\dot{\boldsymbol{\sigma}}_Y$. The formulation of the entire incremental boundary value problem, based on an appropriate functional with the specified rate potential function and the prescribed boundary conditions on $\dot{u}_i$ and $\dot{e}_p$, is discussed by Hutchinson (2012).

11.6 Recoverable and Dissipative Parts of $\bar{q}$ and $\bar{\tau}_k$

The rate of plastic part of the free energy is the non-dissipated portion of the rate of plastic work, which is given by [147]. This can be rewritten as

$$\dot{\psi}_p = \bar{q}^{\mathrm{rec}}\dot{e}_p + \bar{\boldsymbol{\tau}}_k^{\mathrm{rec}}\dot{e}_{p,k} \qquad [156]$$

where the recoverable potions of $\bar{q}$ and $\bar{\tau}_k$ are

$$\bar{q}^{\mathrm{rec}} = \bar{q} - f = \boldsymbol{\sigma}_0(E_p)\frac{e_p}{E_p} - f, \quad \bar{\boldsymbol{\tau}}_k^{\mathrm{rec}} = \bar{\boldsymbol{\tau}}_k = l^2\boldsymbol{\sigma}_0(E_p)\frac{e_{p,k}}{E_p} \qquad [157]$$

The dissipative parts are

$$\bar{q}^{\mathrm{dis}} = f = \eta(e_p)\boldsymbol{\sigma}_0(e_p), \quad \bar{\boldsymbol{\tau}}_k^{\mathrm{dis}} = 0 \qquad [158]$$

If it is assumed that $\eta = 1$ throughout the plastic deformation, the expressions [157] and [158] reduce to those first proposed by Hutchinson (2012).

The partition of q_{ij} and τ_{ijk} proceeds similarly. Since

$$\dot{\psi}_p = q_{ij}\dot{\varepsilon}_{ij}^p + \boldsymbol{\tau}_{ijk}\dot{\varepsilon}_{ij,k}^p - f\dot{e}_p - g_k\dot{e}_{p,k}, \quad g_k = 0 \qquad [159]$$

and since $\dot{\varepsilon}_{ij}^{\mathrm{p}} = \dot{e}_{\mathrm{p}} m_{ij}$ implies $\dot{e}_{\mathrm{p}} = (2/3) m_{ij} \dot{\varepsilon}_{ij}^{\mathrm{p}}$, there follows

$$\dot{\psi}_{\mathrm{p}} = \left(q_{ij} - \frac{2}{3} f m_{ij} \right) \dot{\varepsilon}_{ij}^{\mathrm{p}} + \boldsymbol{\tau}_{ijk} \dot{\varepsilon}_{ij,k}^{\mathrm{p}} \tag{160}$$

Expressing the rate of plastic part of the free energy as

$$\dot{\psi}_{\mathrm{p}} = q_{ij}^{\mathrm{rec}} \dot{\varepsilon}_{ij}^{\mathrm{p}} + \boldsymbol{\tau}_{ijk}^{\mathrm{rec}} \dot{\varepsilon}_{ij,k}^{\mathrm{p}} \tag{161}$$

the comparison of [160] and [161] yields

$$q_{ij}^{\mathrm{rec}} = q_{ij} - \frac{2}{3} f m_{ij}, \quad \boldsymbol{\tau}_{ijk}^{\mathrm{rec}} = \boldsymbol{\tau}_{ijk} \tag{162}$$

up to their immaterial (workless) parts. The dissipative parts are accordingly

$$q_{ij}^{\mathrm{dis}} = \frac{2}{3} f m_{ij}, \quad \boldsymbol{\tau}_{ijk}^{\mathrm{dis}} = 0 \tag{163}$$

11.7 Yield Surface in q_{ij}^{dis} Space

In view of [163], the plastic yield surface can also be defined in the q_{ij}^{dis} space,

$$\Phi = \left(\frac{3}{2} q_{ij}^{\mathrm{dis}} q_{ij}^{\mathrm{dis}} \right)^{1/2} - f = 0, \quad f = \eta(e_{\mathrm{p}}) \boldsymbol{\sigma}_0(e_{\mathrm{p}}) \tag{164}$$

such that the plastic strain rate obeys the normality rule

$$\dot{\varepsilon}_{ij}^{\mathrm{p}} = \dot{e}_{\mathrm{p}} \frac{\partial \Phi}{\partial q_{ij}^{\mathrm{dis}}} \tag{165}$$

Since for the hardening material, $\sigma_0 = \sigma_0(e_{\mathrm{p}})$ is a monotonically increasing function, the yield surface [164] expands isotropically in the q_{ij}^{dis} space during plastic deformation, similarly as the yield surface $\sigma_{\mathrm{eq}} = \sigma_0(e_{\mathrm{p}})$ expands in the Cauchy stress space in the case of classical von Mises plasticity (Fleck *et al.*, 2014).

11.8 Proportional Loading

In the case of proportional loading, all stress components increase in proportion to a single scalar parameter, and the rate-type analysis simplifies to the strain gradient J_2 deformation theory of plasticity. The total plastic strain is itself codirectional with the deviatoric stress,

$$\varepsilon_{ij}^{\mathrm{p}}(t) = e_{\mathrm{p}}(t) m_{ij}, \quad m_{ij} = \frac{3}{2} \frac{\boldsymbol{\sigma}'_{ij}}{\boldsymbol{\sigma}_{\mathrm{eq}}} \tag{166}$$

while the effective plastic strain can be expressed as

$$e_{\mathrm{p}}(t) = \left(\frac{2}{3} \varepsilon_{ij}^{\mathrm{p}} \varepsilon_{ij}^{\mathrm{p}} \right)^{1/2} > 0 \tag{167}$$

It readily follows that

$$q_{ij} = \frac{2}{3} \boldsymbol{\sigma}_0(E_{\mathrm{p}}) \frac{e_{\mathrm{p}}}{E_{\mathrm{p}}} m_{ij}, \quad \boldsymbol{\tau}_{ijk} = \frac{2}{3} l^2 \boldsymbol{\sigma}_0(E_{\mathrm{p}}) \frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}} m_{ij} \tag{168}$$

and

$$\bar{q} = q_{ij} m_{ij} = \boldsymbol{\sigma}_0(E_{\mathrm{p}}) \frac{e_{\mathrm{p}}}{E_{\mathrm{p}}}, \quad \bar{\boldsymbol{\tau}}_k = \boldsymbol{\tau}_{ijk} m_{ij} = l^2 \boldsymbol{\sigma}_0(E_{\mathrm{p}}) \frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}} \tag{169}$$

In the variational formulation, the actual solution (u_i, e_p) minimizes the potential energy functional Hutchinson (2012)

$$\pi(u_i, e_\mathrm{p}) = \int_V [w_\mathrm{e}(\varepsilon_{ij}^\mathrm{e}) + w_\mathrm{p}(e_\mathrm{p}, e_{\mathrm{p},k})]\mathrm{d}V - \int_{S_T} (T_i u_i + \bar{t} e_\mathrm{p})\mathrm{d}S \qquad [170]$$

where T_i and $\bar{t}$ are prescribed on the portion S_T of the boundary, $w_\mathrm{e}(\varepsilon_{ij}^\mathrm{e})$ is defined by [132], and $w_\mathrm{p}(e_\mathrm{p}, e_{\mathrm{p},k})$ by [131]. If e_p is unconstrained on the portion of the boundary, then $\bar{t} = 0$ over that portion; if e_p is constrained to be zero, $\bar{t}$ will generally be non-zero. Further analysis of the boundary conditions and the conditions at the interface between elastically and plastically deformed parts of the body can be found in Gudmundson (2004), Fleck and Willis (2009), and Hutchinson (2012).

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