TOPICS IN PLASTICITY

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Professor E. H. Lee

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Wei H. Yang
Co-author, Editor

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By using Lee's intermediate configuration, partition of the strain-rate into elastic and plastic parts is analysed, associated with the Jaumann and convected rates of stress, as well as the corresponding constitutive structures.

1. Introduction

Let $E$ be an objective, symmetric strain tensor and $T$ its conjugate stress, $T : dE$ being increment of work per unit volume in the reference state from where $E$ is measured ($:$ denotes the trace), [HR]. Let further $L^e$ be the corresponding instantaneous elastic moduli tensor, such that $L^e : dE$ is the stress increment that would result if the response on arbitrary strain variation $dE$ were purely elastic. With $M^e = L^e^{-1}$ as the instantaneous elastic compliance tensor, $M^e : dT$ is the strain increment that would result from purely elastic response corresponding to stress increment $dT$, and subtraction from the actual strain increment $dE$ gives the plastic part $d^p E = dE - M^e : dT$. This is the residual strain increment in an infinitesimal loading-unloading cycle of stress $T$, which can be shown to be governed by the plastic potential, i.e. codirectional with the outward normal to a locally smooth yield surface in stress $T$ space. For example, by identifying $E$ as the logarithmic strain, and with current state as a reference: $T = \sigma$ (Cauchy stress), $dE = D \, dt$ ($D$ is the strain-rate and $t$ a timelike parameter), and $dT = \dot{T} \, dt$, where $\dot{T} = \dot{\sigma} + (\text{tr} \, D) \sigma$ is the Jaumann rate of the Kirchhoff
stress $\tau = (\det F)\sigma$ at current state as a reference (deformation gradient $F$ being identity, $F \equiv I$). Plastic contribution $D^p$ to total strain-rate $D$ is therefore

$$D^p = D - M^e : \dot{\tau}.$$  \hspace{0.5cm} (1)

With different choice of conjugate variables and, therefore, different objective stress-rate, elastic compliance changes and what appears to be elastic and plastic strain-rates depends on the chosen stress-rate. Indeed, let $E$ be the Lagrange strain, the conjugate stress is then symmetric Piola–Kirchhoff stress, and by taking the reference state coincident with the current state, we have: $dE = D dt$ and $dT = \ddot{\tau} dt$, where $\ddot{\tau} = \ddot{\tau} - D\sigma - \sigma D$ is convected derivative of $\tau$ at $F = I$. Hence, in this case

$$\dot{D}^p = D - \dot{M}^e : \ddot{\tau},$$ \hspace{0.5cm} (2)

To obtain, in a consistent manner, instantaneous elastic compliance $(M^e$ or $\dot{M}^e)$, the finite elasticity law

$$\tau = 2F^e \frac{\partial \omega(C^e)}{\partial C^e} F^e T$$ \hspace{0.5cm} (3)

can conveniently be used. $F^e$ is the elastic part of total deformation gradient, corresponding to elastic stressing from intermediate (stress free, relaxed) to current configuration, as given by [Lee] multiplicative decomposition

$$F = F^e F^p,$$ \hspace{0.5cm} (4)

$F^p$ denoting the plastic part. In (3) plastic incompressibility is assumed, $C^e$ is the right Cauchy–Green deformation tensor, while $\omega$ is the strain energy per unit unstressed volume. For example, assuming elastic isotropy, which remains preserved during elasto–plastic deformation, $\omega$ is an isotropic function of $C^e$. By using (4), the velocity gradient $L = \dot{F} F^{-1}$ can be expressed as
\[ L = \dot{F}^e F^{-1} + F^e(\dot{F}^p F^p - 1) F^{-1}, \]  
(5)

and by taking symmetric and antisymmetric parts, the velocity strain and spin tensors are:

\[ D = (\dot{F}^e F^{-1})_a + [F^e(\dot{F}^p F^p - 1) F^{-1}]_a, \]  
(6)

\[ W = (\dot{F}^e F^{-1})_a + [F^e(\dot{F}^p F^p - 1) F^{-1}]_a. \]  
(7)

Identification (separation) of elastic and plastic contributions on the right hand side of (6) has been the subject of many papers. We address this issue here from two perspectives: based on the Jaumann and convected derivatives used to define elastic strain rate.

2. Analysis based on the Jaumann derivative

As discussed in [Lub], any intermediate configuration can be used in the analysis. Often utilized choice is based on destressing without rotation ([Lee], [LuL]). A convenient choice is also obtained by destressing such that

\[ [F^e(\dot{F}^p F^p - 1) F^{-1}]_a = 0, \]  
(8)

as it turns out that in this case elastic and plastic parts of \( D \) are:

\[ D^e = (\dot{F}^e F^{-1})_a, \]  
(9)

\[ D^p = F^e(\dot{F}^p F^p - 1) F^{-1}. \]  
(10)

Indeed, by differentiating (3)
\[
\dot{\tau} = (\dot{F}^* F^{-1})^* \sigma + \sigma (F^{-T} \dot{F}^* T) + \frac{2}{\det F^* F^{-1}} \left[ \frac{\partial^2 w(C^*)}{\partial C^* \otimes \partial C^*} \cdot \mathcal{C}^* \right] F^* T,
\]

(11)

we recognize (9), as equation (11) can then be written as

\[
\dot{\tau} = D^* \sigma + \sigma D^* + \frac{4}{\det F^* F^{-1}} \left[ \frac{\partial^2 w(C^*)}{\partial C^* \otimes \partial C^*} \cdot (F^* T D^* F^*) \right] F^* T \equiv \mathcal{L}^* : D^*.
\]

(12)

This identifies elastic moduli tensor \( \mathcal{L}^* \), or its inverse \( \mathcal{M}^* \), appearing in (1) and corresponding to Jaumann derivative \( \dot{\tau} \). Note that for any other choice of intermediate configuration (not defined by (8)), symmetric part of \( \dot{F}^* F^{-1} \) is not all, but just a portion of the elastic strain-rate \( D^* \). Elastic moduli \( \mathcal{L}^* \) and elastic strain-rate \( D^* \) are, of course, independent of the selected intermediate configuration. These issues are in detail discussed in [Lub].

3. Analysis based on the convected derivative

3.1 Geometric preliminaries

Introduce the convected coordinates \( \theta^i \), superscripted indices denote contravariant components and subscripted the covariant components [Hut]. Denote base vectors in the undeformed state by \( \mathbf{g}_i \) and their reciprocals by \( \mathbf{g}^i \). The base vectors in the deformed (current and intermediate) states are denoted by \( \hat{\mathbf{g}}_i \) and \( \hat{\mathbf{g}}_i \), respectively, their reciprocals being \( \hat{\mathbf{g}}^i \) and \( \hat{\mathbf{g}}^i \). The metric tensors of the three configurations are therefore: \( g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \hat{g}_{ij} = \hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_j \) and \( \hat{g}_{ij} = \hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_j \), and let \( g^{ij}, \hat{g}^{ij} \) and \( \hat{g}^{ij} \) be their respective inverses. From the definition of the intermediate configuration: \( \hat{\mathbf{g}}_i = F \mathbf{g}_i = F^* \hat{\mathbf{g}}_i \) and \( \hat{\mathbf{g}}_i = F^p \mathbf{g}_i \).

The Lagrangian \( \eta \) strain tensor is defined by:

\[
\eta = \eta_{ij} g^i \otimes g^j, \quad \eta_{ij} = \mathbf{g}_i \cdot \eta \cdot \mathbf{g}_j = \frac{1}{2} (\hat{g}_{ij} - g_{ij})
\]

(13)

which also can be expressed via deformation tensor \( C = F^T F \) as \( \eta = \frac{1}{2} (C - I) \).

Now, if \( ds, d\hat{s} \) and \( d\hat{s} \) are the lengths of an infinitesimal material element in the initial, current and intermediate state, it follows:
\[ ds^2 - dS^2 = (\hat{g}_{ij} - g_{ij}) d\theta^i d\theta^j = (\hat{g}_{ij} - \hat{g}_{ij}) d\theta^i d\theta^j + (\hat{g}_{ij} - g_{ij}) d\theta^i d\theta^j. \] (14)

Consequently, we define elastic and plastic parts of Lagrange strain by:

\[ \eta^e = \eta^e_{ij} g^i \otimes g^j, \quad \eta^p_{ij} = \frac{1}{2}(\hat{g}_{ij} - \hat{g}_{ij}) \]
\[ \eta^e = \eta^p_{ij} g^i \otimes g^j, \quad \eta^p_{ij} = \frac{1}{2}(\hat{g}_{ij} - g_{ij}) \] (15)

and from (13) and (15), \( \eta = \eta^e + \eta^p \). On the other hand

\[ ds^2 - d\hat{s}^2 = dx \cdot dx - d\hat{x} \cdot d\hat{x} = d\hat{x} \cdot (C^e - I) \cdot d\hat{x} = \hat{g}_i \cdot (C^e - I) \cdot \hat{g}_j d\theta^i d\theta^j, \]

hence, in view of \( ds^2 - d\hat{s}^2 = 2\eta^e_{ij} d\theta^i d\theta^j \),

\[ \eta^e_{ij} = \hat{g}_i \cdot \eta^e \cdot \hat{g}_j, \quad \eta^e = \frac{1}{2}(C^e - I). \] (17)

Expressing \( \eta^e \) via its components on \( \hat{g}^i \) reciprocal basis

\[ \eta^e = \eta^e_{ij} \hat{g}^i \otimes \hat{g}^j = \frac{1}{2}(C^e_{ij} - \hat{g}_{ij}) \hat{g}^i \otimes \hat{g}^j \] (18)

and comparison with (17) gives

\[ \eta^e_{ij} \equiv \eta^e_\delta = \frac{1}{2}(C^e_{ij} - \hat{g}_{ij}). \] (19)

Introducing further the strain-rate \( D \), it follows that

\[ \hat{g}_i \cdot D \cdot \hat{g}_j \equiv \hat{g}_i \cdot \dot{\eta} \cdot \hat{g}_j = \dot{\eta}_{ij} \] (20)

and, therefore, by writing \( D = D_{ij} \hat{g}^i \otimes \hat{g}^j \), we have \( D_{ij} \equiv \eta_{ij} \). Substitution of \( D = \dot{\eta}^e + \dot{\eta}^p \) into (20) then gives
In the following subsection we shall show that elastic and plastic strain-rates, corresponding to convected stress rate \( \mathbf{\bar{\sigma}} \), turn out to be:

\[
\mathbf{\dot{D}}^e = \dot{\eta}_{ij}^e \mathbf{g}^i \otimes \mathbf{g}^j
\]

\[
\mathbf{\dot{D}}^p = \dot{\eta}_{ij}^p \mathbf{g}^i \otimes \mathbf{g}^j.
\]

### 3.2 Constitutive expression for elastic strain-rate

The elastic part of deformation gradient \( F^e \) can be expressed in dyadic form with respect to intermediate basis \( \mathbf{\bar{g}}_i \) as

\[
F^e = (F^e)^{ij} \mathbf{\bar{g}}_i \otimes \mathbf{\bar{g}}_j,
\]

and substitution into (3) gives, after easy manoeuvres,

\[
J \sigma^{ij} \mathbf{\bar{g}}_i \otimes \mathbf{\bar{g}}_j = 2 \frac{\partial w}{\partial \epsilon_{ij}^e} \mathbf{\bar{g}}_i \otimes \mathbf{\bar{g}}_j,
\]

i.e. \( J \sigma^{ij} = 2 \frac{\partial w}{\partial \epsilon_{ij}^e} \)

with \( J \) standing for \( \text{det} \ F^e \), while \( \sigma^{ij} \) are the contravariant components of the Cauchy stress on the current basis \( \mathbf{\bar{g}}_i \). Utilizing Eq. (19), above can be rewritten in the form

\[
J \sigma^{ij} = \frac{\partial w}{\partial \eta_{ij}^e},
\]

and differentiation gives

\[
(\text{tr} \ D) \sigma^{ij} + \dot{\sigma}^{ij} = \frac{1}{J} \frac{\partial w}{\partial \eta_{ij}^e} \eta_{ki}^e.
\]

Differentiating the Kirchoff stress

Next, by
\[ \tau^{ij} \hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j \]  

(28)

and by using \( \hat{\mathbf{g}}_i = L \cdot \mathbf{g}_i \), \( L = \hat{F} F^{-1} = \hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}^k \) being the velocity gradient, we obtain

\[ \tau^{ij} \hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = \tau - L \tau - \tau L^T, \]  

(29)

which is the convected derivative of the Kirchhoff stress, associated with the choice \( F = I \). Hence, we recognize (22), since expression (27) then becomes

\[ \bar{\tau} = \bar{\mathcal{D}}^e : \bar{\mathcal{D}}^e, \quad (\bar{\mathcal{D}}^e)^{ijkl} = \frac{1}{J} \frac{\partial^2 w}{\partial \eta^e_{ij} \partial \eta^e_{kl}} \]  

(30)

or by inversion

\[ \bar{\mathcal{D}}^e = \dot{\mathcal{M}}^e : \bar{\tau}, \quad \dot{\mathcal{M}}^e = \bar{\mathcal{D}}^{-1}. \]  

(31)

This establishes the elastic compliance tensor appearing in (2), corresponding to convected derivative \( \bar{\tau} \). If elastic component of strain is small, \( \mathcal{L}^e \approx \bar{\mathcal{L}}^e \) and \( \mathcal{M}^e \approx \dot{\mathcal{M}}^e \).

4. Discussion

Having established constitutive expression for the elastic strain-rate, one needs to derive expression for the associated plastic strain-rate. When the Jaumann derivative \( \bar{\tau} \) is utilized, development goes in a straightforward manner. For example, in the case of isotropic hardening and Mises yield condition, one obtains

\[ D^p = \hat{\lambda} \sigma', \quad \dot{\hat{\lambda}} = \frac{1}{h} \sigma' : \bar{\tau} \]  

(32)

(\( h \) is a hardening parameter), which expresses the normality to Mises yield surface and coaxiality of \( D^p \) with the deviatoric part of stress \( \sigma' \). Rewriting (32)
in the form $D^p = M^p : \dot{\tau}$ and combining with $D^e = M^e : \dot{\tau}$ one arrives at the well known elasto–plastic constitutive structure, such as for example given in [MR]. However, when the convected derivative $\dot{\tau}$ is utilized in the partition of the strain-rate into corresponding elastic and plastic parts, an explicit development of the constitutive expression for strain-rate $\dot{D}^p$ becomes more difficult. For example, $\dot{D}^p$ is not normal to Mises yield surface, so that modification of the associated flow rule or the yield condition is needed in order to obtain a physically plausible constitutive law. Of course, the transition from the final elasto–plastic constitutive structure with the Jaumann derivative $\dot{\tau} = L : D$, to one with the convected derivative is direct via $\dot{\tau} = \overset{\circ}{\tau} + D\sigma + \sigma D$. 