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On the Partition of the Strain–Rate into Elastic and Plastic Parts and Corresponding Constitutive Expressions

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By using Lee's intermediate configuration, partition of the strain–rate into elastic and plastic parts is analysed, associated with the Jaumann and convected rates of stress, as well as the corresponding constitutive structures.

1. Introduction

Let E be an objective, symmetric strain tensor and T its conjugate stress, $T : dE$ being increment of work per unit volume in the reference state from where E is measured (: denotes the trace), [HR]. Let further $\mathcal{L}^e$ be the corresponding instanteneous elastic moduli tensor, such that $\mathcal{L}^e : dE$ is the stress increment that would result if the response on arbitrary strain variation dE were purely elastic. With $\mathcal{M}^e = \mathcal{L}^{e\,-1}$ as the instanteneous elastic compliance tensor, $\mathcal{M}^e : dT$ is the strain increment that would result from purely elastic response corresponding to stress increment dT, and subtraction from the actual strain increment dE gives the plastic part $d^pE = dE - \mathcal{M}^e : dT$. This is the residual strain increment in an infinitesimal loading–unloading cycle of stress T, which can be shown to be governed by the plastic potential, i.e. codirectional with the outward normal to a locally smooth yield surface in stress T space. For example, by identifying E as the logarithmic strain, and with current state as a reference: $T = \sigma$ (Cauchy stress), $dE = D\,dt$ (D is the strain–rate and t a timelike parameter), and $dT = \overset{\circ}{\tau}\,dt$, where $\overset{\circ}{\tau} = \overset{\circ}{\sigma} + (\operatorname{tr} D)\sigma$ is the Jaumann rate of the Kirchhoff stress $\tau = (\det F)\sigma$ at current state as a reference (deformation gradient F being identity, $F \equiv I$). Plastic contribution D^p to total strain-rate D is therefore

$$D^p = D - \mathcal{M}^e : \overset{\circ}{\tau}. \tag{1}$$

With different choice of conjugate variables and, therefore, different objective stress-rate, elastic compliance changes and what appears to be elastic and plastic strain-rates depends on the chosen stress-rate. Indeed, let E be the Lagrange strain, the conjugate stress is then symmetric Piola-Kirchhoff stress, and by taking the reference state coincident with the current state, we have: $dE = D\,dt$ and $dT = \overset{\square}{\tau}\,dt$, where $\overset{\square}{\tau} = \overset{\circ}{\tau} - D\sigma - \sigma D$ is convected derivative of τ at $F = I$. Hence, in this case

$$\tilde{D}^p = D - \tilde{\mathcal{M}}^e : \overset{\square}{\tau}, \tag{2}$$

To obtain, in a consistent manner, instanteneous elastic compliance ($\mathcal{M}^e$ or $\tilde{\mathcal{M}}^e$), the finite elasticity law

$$\tau = 2F^e \frac{\partial w(C^e)}{\partial C^e} F^{e\,T} \tag{3}$$

can conveniently be used. F^e is the elastic part of total deformation gradient, corresponding to elastic stressing from intermediate (stress free, relaxed) to current configuration, as given by [Lee] multiplicative decomposition

$$F = F^e F^p, \tag{4}$$

F^p denoting the plastic part. In (3) plastic incompressibility is assumed, C^e is the right Cauchy-Green deformation tensor, while w is the strain energy per unit unstressed volume. For example, assuming elastic isotropy, which remains preserved during elasto-plastic deformation, w is an isotropic function of C^e. By using (4), the velocity gradient $L = \dot{F}F^{-1}$ can be expressed as

$$L = \dot{F}^e F^{e\,-1} + F^e(\dot{F}^p F^{p\,-1})F^{e\,-1}, \tag{5}$$

and by taking symmetric and antisymmetric parts, the velocity strain and spin tensors are:

$$D = (\dot{F}^e F^{e\,-1})_s + [F^e(\dot{F}^p F^{p\,-1})F^{e\,-1}]_s, \tag{6}$$

$$W = (\dot{F}^e F^{e\,-1})_a + [F^e(\dot{F}^p F^{p\,-1})F^{e\,-1}]_a. \tag{7}$$

Identification (separation) of elastic and plastic contributions on the right hand side of (6) has been the subject of many papers. We adress this issue here from two perspectives: based on the Jaumann and convected derivatives used to define elastic strain rate.

2. Analysis based on the Jaumann derivative

As discussed in [Lub] any intermediate configuration can be used in the analysis. Often utilized choice is based on destressing without rotation ([Lee], [LuL]). A convenient choice is also obtained by destressing such that

$$[F^e(\dot{F}^p F^{p\,-1})F^{e\,-1}]_a = 0, \tag{8}$$

as it turns out that in this case elastic and plastic parts of D are:

$$D^e = (\dot{F}^e F^{e\,-1})_s, \tag{9}$$

$$D^p = F^e(\dot{F}^p F^{p\,-1})F^{e\,-1}. \tag{10}$$

Indeed, by differentiating (3)

$$\dot{\tau} = (\dot{F}^e F^{e\,-1})\sigma + \sigma(F^{e\,-T}\dot{F}^{e\,T}) + \frac{2}{\det F^e} F^e [\frac{\partial^2 w(C^e)}{\partial C^e \otimes \partial C^e} : \dot{C}^e] F^{e\,T}, \qquad (11)$$

we recognize (9), as equation (11) can then be written as

$$\overset{\circ}{\tau} = D^e \sigma + \sigma D^e + \frac{4}{\det F^e} F^e [\frac{\partial^2 w(C^e)}{\partial C^e \otimes \partial C^e} : (F^{e\,T} D^e F^e)] F^{e\,T} \equiv \mathcal{L}^e : D^e. \quad (12)$$

This identifies elastic moduli tensor $\mathcal{L}^e$, or its inverse $\mathcal{M}^e$, appearing in (1) and corresponding to Jaumann derivative $\overset{\circ}{\tau}$. Note that for any other choice of intermediate configuration (not defined by (8)), symmetric part of $\dot{F}^e F^{e\,-1}$ is not all, but just a portion of the elastic strain-rate D^e. Elastic moduli $\mathcal{L}^e$ and elastic strain-rate D^e are, of course, independent of the selected intermediate configuration . These issues are in detail discussed in [Lub] .

3. Analysis based on the convected derivative

3.1 Geometric preliminaries

Introduce the convected coordinates θ^i, superscripted indices denote contravariant components and subscripted the covariant components [Hut]. Denote base vectors in the undeformed state by $\mathbf{g}_i$ and their reciprocals by $\mathbf{g}^i$. The base vectors in the deformed (current and intermediate) states are denoted by $\bar{\mathbf{g}}_i$ and $\hat{\mathbf{g}}_i$, respectively, their reciprocals being $\bar{\mathbf{g}}^i$ and $\hat{\mathbf{g}}^i$. The metric tensors of the three configurations are therefore: $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$, $\bar{g}_{ij} = \bar{\mathbf{g}}_i \cdot \bar{\mathbf{g}}_j$ and $\hat{g}_{ij} = \hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_j$, and let g^{ij}, $\bar{g}^{ij}$ and $\hat{g}^{ij}$ be their respective inverses. From the definition of the intermediate configuration: $\bar{\mathbf{g}}_i = F\mathbf{g}_i = F^e\hat{\mathbf{g}}_i$ and $\hat{\mathbf{g}}_i = F^p\mathbf{g}_i$.

The Lagrangian (η) strain tensor is defined by:

$$\eta = \eta_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \qquad \eta_{ij} = \mathbf{g}_i \cdot \eta \cdot \mathbf{g}_j = \frac{1}{2}(\bar{g}_{ij} - g_{ij}) \qquad (13)$$

which also can be expressed via deformation tensor $C = F^T F$ as $\eta = \frac{1}{2}(C - I)$. Now, if dS, ds and $d\hat{s}$ are the lengths of an infinitesimal material element in the initial, current and intermediate state, it follows:

$$ds^2 - dS^2 = (\bar{g}_{ij} - g_{ij})d\theta^i\, d\theta^j = (\bar{g}_{ij} - \hat{g}_{ij})d\theta^i\, d\theta^j + (\hat{g}_{ij} - g_{ij})d\theta^i\, d\theta^j. \quad (14)$$

Consequently, we define elastic and plastic parts of Lagrange strain by:

$$\begin{aligned} \eta^e &= \eta^e_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j, \qquad & \eta^e_{ij} &= \frac{1}{2}(\bar{g}_{ij} - \hat{g}_{ij}) \\ \eta^p &= \eta^p_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j, \qquad & \eta^p_{ij} &= \frac{1}{2}(\hat{g}_{ij} - g_{ij}) \end{aligned} \quad (15)$$

and from (13) and (15), $\eta = \eta^e + \eta^p$. On the other hand

$$ds^2 - d\hat{s}^2 = d\mathbf{x} \cdot d\mathbf{x} - d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}} = d\hat{\mathbf{x}} \cdot (C^e - I) \cdot d\hat{\mathbf{x}} = \hat{\mathbf{g}}_i \cdot (C^e - I) \cdot \hat{\mathbf{g}}_j\, d\theta^i\, d\theta^j, \quad (16)$$

hence, in view of $ds^2 - d\hat{s}^2 = 2\eta^e_{ij}\, d\theta^i\, d\theta^j$,

$$\eta^e_{ij} = \hat{\mathbf{g}}_i \cdot \hat{\eta}^e \cdot \hat{\mathbf{g}}_j, \qquad \hat{\eta}^e = \frac{1}{2}(C^e - I). \quad (17)$$

Expressing $\hat{\eta}^e$ via its components on $\hat{\mathbf{g}}^i$ reciprocal basis

$$\hat{\eta}^e = \hat{\eta}^e_{ij}\, \hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j = \frac{1}{2}(C^e_{ij} - \hat{g}_{ij})\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j \quad (18)$$

and comparision with (17) gives

$$\eta^e_{ij} \equiv \hat{\eta}^e_{ij} = \frac{1}{2}(C^e_{ij} - \hat{g}_{ij}). \quad (19)$$

Introducing further the strain–rate D, it follows that

$$\bar{\mathbf{g}}_i \cdot D \cdot \bar{\mathbf{g}}_j \equiv \mathbf{g}_i \cdot \dot{\eta} \cdot \mathbf{g}_j = \dot{\eta}_{ij} \quad (20)$$

and, therefore, by writing $D = D_{ij}\, \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j$, we have $D_{ij} \equiv \dot{\eta}_{ij}$. Substitution of $D = \tilde{D}^e + \tilde{D}^p$ into (20) then gives

$$\bar{\mathbf{g}}_i \cdot (\tilde{D}^e + \tilde{D}^p) \cdot \bar{\mathbf{g}}_j = \dot{\eta}^e_{ij} + \dot{\eta}^p_{ij}. \tag{21}$$

In the following subsection we shall show that elastic and plastic strain-rates , corresponding to convected stress rate $\overset{\square}{\tau}$, turn out to be:

$$\tilde{D}^e = \dot{\eta}^e_{ij}\, \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j \tag{22}$$

$$\tilde{D}^p = \dot{\eta}^p_{ij}\, \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j. \tag{23}$$

3.2 Constitutive expression for elastic strain-rate

The elastic part of deformation gradient F^e can be expressed in dyadic form with respect to intermediate basis $\hat{\mathbf{g}}_i$ as

$$F^e = (F^e)^{ij}\, \hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j, \tag{24}$$

and substitution into (3) gives, after easy manouvers,

$$J\sigma^{ij}\, \bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j = 2\frac{\partial w}{\partial C^e_{ij}} \bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j, \qquad \text{i.e.} \qquad J\sigma^{ij} = 2\frac{\partial w}{\partial C^e_{ij}} \tag{25}$$

with J standing for $\det F$, while σ^{ij} are the contravariant components of the Cauchy stress on the current basis $\bar{\mathbf{g}}_i$. Utilizing Eq. (19), above can be rewritten in the form

$$J\sigma^{ij} = \frac{\partial w}{\partial \eta^e_{ij}} \tag{26}$$

and differentation gives

$$(\operatorname{tr} D)\sigma^{ij} + \dot{\sigma}^{ij} = \frac{1}{J} \frac{\partial w^2}{\partial \eta^e_{ij} \partial \eta^e_{kl}} \dot{\eta}^e_{kl}. \tag{27}$$

differentiating the Kirchhoff stress Next, by

$$\tau = \tau^{ij}\,\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j \tag{28}$$

and by using $\dot{\bar{\mathbf{g}}}_i = L \cdot \bar{\mathbf{g}}_i$, $L = \dot{F}F^{-1} = \dot{\bar{\mathbf{g}}}_k \otimes \bar{\mathbf{g}}^k$ being the velocity gradient, we obtain

$$\dot{\tau}^{ij}\,\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j = \dot{\tau} - L\tau - \tau L^T, \tag{29}$$

which is the convected derivative of the Kirchhoff stress, associated with the choice $F = I$. Hence, we recognize (22), since expression (27) then becomes

$$\overset{\square}{\tau} = \tilde{\mathcal{L}}^e : \tilde{D}^e, \qquad (\tilde{\mathcal{L}}^e)^{ijkl} = \frac{1}{J}\frac{\partial^2 w}{\partial \eta^e_{ij}\partial \eta^e_{kl}} \tag{30}$$

or by inversion

$$\tilde{D}^e = \tilde{\mathcal{M}}^e : \overset{\square}{\tau}, \qquad \tilde{\mathcal{M}}^e = \tilde{\mathcal{L}}^{e\,-1}. \tag{31}$$

This establishes the elasic compliance tensor appearing in (2), corresponding to convected derivative $\overset{\square}{\tau}$. If elastic component of strain is small, $\mathcal{L}^e \approx \tilde{\mathcal{L}}^e$ and $\mathcal{M}^e \approx \tilde{\mathcal{M}}^e$.

4. Discussion

Having established constitutive expression for the elastic strain–rate, one needs to derive expression for the associated plastic strain–rate. When the Jaumann derivative $\overset{\circ}{\tau}$ is utilized, developement goes in a straightforward manner. For example, in the case of isotropic hardening and Mises yield condition, one obtains

$$D^p = \dot{\lambda}\sigma', \qquad \dot{\lambda} = \frac{1}{h}\sigma' : \overset{\circ}{\tau} \tag{32}$$

(h is a hardening parameter), which expresses the normality to Mises yield surface and coaxiality of D^p with the deviatoric part of stress σ'. Rewriting (32) in the form $D^p = \mathcal{M}^p : \overset{\circ}{\tau}$ and combining with $D^e = \mathcal{M}^e : \overset{\circ}{\tau}$ one arrives at the well known elasto– plastic constitutive structure, such as for example given in [MR]. However, when the convected derivative $\overset{\square}{\tau}$ is utilized in the partition of the strain-rate into corresponding elastic and plastic parts, an explicit developement of the constitutive expression for strain-rate $\tilde{D}^p$ becomes more difficult. For example, $\tilde{D}^p$ is not normal to Mises yield surface, so that modification of the associated flow rule or the yield condition is needed in order to obtain a physically plausible constitutive law. Of course, the transition from the final elasto-plastic constitutive structure with the Jaumann derivative $\overset{\circ}{\tau} = \mathcal{L} : D$, to one with the convected derivative is direct via $\overset{\circ}{\tau} = \overset{\square}{\tau} + D\sigma + \sigma D$.