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A note on the compatibility equations for three-dimensional axisymmetric problems

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Abstract
For three-dimensional axisymmetric problems, the six compatibility equations for infinitesimal strains reduce to four, but these can be further reduced to only three equations. This is illustrated for both the Saint-Venant and the Beltrami–Michell compatibility equations. As a consequence, there is only one nontrivial differential relationship among the three compatibility equations for three-dimensional axisymmetric problems.

Keywords
axisymmetric problems, Beltrami–Michell equations, Bianchi identities, cylindrical coordinates, elasticity, Love's potential, Saint-Venant's compatibility equations

1. Introduction

The six Saint-Venant's compatibility equations for the six infinitesimal strain components in cylindrical coordinates (r, θ, z) are (e.g., Lurie [1, p. 33] and Saada [2, p. 142])

$$S_{rr} = \frac{1}{r^2}\frac{\partial^2 \epsilon_{zz}}{\partial \theta^2} + \frac{\partial^2 \epsilon_{\theta\theta}}{\partial z^2} - \frac{2}{r}\frac{\partial^2 \epsilon_{\theta z}}{\partial \theta \partial z} - \frac{2}{r}\frac{\partial \epsilon_{zr}}{\partial z} + \frac{1}{r}\frac{\partial \epsilon_{zz}}{\partial r} = 0, \tag{1}$$

$$S_{\theta\theta} = \frac{\partial^2 \epsilon_{zz}}{\partial r^2} + \frac{\partial^2 \epsilon_{rr}}{\partial z^2} - 2\frac{\partial^2 \epsilon_{zr}}{\partial z \partial r} = 0, \tag{2}$$

$$S_{zz} = \frac{1}{r^2}\frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \epsilon_{\theta\theta}}{\partial r^2} - \frac{2}{r}\frac{\partial^2 \epsilon_{r\theta}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \epsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \epsilon_{\theta\theta}}{\partial r} - \frac{2}{r^2}\frac{\partial \epsilon_{r\theta}}{\partial \theta} = 0, \tag{3}$$

$$S_{r\theta} = -\frac{1}{r}\frac{\partial^2 \epsilon_{zz}}{\partial r \partial \theta} + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \epsilon_{zr}}{\partial \theta} - \frac{\partial \epsilon_{\theta z}}{\partial r} + \frac{\partial \epsilon_{r\theta}}{\partial z}\right) - \frac{1}{r}\frac{\partial \epsilon_{\theta z}}{\partial z} + \frac{1}{r^2}\frac{\partial \epsilon_{zz}}{\partial \theta} = 0, \tag{4}$$

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$$S_{\theta z} = -\frac{1}{r}\frac{\partial^2 \epsilon_{rr}}{\partial\theta\partial z} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \epsilon_{zr}}{\partial\theta} - \frac{\partial \epsilon_{r\theta}}{\partial z} + \frac{\partial \epsilon_{\theta z}}{\partial r}\right) + \frac{2}{r}\frac{\partial \epsilon_{r\theta}}{\partial z} - \frac{1}{r}\frac{\partial \epsilon_{\theta z}}{\partial r} + \frac{\epsilon_{\theta z}}{r^2} = 0, \tag{5}$$

$$S_{zr} = -\frac{\partial^2 \epsilon_{\theta\theta}}{\partial z\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial \epsilon_{r\theta}}{\partial z} - \frac{\partial \epsilon_{\theta z}}{\partial r} + \frac{1}{r}\frac{\partial \epsilon_{zr}}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial \epsilon_{\theta z}}{\partial\theta} + \frac{1}{r}\frac{\partial(\epsilon_{rr} - \epsilon_{\theta\theta})}{\partial z} = 0. \tag{6}$$

The components of the incompatibility tensor $\mathbf{S} = \nabla \times (\nabla \times \boldsymbol{\epsilon})$ (which vanish for compatible strains) satisfy the three Bianchi-type identities from Riemannian geometry. These follow from the condition of the divergence-free incompatibility tensor ($\nabla \cdot \boldsymbol{S} = 0$), which gives

$$\begin{aligned} &\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial\theta} + \frac{\partial S_{zr}}{\partial z} + \frac{S_{rr} - S_{\theta\theta}}{r} = 0, \\ &\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + \frac{\partial S_{\theta z}}{\partial z} + \frac{2S_{r\theta}}{r} = 0, \\ &\frac{\partial S_{zr}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial\theta} + \frac{\partial S_{zz}}{\partial z} + \frac{S_{zr}}{r} = 0. \end{aligned} \tag{7}$$

Expressions (7) correspond to their Cartesian coordinates representations listed, for example, in Malvern [3, p. 187].

In the case of linearly elastic materials, the Beltrami–Michell compatibility equations expressed in terms of stresses are obtained from (1)–(6) by using the generalized Hooke's law and the Cauchy equilibrium equations. In the absence of body forces, these compatibility equations read (Lurie [1, p. 54])

$$\nabla^2\sigma_{rr} - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) - \frac{4}{r^2}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{1+\nu}\frac{\partial^2\sigma}{\partial r^2} = 0, \tag{8}$$

$$\nabla^2\sigma_{\theta\theta} + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{4}{r^2}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{1+\nu}\left(\frac{1}{r}\frac{\partial\sigma}{\partial r} + \frac{1}{r^2}\frac{\partial^2\sigma}{\partial\theta^2}\right) = 0, \tag{9}$$

$$\nabla^2\sigma_{zz} + \frac{1}{1+\nu}\frac{\partial^2\sigma}{\partial z^2} = 0, \tag{10}$$

$$\nabla^2\sigma_{r\theta} + \frac{2}{r^2}\frac{\partial}{\partial\theta}(\sigma_{rr} - \sigma_{\theta\theta}) - \frac{4}{r^2}\sigma_{r\theta} + \frac{1}{1+\nu}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\sigma}{\partial\theta}\right) = 0, \tag{11}$$

$$\nabla^2\sigma_{\theta z} + \frac{2}{r^2}\frac{\partial\sigma_{rz}}{\partial\theta} - \frac{1}{r^2}\sigma_{\theta z} + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial^2\sigma}{\partial\theta\partial z} = 0, \tag{12}$$

$$\nabla^2\sigma_{zr} - \frac{2}{r^2}\frac{\partial\sigma_{\theta z}}{\partial\theta} - \frac{1}{r^2}\sigma_{zr} + \frac{1}{1+\nu}\frac{\partial^2\sigma}{\partial z\partial r} = 0, \tag{13}$$

where ν is Poisson's ratio, $\sigma = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}$ is the first stress invariant, and the Laplacian operator ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}. \tag{14}$$

2. Compatibility equations for axisymmetric problems

For three-dimensional axisymmetric problems, the stress, strain, and displacement components do not depend on the azimuthal angle θ, the displacement component u_θ is equal to zero, and the shear strain components $\epsilon_{r\theta}$ and $\epsilon_{z\theta}$, and the shear stress components $\sigma_{r\theta}$ and $\sigma_{z\theta}$ all vanish. The nonvanishing strains are related to displacements by

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right). \tag{15}$$

Consequently, the six Saint-Venant compatibility equations (1)–(6) reduce to the following four equations,

$$S_{rr} = \frac{\partial^2 \epsilon_{\theta\theta}}{\partial z^2} - \frac{2}{r}\frac{\partial \epsilon_{zr}}{\partial z} + \frac{1}{r}\frac{\partial \epsilon_{zz}}{\partial r} = 0, \tag{16}$$

$$S_{\theta\theta} = \frac{\partial^2 \epsilon_{zz}}{\partial r^2} + \frac{\partial^2 \epsilon_{rr}}{\partial z^2} - 2\frac{\partial^2 \epsilon_{zr}}{\partial z \partial r} = 0, \tag{17}$$

$$S_{zz} = \frac{\partial^2 \epsilon_{\theta\theta}}{\partial r^2} - \frac{1}{r}\frac{\partial \epsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \epsilon_{\theta\theta}}{\partial r} = 0, \tag{18}$$

$$S_{zr} = -\frac{\partial^2 \epsilon_{\theta\theta}}{\partial z \partial r} + \frac{1}{r}\frac{\partial(\epsilon_{rr} - \epsilon_{\theta\theta})}{\partial z} = 0. \tag{19}$$

The three Bianchi identities (7) reduce to two identities

$$\begin{aligned} &\frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{zr}}{\partial z} + \frac{S_{rr} - S_{\theta\theta}}{r} = 0, \\ &\frac{\partial S_{zr}}{\partial r} + \frac{\partial S_{zz}}{\partial z} + \frac{S_{zr}}{r} = 0. \end{aligned} \tag{20}$$

Similarly, the six Beltrami–Michell compatibility equations (8)–(13) reduce to four equations,

$$\nabla^2 \sigma_{rr} - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial r^2} = 0, \tag{21}$$

$$\nabla^2 \sigma_{\theta\theta} + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial \sigma}{\partial r} = 0, \tag{22}$$

$$\nabla^2 \sigma_{zz} + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial z^2} = 0, \tag{23}$$

$$\nabla^2 \sigma_{zr} - \frac{1}{r^2}\sigma_{zr} + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial r \partial z} = 0. \tag{24}$$

Equations (21)–(24) correspond to Equations (16)–(19) in the exact order, and can be derived from them by using the stress–strain relations for linearly elastic isotropic materials, such as

$$\epsilon_{rr} = \frac{1}{E}[(1+\nu)\sigma_{rr} - \nu\sigma], \quad \epsilon_{\theta\theta} = \frac{1}{E}[(1+\nu)\sigma_{\theta\theta} - \nu\sigma], \quad \epsilon_{zr} = \frac{1+\nu}{E}\sigma_{zr}, \tag{25}$$

and the Cauchy equilibrium equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial \sigma_{zr}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} = 0. \tag{26}$$

In particular, by summing the first three equations in (21)–(24), we obtain the well-known result that, in the absence of body forces, σ is a harmonic function ($\nabla^2 \sigma = 0$).

3. Reduction to three compatibility equations

The last two of the compatibility equations in (16)–(19) can be rewritten as

$$S_{zz} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \epsilon_{\theta\theta}}{\partial r} + \epsilon_{\theta\theta} - \epsilon_{rr}\right) = 0, \tag{27}$$

$$S_{zr} = -\frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial \epsilon_{\theta\theta}}{\partial r} + \epsilon_{\theta\theta} - \epsilon_{rr}\right) = 0, \tag{28}$$

which require that the expression within the brackets in (27) and (28) must be constant. This constant is equal to zero, because, from (15),

$$u_r = r\epsilon_{\theta\theta} \quad \Rightarrow \quad \frac{\partial u_r}{\partial r} = \epsilon_{\theta\theta} + r\frac{\partial \epsilon_{\theta\theta}}{\partial r}. \tag{29}$$

Thus, since $\epsilon_{rr} = \partial u_r/\partial r$, (29) gives

$$r\frac{\partial \epsilon_{\theta\theta}}{\partial r} + \epsilon_{\theta\theta} - \epsilon_{rr} = 0. \tag{30}$$

This is the compatibility equation for axisymmetric problems which replaces both (27) and (28), because if the strain components fulfill (30), they certainly fulfill (27) and (28). The same type of compatibility condition applies to axially symmetric two-dimensional problems, e.g., Wang [4, p. 68] and Barber [5, p. 432].

As a consequence, the four Saint-Venant's compatibility equations (16)–(19) reduce to only three compatibility equations,

$$S_{rr} = \frac{\partial^2 \epsilon_{\theta\theta}}{\partial z^2} - \frac{2}{r}\frac{\partial \epsilon_{zr}}{\partial z} + \frac{1}{r}\frac{\partial \epsilon_{zz}}{\partial r} = 0, \tag{31}$$

$$S_{\theta\theta} = \frac{\partial^2 \epsilon_{zz}}{\partial r^2} + \frac{\partial^2 \epsilon_{rr}}{\partial z^2} - 2\frac{\partial^2 \epsilon_{zr}}{\partial z \partial r} = 0, \tag{32}$$

$$T = r\frac{\partial \epsilon_{\theta\theta}}{\partial r} + \epsilon_{\theta\theta} - \epsilon_{rr} = 0. \tag{33}$$

We used the symbol T to designate the expression on the left-hand side of (30).

The second of the Bianchi identities in (20) is trivially fulfilled, whereas the first one becomes

$$\frac{\partial S_{rr}}{\partial r} + \frac{S_{rr} - S_{\theta\theta}}{r} - \frac{1}{r}\frac{\partial^2 T}{\partial z^2} = 0. \tag{34}$$

This is the only nontrivial differential relationship among the compatibility equations for three-dimensional axisymmetric problems, because there are two equilibrium equations (26) and three compatibility equations (31)–(33), which involve only four stress components (σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, and σ_{zr}). For the sake of comparison, it may be recalled that in two-dimensional plane strain problems, there are two equilibrium equations and one compatibility equation ($S_{zz} = 0$) for three inplane stress components, so that all Bianchi-type identities are trivially fulfilled.

3.1. Reduction to three compatibility equations in terms of stresses

As the last two Beltrami–Michell compatibility equations in (21)–(24) follow from the last two of the Saint-Venant compatibility equations (16)–(19), and since (18) and (19) were replaced with a single equation (33), Equations (23) and (24) can be replaced with a single equation as well. This equation follows from (33) by using Hooke's law and the Cauchy equilibrium equations. Indeed, by substituting, from (25), the stress–strain relationships

$$\epsilon_{\theta\theta} = \frac{1}{E}\left[(1+\nu)\sigma_{\theta\theta} - \nu\sigma\right], \quad \epsilon_{rr} - \epsilon_{\theta\theta} = \frac{1+\nu}{E}(\sigma_{rr} - \sigma_{\theta\theta}) \tag{35}$$

into (33), and by observing that from the first of the equilibrium equations in (26),

$$\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{zr}}{\partial z}\right), \tag{36}$$

we obtain

$$(1+\nu)\left(\frac{\partial \sigma_{\theta\theta}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{zr}}{\partial z}\right) = \nu \frac{\partial \sigma}{\partial r}. \tag{37}$$

Finally, by expressing the hoop stress as $\sigma_{\theta\theta} = \sigma - \sigma_{rr} - \sigma_{zz}$, Equation (37) becomes

$$\frac{\partial \sigma_{zr}}{\partial z} - \frac{\partial \sigma_{zz}}{\partial r} + \frac{1}{1+\nu} \frac{\partial \sigma}{\partial r} = 0, \tag{38}$$

which is a desired form of the compatibility equation.

In summary, the set of compatibility equations expressed in terms of stresses for three-dimensional axisymmetric elasticity problems consists of the following three equations

$$\nabla^2 \sigma_{rr} - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial r^2} = 0, \tag{39}$$

$$\nabla^2 \sigma_{\theta\theta} + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial \sigma}{\partial r} = 0, \tag{40}$$

$$\frac{\partial \sigma_{zr}}{\partial z} - \frac{\partial \sigma_{zz}}{\partial r} + \frac{1}{1+\nu} \frac{\partial \sigma}{\partial r} = 0. \tag{41}$$

In the well-known book by Sneddon [6, p. 451], only Equations (39) and (40) are listed as the compatibility relations in cylindrical coordinates for axisymmetric problems. Timoshenko and Goodier [7, p. 388], refer to all six compatibility equations, three containing only normal stresses and three containing shear stresses as well, but they do not discuss their reduction to three equations in the case of axisymmetry.

If the stress components are expressed in terms of Love's potential function $\Omega = \Omega(r, z)$ (e.g., [7, 8]) as

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu \nabla^2 \Omega - \frac{\partial^2 \Omega}{\partial r^2}\right), \quad \sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu \nabla^2 \Omega - \frac{1}{r}\frac{\partial \Omega}{\partial r}\right),$$
$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 \Omega - \frac{\partial^2 \Omega}{\partial z^2}\right], \quad \sigma_{zr} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 \Omega - \frac{\partial^2 \Omega}{\partial z^2}\right], \tag{42}$$

it readily follows that the first of the equilibrium equations in (26) is identically satisfied, whereas the second equilibrium equation requires that $\nabla^4 \Omega = 0$. Likewise, the compatibility equation (41) is identically satisfied by (42), whereas the compatibility equations (39) and (40) are satisfied provided that $\nabla^4 \Omega = 0$, as already required by the second equilibrium equation in (26). In a classic book of lasting value by Sneddon [6, p. 452], there is an inaccurate statement that the stress expressions in terms of Love's potential automatically satisfy the Cauchy equilibrium equations, whereas the biharmonic equation follows from the Beltrami–Michell compatibility equations; cf. Timoshenko and Goodier [7, p. 381].

4. Conclusions

The set of three compatibility equations in terms of infinitesimal strains has been derived for three-dimensional axisymmetric problems. This set is accompanied by only one differential relationship among the three compatibility equations. The corresponding set of three compatibility equations for linearly elastic isotropic materials in terms of stresses has also been derived and discussed. The presented analysis is believed to be of interest from both conceptual and pedagogical points of view.

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