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Recursive formulas for curvature and internal stresses in thermally loaded multilayer strips

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ABSTRACT
Recursive formulas are derived for the determination of the curvature and internal stresses in a uniformly heated multilayer strip made of n perfectly bonded isotropic layers in terms of the curvature and stresses in the strip with $n-1$ bonded layers. This is accomplished by reducing the problem to solving two linear algebraic equations for two unknowns, the curvature of the strip and the axial force in the n^{th} layer, from which all other forces, moments, and stresses readily follow. The presented analysis complements the authors' previous analysis which is based on a developed matrix algorithm for the simultaneous determination of the curvature and stresses in all layers. The derived formulas are applied to trilayer and quadralayer strips. Simple expressions are obtained in the case when all layers have equal thicknesses and equal moduli of elasticity.

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1. Introduction

The determination of the curvature and internal stresses in a multilayer strip due to a uniform change of its temperature has received considerable attention in the literature. The early work for a bilayer metallic strip, with application to buckling analysis of bimetal termostats, was done in [1]. The subsequent generalization to thermally loaded trilayer and multilayer thermostats was given in [2, 3] and, with included analysis of interface stresses and edge effects in [5, 6]. Numerous other contributions to the analysis of multilayer strips subjected to uniform heating, electric field, and hydration have followed, motivated to large extent by the design of piezoelectric and hygromorphic actuators and other devices of MEMS and bioengineering technology [7–9]; representative references were listed and commented upon in our preceding paper [10]. A number of books and monographs on composite materials also address in great detail the determination of deformation and stresses in multilayer strips, particularly in structurally important cross-ply and angle-ply laminates subjected to in-plane and bending loads and/or a temperature and moisture change [11–17]. This general lamination theory is based on the anisotropic elasticity and the Kirchhoff assumptions for stretching and bending of thin plates.

In our preceding paper, we presented a new matrix algorithm for determining the curvature and internal stresses in isotropic multilayer strips, which complements the matrix algorithm from the general lamination theory, and which is appealing due to its simplicity and fully explicit form. In the present paper we present an alternative appealing approach to the determination of curvature and internal stresses in uniformly heated multilayer strips, which is based on the recursive formulas which express the curvature and internal stresses in the strip with n perfectly

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bonded layers in terms of the curvature and stresses in the strip with $n-1$ layers. The use of the recursive formulas is frequent in the analysis of various problems of engineering science and applied physics, e.g., the multilayer reflectivity calculations [18], the calculation of absorptance, reflectance, and transmittance of parallel surfaces [19], the modeling of the layered diffraction gratings [20], the determination of multilayer magnetic structures by resonant x-ray magnetic scattering [21], the kinematic and kinetic analysis of multibody systems such as robots [22], the analysis of wave propagation in layered media [23], etc. In the approach adopted in our present paper, it is assumed that the solution for the multilayer strip with $n-1$ layers has already been determined and that, based on that solution, the objective is to construct the solution for the multilayer strip with n layers, obtained by attaching the n^{th} layer to the top or bottom surface of the pre-heated $(n-1)$-layer strip. Two equilibrium conditions and only one interface condition, the strain continuity between the top surface of the $(n-1)$-layer strip and the bottom surface of the added n^{th} layer, are needed in this approach. The problem reduces to solving the system of two linear algebraic equations for two unknowns, the curvature of the strip and the axial force in the n^{th} layer, from which all other forces and moments readily follow. The stresses in the first $n-1$ layers of the n-layer strip are expressed as the sum of the corresponding stresses in the $(n-1)$-layer strip, and the additional stresses due to the attachment of the n^{th} layer. This is conceptually different approach from the matrix approach utilized in our previous paper, where the multilayer strips were analyzed by considering simultaneously two equilibrium equations for the entire strip and $n-1$ interface conditions for the continuity of strain between all adjoining layers. To illustrate the application of the derived recursive formulas, they are applied to the analysis of trilayer and quadralayer strips. Particularly simple expressions are deduced for multilayers with equal layer thicknesses and equal moduli of elasticity.

2. Bilayer strip under uniform temperature change

Figure 1 shows a bilayer strip made of two perfectly bonded layers of rectangular cross sections having the same width b and the heights h_1 and h_2. The thermoelastic properties of isotropic layers are $(E_i, \alpha_i,\ i=1,2)$, where E denotes the modulus of elasticity and α the coefficient of linear thermal expansion. If the strip is uniformly heated from the initial temperature T_0 to the final temperature $T = T_0 + \Delta T$, the self-equilibrating state of stress develops in the strip. Away from the ends of the strip these stresses give rise to an axial force and a bending moment in the cross section of each layer, which equilibrate each other (in the absence of external mechanical load), and which are such that the compatibility (no slip) condition of equal strain along the bonded interface between two layers is satisfied. Because of the presence of moments M_1 and M_2, the layers of the strip bend in the vertical plane, such that

$$M_1^{[2]} = \frac{E_1 I_1}{\rho^{[2]}}, \quad M_2^{[2]} = \frac{E_2 I_2}{\rho^{[2]}}, \tag{2.1}$$

where $\rho^{[2]}$ is the radius of curvature of the strip. The superscript [2] designates that the strip is a bilayer strip. The bending stiffnesses of the layers are $E_1 I_1$ and $E_2 I_2$, where $I_i = bh_i^3/12$ $(i=1,\ 2)$.

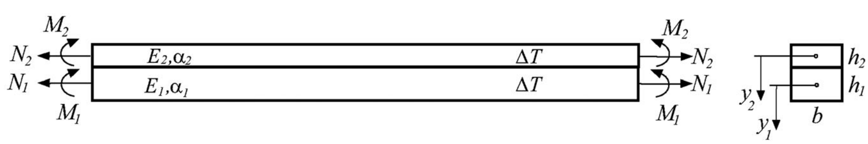

Figure 1. A segment of long bilayer strip away from its ends. The strip consists of two bonded layers with rectangular cross sections of width b and heights h_1 and h_2. The moduli of elasticity of two layers are E_1 and E_2, and their coefficients of thermal expansion α_1 and α_2. The strip is subjected to uniform change of temperature ΔT. Because of their thermal expansion mismatch, the layers stretch and bend. The corresponding axial forces (N_1, N_2) and bending moments (M_1, M_2) are self-equilibrating in each cross section of the strip. The coordinates y_1 and y_2 are measured from the centroids of the cross sections of two layers.

The Euler–Bernoulli assumption is adopted according to which the plane cross-sections of the strip remain plane and become perpendicular to the curved axis of the strip. Denoting the axial forces in the layers by $N_1^{[2]}$ and $N_2^{[2]}$, the condition for the vanishing of the total longitudinal force in the cross section of the strip is

$$N_1^{[2]} + N_2^{[2]} = 0. \tag{2.2}$$

The condition for the vanishing total moment in the cross section of the strip is

$$M_1^{[2]} + M_2^{[2]} + N_1^{[2]} \frac{h_1 + h_2}{2} = 0. \tag{2.3}$$

The remaining equation for determining the five unknown quantities $(\rho^{[2]}, N_1^{[2]}, N_2^{[2]}, M_1^{[2]}, M_2^{[2]})$ is obtained from the interface condition which requires that the longitudinal strain along the interface is equal for both layers. This gives

$$\alpha_1 \Delta T + \frac{N_1^{[2]}}{E_1 A_1} - \frac{h_1}{2\rho^{[2]}} = \alpha_2 \Delta T + \frac{N_2^{[2]}}{E_2 A_2} + \frac{h_2}{2\rho^{[2]}}, \tag{2.4}$$

where $A_1 = bh_1$ and $A_2 = bh_2$ are the cross-sectional areas of two layers. The curvature is assumed to be concave upward, which is the case if $\alpha_1 > \alpha_2$. If not, the calculated value of $\rho^{[2]}$ is negative, which means that the curvature is concave downward.

By substituting $N_1^{[2]} = -N_2^{[2]}$ from (2.2) into (2.4), we obtain

$$N_2^{[2]} = \frac{bE_1E_2h_1h_2}{E_1h_1 + E_2h_2} \left[(\alpha_1 - \alpha_2)\Delta T - (h_1 + h_2) \frac{1}{2\rho^{[2]}} \right]. \tag{2.5}$$

The curvature $1/\rho^{[2]}$ is obtained by substitution of (2.1) and (2.5) into (2.3),

$$\frac{1}{\rho^{[2]}} = \frac{6E_1E_2h_1h_2(h_1 + h_2)(\alpha_1 - \alpha_2)\Delta T}{(E_1h_1 + E_2h_2)(E_1h_1^3 + E_2h_2^3) + 3E_1E_2h_1h_2(h_1 + h_2)^2}, \tag{2.6}$$

which was originally derived in [1]. The forces $N_1^{[2]}$ and $N_2^{[2]}$ follow from (2.6) and (2.5), while the moments $M_1^{[2]}$ and $M_2^{[2]}$ follow from (2.6) into (2.1). The longitudinal stresses in two layers are determined from the beam formulas

$$\sigma_i^{[2]} = \frac{N_i^{[2]}}{A_i} + \frac{M_i^{[2]}}{I_i} y_i, \quad -\frac{h_i}{2} \le y_i \le \frac{h_i}{2}, \quad (i = 1, 2), \tag{2.7}$$

where y_1 and y_2 are the vertical coordinates from the centroids of the cross sections of two layers, measured positive downwards. The curvature (2.6) and the stresses (2.7) are independent of the width b.

Figure 2 shows the stress distribution across the thickness of the strip in the case when two layers are from mild steel and copper (Figure 2a), or mild steel and brass (Figure 2b),

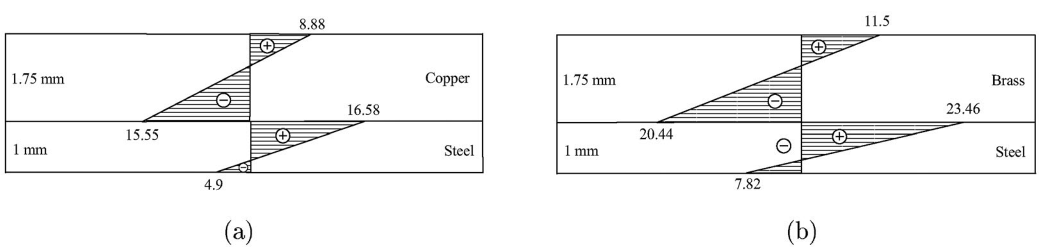

Figure 2. Stress distribution (in MPa) in a bilayer strip made of (a) mild steel and copper, and (b) mild steel and brass, due to uniform rise of temperature $\Delta T = 50\ ^\circ\mathrm{C}$. Thermoelastic properties used are: for mild steel, $E = 180$ GPa, $\alpha = 1.25 \times 10^{-5}\ ^\circ\mathrm{C}^{-1}$; for copper, $E = 117$ GPa, $\alpha = 1.7 \times 10^{-5}\ ^\circ\mathrm{C}^{-1}$; and for brass, $E = 105$ GPa, $\alpha = 1.9 \times 10^{-5}\ ^\circ\mathrm{C}^{-1}$.

corresponding to the temperature rise of 50 °C. The thicknesses of two layers are as shown, and their thermoelastic properties are listed in the figure caption. Since brass has a higher value of the coefficient of thermal expansion than copper, the magnitude of the radius of curvature in the steel/brass bilayer is smaller than in the steel/copper bilayer (5.754 m vs. 8.381 m). The curvature is negative in both cases because the steel layer, with a lower value of α, is placed below the copper and brass layers. The thicknesses of the copper and brass layers in cases (a) and (b) are taken to be the same for the sake of comparison. If one layer had a much smaller thickness than the other layer, a thin layer (say, layer [1]) would be approximately under the state of uniform tension or compression, $\sigma_1 = -E_1(\alpha_1 - \alpha_2)\Delta T$, the curvature of the strip would be very small, and the stress in a thick layer would be much smaller than the stress in a thin layer. For example, if $h_1 = 0.01$ mm and $h_2 = 1.75$ mm, with thermoelastic properties as in the caption of Figure 2, the stresses are $\sigma_1 \approx 39.1$ MPa, $\sigma_2(h_2/2) = -0.9$ MPa, $\sigma_2(-h_2/2) = 0.45$ MPa, while the radius of curvature is $\rho^{[2]} = -151.8$ m (the negative sign indicating the concave downward curvature).

For the derivation in section 3, the expression for the longitudinal strain at the top of the layer [2] is

$$\epsilon_+^{[2]} = \alpha_2\Delta T + \frac{N_2^{[2]}}{E_2A_2} - \frac{h_2}{2\rho^{[2]}}. \tag{2.8}$$

2.1. Bilayer with equal layer thickness and modulus of elasticity

If $E_1 = E_2 = E$ and $h_1 = h_2 = H_{[2]}/2$, where $H_{[2]}$ is the total height of a bilayer strip, it readily follows from (2.5) and (2.6) that

$$N_2^{[2]} = \frac{EbH_{[2]}}{16}(\alpha_1 - \alpha_2)\Delta T, \qquad \frac{1}{\rho^{[2]}} = \frac{3}{2H_{[2]}}(\alpha_1 - \alpha_2)\Delta T, \tag{2.9}$$

while, from (2.8),

$$\epsilon_+^{[2]} = \frac{1}{2}(\alpha_1 + \alpha_2)\Delta T - \frac{H_{[2]}}{2\rho^{[2]}} = \frac{1}{4}(5\alpha_2 - \alpha_1)\Delta T. \tag{2.10}$$

The bending moments are

$$M_1^{[2]} = M_2^{[2]} = \frac{EbH_{[2]}^2}{64}(\alpha_1 - \alpha_2)\Delta T. \tag{2.11}$$

The stresses at the top and bottom of each layer readily follow from (2.7), and are given by

$$\begin{aligned} \sigma_1^{[2]\mathrm{top}} &= -\frac{E}{2}(\alpha_1 - \alpha_2)\Delta T, \quad \sigma_1^{[2]\mathrm{bott}} = \frac{E}{4}(\alpha_1 - \alpha_2)\Delta T, \\ \sigma_2^{[2]\mathrm{top}} &= -\frac{E}{4}(\alpha_1 - \alpha_2)\Delta T, \quad \sigma_2^{[2]\mathrm{bott}} = \frac{E}{2}(\alpha_1 - \alpha_2)\Delta T. \end{aligned} \tag{2.12}$$

These results will be used in the sequel.

3. Trilayer strip

Figure 3 shows a trilayer strip made of three bonded layers with rectangular cross sections whose width is b and heights h_1, h_2, and h_3. The moduli of elasticity of the layers are E_1, E_2, and E_2, and their coefficients of thermal expansion are α_1, α_2, and α_3. The strip is subjected to uniform change of temperature ΔT. Away from the ends of the strip, the layers are under axial forces (N_1, N_2, N_3) and bending moments (M_1, M_2, M_3), which are self-equilibrating in each cross section of the strip. The direct determination of these forces and moments by imposing the

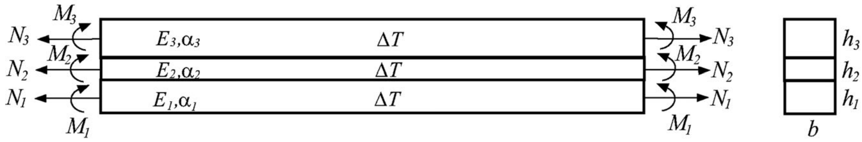

Figure 3. A trilayer strip made of three bonded layers of rectangular cross sections of width b and heights h_1, h_2, and h_3. The moduli of elasticity of three layers are E_1, E_2, and E_3, and their coefficients of thermal expansion α_1, α_2, and α_3. The strip is subjected to uniform change of temperature ΔT. Away from the ends of the strip, the layers are under axial forces (N_1, N_2, N_3) and bending moments (M_1, M_2, M_3), which are self-equilibrating in each cross section of the strip.

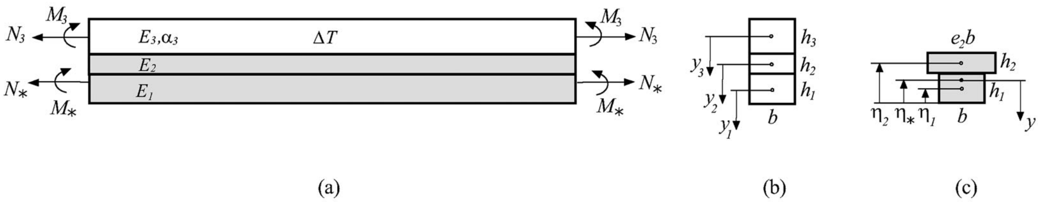

Figure 4. (a) A trilayer strip is made by attaching a layer [3] to the bonded bilayer, consisting of layers [1] and [2], which have already experienced the temperature change ΔT and are under the corresponding states of internal stress. The temperature of the layer [3] is then changed by ΔT. To fulfill the bonded interface condition, the layer [3] experiences the axial force $N_3^{[3]}$ and the bending moment $M_3^{[3]}$, while the bonded bilayer experiences the axial force $N_*^{[3]}$ and the bending moment $M_*^{[3]}$. (b) The cross section of the trilayer strip. (c) The transformed cross section of the bilayer portion of the strip is obtained by changing the width of the layer [2] from b to $e_2 b$, where $e_2 = E_2/E_1$. The coordinate η_* specifies the centroid of the transformed cross section.

equilibrium conditions of zero resulting axial force and zero resulting bending moment in the cross section of the strip, and two interface conditions of equal longitudinal strain between the layers [1] and [2], and the layers [2] and [3], has been reviewed and elaborated upon in our previous paper [10]. In this paper, we follow an alternative approach and consider a trilayer strip to be made by attaching the layer [3] to a bonded bilayer strip, consisting of layers [1] and [2] (shown shaded in Figure 4a), which has already been subjected to the temperature change ΔT and which is thus under the state of internal stress $\sigma_i^{[2]}$ $(i = 1, 2)$, determined in Section 2 of this paper. Upon bonding of the layer [3] to preheated bilayer strip, the temperature of the layer [3] is imagined to be changed by ΔT. Due to thermal mismatch, this produces stresses in the layer [3] and the additional stresses in the bilayer strip. The stresses in the layer [3] are statically equivalent to axial force $N_3^{[3]}$ and bending moment $M_3^{[3]}$, while the additional stresses in the bonded bilayer are statically equivalent to axial force $N_*^{[3]}$ and bending moment $M_*^{[3]}$. The axial force $N_*^{[3]}$ acts at the centroid of the transformed cross section of the bilayer, shown in Figure 4c. The transformation is made by changing the width of the layer [2] from b to $e_2 b$, where $e_2 = E_2/E_1$, and by taking its modulus of elasticity to be equal to E_1, thus preserving the bending stiffness of the layer [2]. This homogenization of the bilayer simplifies the subsequent analysis, and is a well-known procedure from bending analysis of composite beams [24]. The forces $(N_3^{[3]}, N_*^{[3]})$ and moments $(M_3^{[3]}, M_*^{[3]})$ are determined by imposing the equilibrium conditions and the interface condition. The equilibrium conditions require that the forces and moments in each cross section of the strip must be self-equilibrating, i.e.,

$$N_3^{[3]} + N_*^{[3]} = 0, \tag{3.1}$$

$$M_3^{[3]} + M_*^{[3]} - N_3^{[3]}\left(\eta_3 - \eta_*^{[3]}\right) = 0. \tag{3.2}$$

In (3.2), η_3 specifies the centroid of the cross section of the layer [3],

$$\eta_3 = h_1 + h_2 + \frac{h_3}{2}, \tag{3.3}$$

and $\eta_*^{[3]}$ the centroid of the transformed cross section of the bilayer. The latter is determined from

$$\eta_*^{[3]} = \frac{\eta_1 h_1 + \eta_2 e_2 h_2}{h_1 + e_2 h_2}, \tag{3.4}$$

where η_1 and η_2 specify the centroids of the cross sections of layers [1] and [2].

The moment in the layer [3] is related to the curvature of the trilayer strip $\rho^{[3]}$ by

$$M_3^{[3]} = \frac{E_3 I_3}{\rho^{[3]}}, \quad I_3 = \frac{bh_3^3}{12}. \tag{3.5}$$

On the other hand, the moment $M_*^{[3]}$ is related to $\rho^{[3]}$ by

$$M_*^{[3]} = E_1 I_*^{[3]} \left(\frac{1}{\rho^{[3]}} - \frac{1}{\rho^{[2]}} \right), \tag{3.6}$$

because the pre-heated bilayer had a pre-curvature $1/\rho^{[2]}$, determined in section 2. The second areal moment of the transformed cross section of the bilayer is

$$I_*^{[3]} = b \sum_{i=1}^{2} \left[\frac{1}{12} \, e_i h_i^3 + \left(\eta_i - \eta_*^{[3]} \right)^2 e_i h_i \right], \tag{3.7}$$

where $e_1 = 1$ and $e_2 = E_2/E_1$. By substituting (3.5) and (3.6) into (3.2), we obtain the first equation for $\rho^{[3]}$ and $N_3^{[3]}$,

$$\left(E_1 I_*^{[3]} + E_3 I_3 \right) \frac{1}{\rho^{[3]}} - \left(\eta_3 - \eta_*^{[3]} \right) N_3^{[3]} = \frac{E_1 I_*^{[3]}}{\rho^{[2]}}. \tag{3.8}$$

The second equation for determining $(\rho^{[3]}, N_3^{[3]})$ is obtained from the bonded interface condition which requires that the longitudinal strain in the layer [3] along the interface with the bilayer is equal to that of the bilayer itself,

$$\alpha_3 \Delta T + \frac{N_3^{[3]}}{E_3 A_3} + \frac{h_3}{2\rho^{[3]}} = \epsilon_+^{[2]} + \frac{N_*^{[3]}}{E_1 A_*^{[3]}} - \frac{M_*^{[3]} \left(h_1 + h_2 - \eta_*^{[3]} \right)}{E_1 I_*^{[3]}}, \quad A_3 = bh_3. \tag{3.9}$$

This can be rewritten as

$$\left(\eta_3 - \eta_*^{[3]} \right) \frac{1}{\rho^{[3]}} + \left(\frac{1}{E_1 A_*^{[3]}} + \frac{1}{E_3 A_3} \right) N_3^{[3]} = -\alpha_3 \Delta T + \epsilon_+^{[2]} + \frac{h_1 + h_2 - \eta_*^{[3]}}{\rho^{[2]}}, \tag{3.10}$$

which is the desired form of the second equation for the unknown quantities $(\rho^{[3]}, N_3^{[3]})$.

To proceed, we conveniently introduce the parameters

$$d_{11}^{[3]} = E_1 I_*^{[3]} + E_3 I_3, \quad d_{22}^{[3]} = \frac{1}{E_1 A_*^{[3]}} + \frac{1}{E_3 A_3}, \quad d_{12}^{[3]} = \eta_3 - \eta_*^{[3]}, \tag{3.11}$$

$$g_1^{[3]} = \frac{E_1 I_*^{[3]}}{\rho^{[2]}}, \quad g_2^{[3]} = -\alpha_3 \Delta T + \epsilon_+^{[2]} + \frac{h_1 + h_2 - \eta_*^{[3]}}{\rho^{[2]}}, \tag{3.12}$$

and rewrite Eqs. (3.8) and (3.10) as

$$\begin{aligned} d_{11}^{[3]} \frac{1}{\rho^{[3]}} - d_{12}^{[3]} N_3^{[3]} &= g_1^{[3]}, \\ d_{12}^{[3]} \frac{1}{\rho^{[3]}} + d_{22}^{[3]} N_3^{[3]} &= g_2^{[3]}. \end{aligned} \tag{3.13}$$

The solution to this system of two linear algebraic equations for $1/\rho^{[3]}$ and $N_3^{[3]}$ is

$$\frac{1}{\rho^{[3]}} = \frac{g_1^{[3]} d_{22}^{[3]} + g_2^{[3]} d_{12}^{[3]}}{d_{11}^{[3]} d_{22}^{[3]} + d_{12}^{[3]2}}, \quad N_3^{[3]} = \frac{g_2^{[3]} d_{11}^{[3]} - g_1^{[3]} d_{12}^{[3]}}{d_{11}^{[3]} d_{22}^{[3]} + d_{12}^{[3]2}}. \tag{3.14}$$

The strain at the top of the layer [3] is

$$\epsilon_+^{[3]} = \alpha_3 \Delta T + \frac{N_3^{[3]}}{E_3 A_3} - \frac{h_3}{2\rho^{[3]}}. \tag{3.15}$$

The stresses in all three layers can now be determined from

$$\begin{aligned} \sigma_i^{[3]} &= \sigma_i^{[2]} + \frac{E_i}{E_1}\left(\frac{N_*^{[3]}}{A_*^{[3]}} + \frac{M_*^{[3]}}{I_*^{[3]}}\ y^{[3]}\right), \quad (i = 1, 2), \\ \sigma_3^{[3]} &= \frac{N_3^{[3]}}{A_3} + \frac{M_3^{[3]}}{I_3}\ y_3, \end{aligned} \tag{3.16}$$

where, from (3.5) and (3.6),

$$N_*^{[3]} = -N_3^{[3]}, \quad \frac{M_*^{[3]}}{I_*^{[3]}} = E_1\left(\frac{1}{\rho^{[3]}} - \frac{1}{\rho^{[2]}}\right), \quad \frac{M_3^{[3]}}{I_3} = \frac{E_3}{\rho^{[3]}}. \tag{3.17}$$

The coordinates y_i shown in Figure 4b are related to the coordinate $y^{[3]}$ (denoted in Figure 4c simply as y), emanating from the centroid of the transformed cross section, by

$$y_i = y^{[3]} + \eta_i - \eta_*^{[3]}, \quad (i = 1, 2). \tag{3.18}$$

Thus, the stress expressions (3.16) can be rewritten as

$$\begin{aligned} \sigma_i^{[3]} &= \sigma_i^{[2]} + \frac{E_i}{E_1}\left[\frac{N_*^{[3]}}{A_*^{[3]}} + \frac{M_*^{[3]}}{I_*^{[3]}}\left(y_i - \eta_i + \eta_*^{[3]}\right)\right], \quad -\frac{h_i}{2} \le y_i \le \frac{h_i}{2}, \quad (i = 1, 2), \\ \sigma_3^{[3]} &= \frac{N_3^{[3]}}{A_3} + \frac{M_3^{[3]}}{I_3}\ y_3, \quad -\frac{h_3}{2} \le y_3 \le \frac{h_3}{2}, \end{aligned} \tag{3.19}$$

3.1. Expressions for $(N_i^{[3]}, M_i^{[3]})$

One can easily determine $(N_i^{[3]}, M_i^{[3]})$ for $i = 1$, 2 by the integration from stresses (3.19). It follows that

$$\begin{aligned} N_i^{[3]} &= \int_{A_i} \sigma_i^{[3]}\ \mathrm{d}A_i = N_i^{[2]} + \frac{E_i A_i}{E_1}\left[\frac{N_*^{[3]}}{A_*^{[3]}} + \frac{M_*^{[3]}}{I_*^{[3]}}\left(\eta_*^{[3]} - \eta_i\right)\right], \\ M_i^{[3]} &= \int_{A_i} \sigma_i^{[3]} y_i\ \mathrm{d}A_i = M_i^{[2]} + \frac{E_i I_i}{E_1 I_*^{[3]}}\ M_*^{[3]}, \quad (i = 1, 2). \end{aligned} \tag{3.20}$$

The difference $\eta_*^{[3]} - \eta_i$ specifies the distance between the centroid of the transformed cross section and the centroid of the i^{th} layer.

Note that, once $\rho^{[3]}$ has been determined, the moments in all three layers can be also determined directly from

$$M_i^{[3]} = \frac{E_i I_i}{\rho^{[3]}}, \quad (i = 1, 2, 3). \tag{3.21}$$

Furthermore, once $N_3^{[3]}$ and $(M_1^{[3]}, M_2^{[3]}, M_3^{[3]})$ have been determined, the expressions for the forces $N_1^{[3]}$ and $N_2^{[3]}$ can be obtained from the equilibrium conditions applied to the original tri-layer configuration shown in Figure 3, i.e.,

$$\begin{aligned} &N_1^{[3]} + N_2^{[3]} + N_3^{[3]} = 0, \\ &N_1^{[3]} \frac{h_1 + h_2}{2} - N_3^{[3]} \frac{h_2 + h_3}{2} + M_1^{[3]} + M_2^{[3]} + M_3^{[3]} = 0. \end{aligned} \tag{3.22}$$

This gives

$$\begin{aligned} N_1^{[3]} &= \frac{h_2 + h_3}{h_1 + h_2} N_3^{[3]} - \frac{2}{h_1 + h_2}\left(M_1^{[3]} + M_2^{[3]} + M_3^{[3]}\right), \\ N_2^{[3]} &= -\frac{h_1 + 2h_2 + h_3}{h_1 + h_2} N_3^{[3]} + \frac{2}{h_1 + h_2}\left(M_1^{[3]} + M_2^{[3]} + M_3^{[3]}\right). \end{aligned} \tag{3.23}$$

The corresponding stress expressions, alternative to (3.19), are

$$\sigma_i^{[3]} = \frac{N_i^{[3]}}{A_i} + \frac{M_i^{[3]}}{I_i} y_i, \quad (i = 1, 2, 3). \tag{3.24}$$

3.2. Numerical example

Stress distribution across the thickness of a trilayer strip, determined from expressions (3.19), is shown in Figure 5. In part (a) of this figure, a trilayer strip is made by bonding an aluminum layer of thickness $h_1 = 2$ mm to a brass layer of thickness $h_2 = 1.75$ mm, and a brass layer to a copper layer of thickness $h_3 = 1.5$ mm. The width of the strip is $b = 25$ mm. The uniform temperature rise is $\Delta T = 50\ ^\circ$C. Thermoelastic properties used for aluminum are: $E = 69$ GPa, $\alpha = 2.4 \times 10^{-5}\ ^\circ C^{-1}$; for brass: $E = 105$ GPa, $\alpha = 1.9 \times 10^{-5}\ ^\circ C^{-1}$; and for copper: $E = 117$ GPa, $\alpha = 1.7 \times 10^{-5}\ ^\circ C^{-1}$. The geometric properties of the transformed cross section of the aluminum/brass bilayer are $\eta_*^{[3]} = 2.1$ mm, $I_*^{[3]} = 134$ mm^4, and $A_*^{[3]} = 116.6$ mm^2, while $\rho^{[2]} = 10.02$ m and $\epsilon_+^{[2]} = 8.9 \times 10^{-4}$. The solution to the system of equations (3.13) is $N_3^{[3]} = -16.55$ N and $\rho_3^{[3]} = 11.4$ m, while the moments are $M_3^{[3]} = 72.2$ Nmm and $M_*^{[3]} = -112.4$ Nmm. The forces and moments in other layers were calculated by using (3.20), and they are $N_1^{[3]} = -161.46$ N, $N_2^{[3]} = 178.01$ N, $M_1^{[3]} = 100.9$ Nmm, and $M_2^{[3]} = 102.8$ Nmm. The longitudinal strain at the top of the trilayer is $\epsilon_+^{[3]} = 7.81 \times 10^{-4}$, as determined from (3.15).

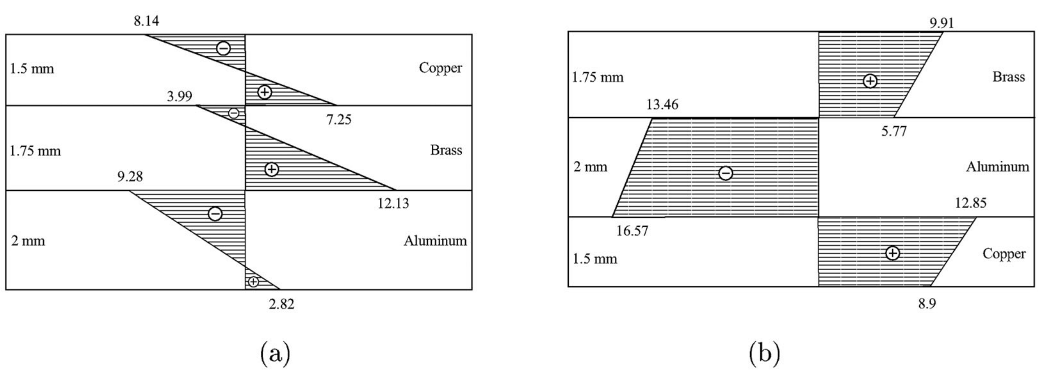

Figure 5. (a) Stress distribution across the thickness of a trilayer strip made of an aluminum layer of thickness $h_1 = 2$ mm, a brass layer of thickness $h_2 = 1.75$ mm, and a copper layer of thickness $h_3 = 1.5$ mm, corresponding to uniform temperature rise $\Delta T = 50\ ^\circ$C. (b) Stress distribution for a trilayer in which the copper layer is placed below the aluminum/brass bilayer, rather than above it, as in part (a).

In Figure 5b the stress distribution is shown for a trilayer in which the copper layer is placed below the aluminum/brass bilayer, rather than above it, as in Figure 5a. Because the aluminum layer, which has the greatest value of the coefficient of thermal expansion, is now in the middle of the strip, and the outer copper and brass layers have their coefficients of thermal expansion not very different from each other, the self-equilibrating stress distribution is much closer to a symmetric stress distribution, and the resulting curvature is consequently much smaller. Indeed, in this case $\rho_3^{[3]} = -44.38$ m, corresponding to negative (concave downward) curvature, because the bottom layer is copper with smaller value of $\alpha = 1.7 \times 10^{-5}\ ^\circ\mathrm{C}^{-1}$ than $\alpha = 1.9 \times 10^{-5}\,^\circ\mathrm{C}^{-1}$ of the brass layer at the top of the strip. The values of $N_*^{[3]}$ and $M_*^{[3]}$ are -342.9 N and 1,005 Nmm, respectively.

3.3. Trilayer with equal layer thickness and modulus of elasticity

If $E_1 = E_2 = E_3 = E$ and $h_1 = h_2 = h_3 = H_{[3]}/3$, where $H_{[3]}$ is the total height of a trilayer strip, from (3.4) and (3.7) we obtain

$$\eta_*^{[3]} = \frac{H_{[3]}}{3}, \quad I_*^{[3]} = \frac{2bH_{[3]}^3}{81}, \tag{3.25}$$

while, from (3.11),

$$d_{11}^{[3]} = \frac{EbH_{[3]}^3}{36}, \quad d_{22}^{[3]} = \frac{9}{2EbH_{[3]}}, \quad d_{12}^{[3]} = \frac{H_{[3]}}{2}. \tag{3.26}$$

Furthermore, from (2.9) and (2.10) with $H_{[2]} = 2H_{[3]}/3$, we have

$$N_2^{[2]} = \frac{EbH_{[3]}}{24}(\alpha_1 - \alpha_2)\Delta T, \quad \frac{1}{\rho^{[2]}} = \frac{9}{4H_{[3]}}(\alpha_1 - \alpha_2)\Delta T, \tag{3.27}$$

and

$$\epsilon_+^{[2]} = \frac{1}{4}(5\alpha_2 - \alpha_1)\Delta T. \tag{3.28}$$

Consequently, from (3.12),

$$g_1^{[3]} = \frac{EbH_{[3]}^2}{18}(\alpha_1 - \alpha_2)\Delta T, \quad g_2^{[3]} = \frac{1}{2}(\alpha_1 + \alpha_2 - 2\alpha_3)\Delta T. \tag{3.29}$$

By substitution of (3.26) and (3.29) into (3.14), we then obtain

$$\frac{1}{\rho^{[3]}} = \frac{4}{3H_{[3]}}(\alpha_1 - \alpha_3)\Delta T, \quad N_3^{[3]} = -\frac{EbH_{[3]}}{27}(\alpha_1 - 3\alpha_2 + 2\alpha_3)\Delta T. \tag{3.30}$$

The strain at the top of the layer [3] is, from (3.15) and (3.30),

$$\epsilon_+^{[3]} = \frac{1}{3}(-\alpha_1 + \alpha_2 + 3\alpha_3)\Delta T. \tag{3.31}$$

It is also noted that

$$\frac{1}{\rho^{[3]}} - \frac{1}{\rho^{[2]}} = -\frac{1}{12H_{[3]}}(11\alpha_1 - 27\alpha_2 + 16\alpha_3)\Delta T. \tag{3.32}$$

The stresses at the top and bottom of layers [1] and [2] are, from (3.19),

$$
\begin{aligned}
\sigma_1^{[3]\mathrm{top}} &= \sigma_1^{[2]\mathrm{top}} + \frac{E}{18}(\alpha_1 - 3\alpha_2 + 2\alpha_3)\Delta T = -\frac{E}{9}(4\alpha_1 - 3\alpha_2 - \alpha_3)\Delta T,\\
\sigma_1^{[3]\mathrm{bott}} &= \sigma_1^{[2]\mathrm{bott}} - \frac{E}{12}(3\alpha_1 - 7\alpha_2 + 4\alpha_3)\Delta T = \frac{E}{3}(\alpha_2 - \alpha_3)\Delta T,\\
\sigma_2^{[3]\mathrm{top}} &= \sigma_2^{[2]\mathrm{top}} + \frac{E}{36}(13\alpha_1 - 33\alpha_2 + 20\alpha_3)\Delta T = \frac{E}{9}(\alpha_1 - 6\alpha_2 + 5\alpha_3)\Delta T,\\
\sigma_2^{[3]\mathrm{bott}} &= \sigma_2^{[2]\mathrm{bott}} + \frac{E}{18}(\alpha_1 - 3\alpha_2 + 2\alpha_3)\Delta T = \frac{E}{9}(5\alpha_1 - 6\alpha_2 + \alpha_3)\Delta T.
\end{aligned} \tag{3.33}
$$

The stresses in the layer [3] are

$$
\sigma_3^{[3]\mathrm{top}} = -\frac{E}{3}(\alpha_1 - \alpha_2)\Delta T, \quad \sigma_3^{[3]\mathrm{bott}} = \frac{E}{9}(\alpha_1 + 3\alpha_2 - 4\alpha_3)\Delta T. \tag{3.34}
$$

The obtained stress expressions are identical to those in section 6.1 of [10], which were derived in that paper by using the matrix algorithm developed by the consideration of all layers separately.

4. Recursive formula for $n \geq 3$

In this section we generalize the analysis from section 3 and derive recursive formulas for the curvature and internal stresses in the multilayer strip with n layers in terms of the corresponding quantities for the multilayer strip with $n-1$ layers, without the n^{th} layer. Figure 6 is the generalization of Figure 4. The n^{th} layer is imagined to be attached to the multilayer strip with $n-1$ layers (shown shaded in Figure 6a), which have already been subjected to the temperature change ΔT and which are, thus, already under the states of internal stress $\sigma_i^{[n-1]}$ $(i = 1, 2, ..., n-1)$, assumed to be determined previously. Upon bonding of the n^{th} layer, the temperature of the layer $[n]$ is changed by ΔT, which produces stresses in the layer $[n]$ and the additional stresses in the bonded $(n-1)$ layers. The stresses in the layer $[n]$ are statically equivalent to axial force $N_n^{[n]}$ and bending moment $M_n^{[n]}$, while the additional stresses in the $(n-1)$-multilayer strip are statically equivalent to axial force $N_*^{[n]}$ and bending moment $M_*^{[n]}$. The axial force $N_*^{[n]}$ acts at the centroid of the transformed cross section of the $(n-1)$-multilayer strip, as sketched in Figure 6c. The transformation is made by changing the width of the layer $[i]$ from b to $e_i b$, where $e_i = E_i/E_1$, and by taking its modulus of elasticity to be equal to E_1, thus preserving the bending stiffness of the layer $[i]$, for each $i = 1, 2, ..., n-1$. The forces $(N_n^{[n]}, N_*^{[n]})$ and moments $(M_n^{[n]}, M_*^{[n]})$ are determined by imposing the equilibrium conditions and the interface condition. The equilibrium

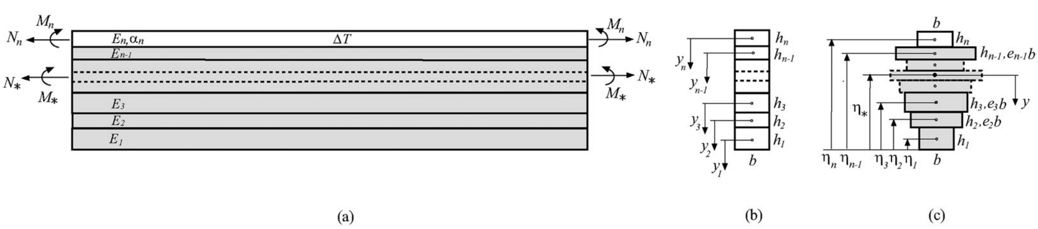

Figure 6. (a) A multilayer strip is made by attaching the layer $[n]$ to a bonded strip consisting of $n-1$ layers, which have already experienced the temperature change ΔT and are under the corresponding state of internal stress. The temperature of the layer $[n]$ is subsequently changed by ΔT. To fulfill the bonded interface condition, the layer $[n]$ experiences the axial force $N_n^{[n]}$ and the bending moment $M_n^{[n]}$, while the bonded $(n-1)$-layer strip experiences the axial force $N_*^{[n]}$ and the bending moment $M_*^{[n]}$. (b) The cross section of the multilayer strip. (c) The transformed cross section of the $(n-1)$-layer strip is made by changing the width of the layer $[i]$ from b to e_{ib}, where $e_i = E_i/E_1$ $(i = 1, 2, ..., n-1)$.

conditions require that the forces and moments in each cross section of the entire n-layer strip must be self-equilibrating, i.e.,

$$N_n^{[n]} + N_*^{[n]} = 0, \tag{4.1}$$

while the condition for the vanishing total moment in the cross section of the strip is

$$M_n^{[n]} + M_*^{[n]} - N_n^{[n]}\left(\eta_n - \eta_*^{[n]}\right) = 0. \tag{4.2}$$

The coordinate η_n specifies the centroid of the cross section of the n^{th} layer,

$$\eta_n = \sum_{i=1}^{n-1} h_i + \frac{h_n}{2}, \tag{4.3}$$

while $\eta_*^{[n]}$ specifies the centroid of the transformed cross section, which is determined from

$$\eta_*^{[n]} = \frac{\sum_{i=1}^{n-1} \eta_i e_i h_i}{\sum_{i=1}^{n-1} e_i h_i}. \tag{4.4}$$

The coordinates η_i $(i = 1, 2, ..., n)$ specify the centroids of the cross sections of each layer.

The moment in the layer $[n]$ is related to the curvature of the n-layer strip $\rho^{[n]}$ by

$$M_n^{[n]} = \frac{E_n I_n}{\rho^{[n]}}, \quad I_n = \frac{b h_n^3}{12}. \tag{4.5}$$

On the other hand, the moment $M_*^{[n]}$ is related to $\rho^{[n]}$ by

$$M_*^{[n]} = E_1 I_*^{[n]}\left(\frac{1}{\rho^{[n]}} - \frac{1}{\rho^{[n-1]}}\right), \tag{4.6}$$

in analogy to (3.6). The second areal moment of the transformed cross section of the $(n-1)$-layer strip is

$$I_*^{[n]} = b\sum_{i=1}^{n-1}\left[\frac{1}{12}\, e_i h_i^3 + \left(\eta_i - \eta_*^{[n]}\right)^2 e_i h_i\right]. \tag{4.7}$$

By substituting (4.5) and (4.6) into (4.2), we obtain the first equation for $\rho^{[n]}$ and $N_n^{[n]}$,

$$\left(E_1 I_*^{[n]} + E_n I_n\right)\, \frac{1}{\rho^{[n]}} - \left(\eta_n - \eta_*^{[n]}\right) N_n^{[n]} = \frac{E_1 I_*^{[n]}}{\rho^{[n-1]}}. \tag{4.8}$$

The second equation for determining the unknown quantities $(\rho^{[n]}, N_n^{[n]})$ is obtained from the bonded interface condition which requires that the longitudinal strain in the layer $[n]$ along the interface with the $(n-1)$-layer strip is equal to that of the $(n-1)$-layer strip itself, i.e.,

$$\alpha_n \Delta T + \frac{N_n^{[n]}}{E_n A_n} + \frac{h_n}{2\rho^{[n]}} = \epsilon_+^{[n-1]} + \frac{N_*^{[n]}}{E_1 A_*^{[n]}} - \frac{M_*^{[n]}\left(\sum_{i=1}^{n-1} h_i - \eta_*^{[n]}\right)}{E_1 I_*^{[n]}}, \quad A_n = b h_n, \tag{4.9}$$

where

$$A_*^{[n]} = b\sum_{i=1}^{n-1} e_i h_i, \quad e_i = \frac{E_i}{E_1}, \quad (i = 1, 2, ..., n-1) \tag{4.10}$$

is the area of the transformed cross section. Expression (4.9) can be rewritten as

$$\left(\eta_n - \eta_*^{[n]}\right)\frac{1}{\rho^{[n]}} + \left(\frac{1}{E_1 A_*^{[n]}} + \frac{1}{E_n A_n}\right)N_n^{[n]} = -\alpha_n \Delta T + \epsilon_+^{[n-1]} + \frac{\sum_{i=1}^{n-1} h_i - \eta_*^{[n]}}{\rho^{[n-1]}}, \tag{4.11}$$

which is the second equation for determining the unknown quantities $(\rho^{[n]}, N_n^{[n]})$.

As in section 3, we introduce the parameters

$$d_{11}^{[n]} = E_1 I_*^{[n]} + E_n I_n, \quad d_{22}^{[n]} = \frac{1}{E_1 A_*^{[n]}} + \frac{1}{E_n A_n}, \quad d_{12}^{[n]} = \eta_n - \eta_*^{[n]}, \tag{4.12}$$

$$g_1^{[n]} = \frac{E_1 I_*^{[n]}}{\rho^{[n-1]}}, \quad g_2^{[n]} = -\alpha_n \Delta T + \epsilon_+^{[n-1]} + \frac{\sum_{i=1}^{n-1} h_i - \eta_*^{[n]}}{\rho^{[n-1]}}, \tag{4.13}$$

and rewrite Eqs. (4.8) and (4.11) compactly as

$$\begin{aligned} d_{11}^{[n]} \frac{1}{\rho^{[n]}} - d_{12}^{[n]} N_n^{[n]} &= g_1^{[n]}, \\ d_{12}^{[n]} \frac{1}{\rho^{[n]}} + d_{22}^{[n]} N_n^{[n]} &= g_2^{[n]}. \end{aligned} \tag{4.14}$$

The solution to this system of two linear algebraic equations for $1/\rho^{[n]}$ and $N_n^{[n]}$ is

$$\frac{1}{\rho^{[n]}} = \frac{g_1^{[n]} d_{22}^{[n]} + g_2^{[n]} d_{12}^{[n]}}{d_{11}^{[n]} d_{22}^{[n]} + d_{12}^{[n]2}}, \quad N_n^{[n]} = \frac{g_2^{[n]} d_{11}^{[n]} - g_1^{[n]} d_{12}^{[n]}}{d_{11}^{[n]} d_{22}^{[n]} + d_{12}^{[n]2}}, \tag{4.15}$$

The stresses in all layers can then be determined from

$$\begin{aligned} \sigma_i^{[n]} &= \sigma_i^{[n-1]} + \frac{E_i}{E_1}\left(\frac{N_*^{[n]}}{A_*^{[n]}} + \frac{M_*^{[n]}}{I_*^{[n]}} \; y^{[n]}\right), \quad (i = 1, 2, \ldots, n-1), \\ \sigma_n^{[n]} &= \frac{N_n^{[n]}}{A_n} + \frac{M_n^{[n]}}{I_n} \; y_n, \end{aligned} \tag{4.16}$$

where, from (4.5) and (4.6),

$$N_*^{[n]} = -N_n^{[n]}, \quad \frac{M_*^{[n]}}{I_*^{[n]}} = E_1\left(\frac{1}{\rho^{[n]}} - \frac{1}{\rho^{[n-1]}}\right), \quad \frac{M_n^{[n]}}{I_n} = \frac{E_n}{\rho^{[n]}}. \tag{4.17}$$

Furthermore, since by geometric considerations, the coordinates y_i shown in Figure 6b are related to the coordinate $y^{[n]}$ (denoted in Figure 6c simply as y), emanating from the centroid of the transformed cross section, by

$$y_i = y^{[n]} + \eta_i - \eta_*^{[n]}, \quad i = 1, 2, \ldots, n-1, \tag{4.18}$$

the stress expressions (4.16) can be rewritten as

$$\begin{aligned} \sigma_i^{[n]} &= \sigma_i^{[n-1]} + \frac{E_i}{E_1}\left[\frac{N_*^{[n]}}{A_*^{[n]}} + \frac{M_*^{[n]}}{I_*^{[n]}}\left(y_i - \eta_i + \eta_*^{[n]}\right)\right], \quad -\frac{h_i}{2} \le y_i \le \frac{h_i}{2}, \quad (i = 1, 2, \ldots, n-1), \\ \sigma_n^{[n]} &= \frac{N_n^{[n]}}{A_n} + \frac{M_n^{[n]}}{I_n} \; y_n, \quad -\frac{h_n}{2} \le y_n \le \frac{h_n}{2}. \end{aligned} \tag{4.19}$$

4.1. Expressions for $(N_i^{[n]}, M_i^{[n]})$

Similarly as in section 3, the axial forces and moments $(N_i^{[n]}, M_i^{[n]})$ for $i = 1, 2, \ldots, n-1$ can be determined by integration from stresses (4.19). The resulting expressions are

$$\begin{aligned} N_i^{[n]} &= \int_{A_i} \sigma_i^{[n]}\, dA_i = N_i^{[n-1]} + \frac{E_i A_i}{E_1}\left[\frac{N_*^{[n]}}{A_*^{[n]}} + \frac{M_*^{[n]}}{I_*^{[n]}}\left(\eta_*^{[n]} - \eta_i\right)\right], \\ M_i^{[n]} &= \int_{A_i} \sigma_i^{[n]} y_i\, dA_i = M_i^{[n-1]} + \frac{E_i I_i}{E_1 I_*^{[n]}}\, M_*^{[n]}, \quad (i = 1, 2, \ldots, n-1). \end{aligned} \tag{4.20}$$

Geometrically, $\eta_*^{[n]} - \eta_i$ is the distance between the centroid of the transformed cross section and the centroid of the i^{th} layer whose cross-sectional area is A_i.

Alternatively, once $\rho^{[n]}$ has been determined, the moments in all n layers can be determined from

$$M_i^{[n]} = \frac{E_i I_i}{\rho^{[n]}}, \quad (i = 1, 2, \ldots, n). \tag{4.21}$$

4.2. Numerical example

The formulas derived in this section are applied to determine the curvature and stresses in the quadralayer strip obtained by attaching to the trilayer aluminum/brass/copper strip shown in Figure 5a from section 3 the fourth layer, the 1 mm-thick layer from mild steel, bonded to copper layer, as shown in Figure 7. The thicknesses of other layers are $h_{\text{Al}} = 2$ mm, $h_{\text{brass}} = 1.75$ mm, and $h_{\text{Cu}} = 1.5$ mm, and the width of all layers is $b = 25$ mm. The thermoelastic properties of aluminum, brass, and copper are specified in the caption of Figure 5, while the thermoelastic properties of mild steel are $E = 180$ GPa and $\alpha = 1.25 \times 10^{-5}\ ^\circ\text{C}^{-1}$. The uniform temperature rise in the entire strip is $\Delta T = 50\ ^\circ\text{C}$. The curvature of the trilayer aluminum/brass/copper strip $\rho^{[3]} = 11.4$ m and the strain $\epsilon_+^{[3]} = 7.8 \times 10^{-4}$ were determined in the example of section 3 corresponding to Figure 5a. Geometric properties of the transformed cross section of the aluminum/brass/copper trilayer (Figure 7c) are $\eta_*^{[4]} = 2.9$ mm, $I_*^{[4]} = 389$ mm^4, and $A_*^{[4]} = 180.2$ mm^2. From (4.13) we then find $N_4^{[4]} = -N_*^{[4]} = 193.2$ N and $\rho^{[4]} = 9.39$ m, while the moments are $M_4^{[4]} = 40$ Nmm and $M_*^{[4]} = 505$ Nmm. The forces and moments in other individual layers are calculated by using (4.18), and are $N_1^{[4]} = -89.8$ N, $N_2^{[4]} = 111.2$ N, $N_3^{[4]} = -214.6$ N, $M_1^{[4]} = 122.5$ Nmm, $M_2^{[4]} = 124.9$ Nmm, and $M_3^{[4]} = 87.6$ Nmm. The stress distribution along the hight of the quadralayer strip is shown in Figure 8. By comparing the stress distributions in Figures 5a and 8 it is observed that by adding the steel layer the maximum stresses in aluminum, brass, and copper

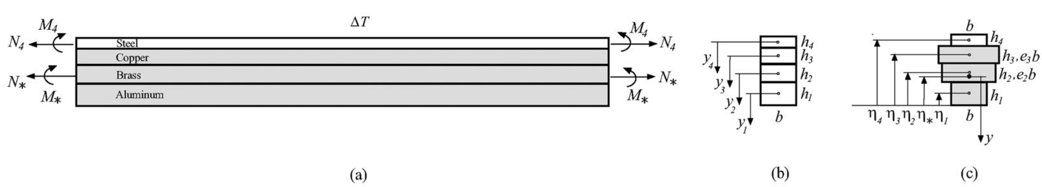

Figure 7. (a) A quadralayer strip is made by attaching the layer [4] to a bonded trilayer, consisting of layers [1], [2], and [3], which have already experienced the temperature change ΔT and are under the corresponding state of internal stress. The temperature of the layer [4] is subsequently changed by ΔT. The layer [4] experiences the axial force $N_4^{[4]}$ and the bending moment $M_4^{[4]}$, while the bonded trilayer experiences the axial force $N_*^{[4]}$ and the bending moment $M_*^{[4]}$. (b) The cross section of the quadralayer strip. (c) The transformed cross section of the trilayer portion of the strip is made by changing the width of the layer [i] from b to $e_i b$, where $e_i = E_i/E_1$ $(i = 1, 2, 3)$.

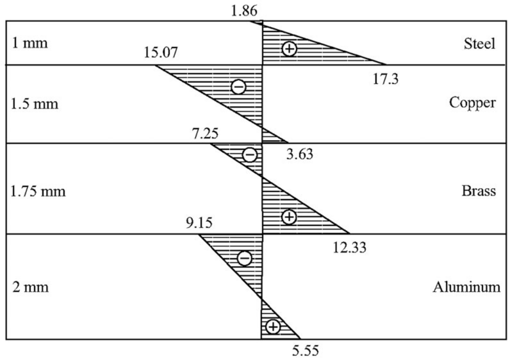

Figure 8. Stress distribution (in MPa) in a quadralayer strip made of aluminum, brass, copper, and mild steel due to uniform rise of temperature $\Delta T = 50\ ^{\circ}$C. The thicknesses of the layers are as shown, while their thermoelastic properties are given in the text.

layers have increased, as expected because the curvature of the quadralayer strip $1/\rho^{[4]} = 0.1065\ \mathrm{m}^{-1}$ has increased relative to the curvature of the trilayer strip $1/\rho^{[3]} = 0.0877\ \mathrm{m}^{-1}$ by the addition of the 1 mm-thick steel layer. Similar analysis can be performed by adding the steel layer to the bottom of the trilayer aluminum/brass/copper strip. The details of this calculations are similar to those from the example corresponding to Figure 5b in section 3 and are omitted here. Other geometric aspects of the analysis can be addressed, such as the effects of the heights of the layers on the curvature and stress. For example, if one layer is much thinner than the other layers, the stress in that layer is approximately uniform (tension or compression). Indeed, if, in the above example, the copper layer has the thickness $h_3 = 0.1$ mm, the radius of curvature of the strip is $\rho^{[4]} = 6.423$ m, while $\sigma_3^{[4]}(-h_3/2) = -10.34$ MPa and $\sigma_3^{[4]}(h_3/2) = -10.16$ MPa. The stresses in other three layers redistribute such that $\sigma_1^{[4]}(-h_1/2) = -11.34$ MPa, $\sigma_1^{[4]}(h_1/2) = 10.15$ MPa, $\sigma_2^{[4]}(-h_2/2) = -19.61$ MPa, $\sigma_2^{[4]}(h_2/2) = 9$ MPa, and $\sigma_4^{[4]}(-h_4/2) = -3.43$ MPa, $\sigma_4^{[4]}(h_4/2) = 24.6$ MPa. The dominant effect on the response of a heated multilayer is still the mismatch of thermal expansion coefficients between the layers, and by tuning them through material selection and sequencing of the layers one can achieve the desired curvature and the range of stresses produced by the given temperature change.

4.3. Multilayer with equal layer thickness and modulus of elasticity

If $E_i = E$ and $h_i = H_{[n]}/n$ for all $i = 1, 2, \ldots, n$, where $H_{[n]}$ is the total height of the strip with n layers, from (4.4) and (4.7) we obtain

$$\eta_*^{[n]} = \frac{n-1}{2n}\,H_{[n]}, \quad I_*^{[n]} = \frac{(n-1)^3}{n^3}\,\frac{bH_{[n]}^3}{12}, \quad A_*^{[n]} = \frac{n-1}{n}\,bH_{[n]}, \tag{4.22}$$

while, from (4.12) and (4.13), the parameters which appear in the system of linear algebraic equations (4.14) for $1/\rho^{[n]}$ and $N_n^{[n]}$ are

$$d_{11}^{[n]} = \frac{n^2 - 3n + 3}{n^2}\,\frac{EbH_{[n]}^3}{12}, \quad d_{22}^{[n]} = \frac{n^2}{n-1}\,\frac{1}{EbH_{[n]}}, \quad d_{12}^{[n]} = \frac{H_{[n]}}{2}, \tag{4.23}$$

and

$$g_1^{[n]} = \frac{(n-1)^3}{n^3}\,\frac{EbH_{[n]}^3}{12}\,\frac{1}{\rho^{[n-1]}}, \quad g_2^{[n]} = -\alpha_n \Delta T + \epsilon_+^{[n-1]} + \frac{n-1}{2n}\,\frac{H_{[n]}}{\rho^{[n-1]}}, \tag{4.24}$$

where $\epsilon_+^{[n-1]}$ and $\rho^{[n-1]}$ are known from the analysis of the $(n-1)$-layer strip.

For example, by using expressions for $\epsilon_+^{[3]}$ and $\rho^{[3]}$ from Section 3.3, we obtain for the quadra-layer strip of height $H_{[4]}$,

$$\frac{1}{\rho^{[4]}} = \frac{3}{8H_{[4]}}(3\alpha_1 + \alpha_2 - \alpha_3 - 3\alpha_4)\Delta T, \quad N_4^{[4]} = -\frac{EbH_{[4]}}{64}(5\alpha_1 - \alpha_2 - 7\alpha_3 + 3\alpha_4). \tag{4.25}$$

Similarly, for a quintalayer strip of height $H_{[5]}$, we have

$$\frac{1}{\rho^{[5]}} = \frac{12}{25H_{[5]}}(2\alpha_1 + \alpha_2 - \alpha_4 - 2\alpha_5)\Delta T, \quad N_5^{[5]} = -\frac{EbH_{[5]}}{125}(7\alpha_1 + \alpha_2 - 5\alpha_3 - 6\alpha_4 + 3\alpha_5). \tag{4.26}$$

5. Conclusions

We have derived in this paper the recursive formulas for the determination of the curvature and internal stresses in thermally loaded multilayer strips which are made of isotropic perfectly bonded layers with different coefficients of thermal expansion and different moduli of elasticity. The width of all layers is assumed to be the same but their heights are different. The formulas express the curvature and internal stresses in the strip with n perfectly bonded layers in terms of the curvature and internal stresses in the strip with $n-1$ layers. The latter are assumed to be known. This is accomplished by performing the thermoelastic analysis of the process of heating the n^{th} layer attached to the top or bottom of the pre-heated $(n-1)$-layer strip. Two equilibrium conditions and only one interface condition, the continuity of strain between the top surface of the $(n-1)$-layer strip and the bottom surface of the added n^{th} layer, are needed in this approach. The problem is reduced to solving two linear algebraic equations for two unknowns, the curvature of the strip and the axial force in the n^{th} layer, from which all other forces and moments, as well as stresses, readily follow. The stresses in the first $n-1$ layers of the n-layer strip are expressed as the sum of the stresses in the pre-heated $(n-1)$-layer strip and additional terms due to the attached and heated n^{th} layer. The derived formulas are applied to trilayer and quadra-layer strips. Particularly simple expressions are deduced in the case when all layers have the same thicknesses and the same moduli of elasticity. The presented analysis can be extended to piezo-electric multilayers subjected to electric field and hygromorphic multilayers subjected to uniform change of relative moisture, along similar lines to those used in our previous paper [10].

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References

[1] S. Timoshenko, "Analysis of bi-metal thermostats," *J. Opt. Soc. Am.*, vol. 11, no. 3, pp. 233–256, 1925. DOI: 10.1364/JOSA.11.000233.

[2] M. Vasudevan and W. Johnson, "Thermal bending of a tri-metal strip," *J. R. Aeronaut. Soc.*, vol. 65, no. 607, pp. 507–509, 1961. DOI: 10.1017/S0368393100074976.

[3] M. Vasudevan and W. Johnson, "On multi-metal thermostats," *Appl. sci. Res.*, vol. 9, no. 6, pp. 420–430, 1961. DOI: 10.1007/BF02921840.

[4] D. Chen, S. Cheng, and T. D. Gerhardt, "Thermal stresses in laminated beams," *J. Therm. Stresses*, vol. 5, no. 1, pp. 67–84, 1982. DOI: 10.1080/01495738208942136.

[5] E. Suhir, "Stresses in bi-metal thermostats," *ASME J. Appl. Mech.*, vol. 53, no. 3, pp. 657–660, 1986. DOI: 10.1115/1.3171827.

[6] E. Suhir, "An approximate analysis of stresses in multilayered elastic thin films," *ASME J. Appl. Mech.*, vol. 55, no. 1, pp. 143–148, 1988. DOI: 10.1115/1.3173620.

[7] A. Dasgupta, "Thermo-mechanical analysis and design," in *Handbook of Electronic Package Design*, [M. Pecht, Ed. New York: Marcal Dekker, 1991, pp. 477–528.

[8] D. L. DeVoe and A. P. Pisano, "Modeling and optimal design of piezoelectric cantilever microactuators," *J. Microelectromech. Syst.*, vol. 6, no. 3, pp. 266–270, 1997. DOI: 10.1109/84.623116.

[9] M. Safaei, H. A. Sodano, and S. R. Anton, "A review of energy harvesting using piezoelectric materials: State-of-the-art a decade later (2008–2018," *Smart Mater. Struct.*, vol. 28, no. 11, pp. 113001, 2019. DOI: 10.1088/1361-665X/ab36e4.

[10] V. A. Lubarda and M. V. Lubarda, "On the curvature and internal stresses in a multilayer strip due to uniform heating, electric field, or hydration," *J. Therm. Stresses*, 2021.

[11] C. T. Herakovich, *Mechanics of Fibrous Composite*. New York: Wiley, 1998.

[12] R. M. Jones, *Mechanics of Composites Materials* (2nd ed.). New York: Brunner–Routledge, 1999.

[13] J. N. Reddy, *Mechanics of Laminated Composite Plates: Theory and Analysis*. Boca Raton, FL: CRC Press, 1997.

[14] G. H. Staab, *Laminar Composites* (2nd ed). Butterworth-Heinemann, Elsevier, 2015.

[15] S. W. Tsai and H. T. Hahn, "Introduction to Composite Materials," 2nd ed. Boca Raton, FL: CRC, Routledge, 2018.

[16] V. V. Vasiliev and E. V. Morozov, *Advanced Mechanics of Composite Materials and Structures*. Amsterdam, Netherlands: Elsevier, 2018.

[17] G. Z. Voyiadjis and P. I. Kattan, *Mechanics of Composite Materials with MATLAB*. Berlin: Springer, 2005.

[18] L. G. Parratt, "Surface studies of solids by total reflection of x-rays," *Phys. Rev*, vol. 95, no. 2, pp. 359–369, 1954. DOI: 10.1103/PhysRev.95.359.

[19] T. W. Cadman and D. Sadowski, "Generalized equations for the calculation of absorptance, reflectance, and transmittance of a number of parallel surfaces," *Appl. Opt*, vol. 17, no. 4, pp. 531–537, 1978. DOI: 10.1364/AO.17.000531.

[20] L. Li, "Formulation and comparison of two recursive matrix algorithms for modeling layered diffraction gratings," *J. Opt. Soc. Am. A*, vol. 13, no. 5, pp. 1024–1035, 1996. DOI: 10.1364/JOSAA.13.001024.

[21] N. Ishimatsu *et al.*, "Magnetic structure of Fe/Gd multilayers determined by resonant x-ray magnetic scattering," *Phys. Rev. B*, vol. 60, no. 13, pp. 9596–9606, 1999. DOI: 10.1103/PhysRevB.60.9596.

[22] S. K. Saha, H. Khas, and W. O. Schiehlen, "Recursive kinematics and dynamics for closed loop multibody systems," *Int. J. Mech. Struct. Mach.*, vol. 29, no. 2, pp. 143–175, 2001. DOI: 10.1081/SME-100104478.

[23] S. I. Rokhlin and L. Wang, "Stable recursive algorithm for elastic wave propagation in layered anisotropic media: Stiffness matrix method," *J. Acoust. Soc. Am.*, vol. 112, no. 3, pp. 822–834, 2002. DOI: 10.1121/1.1497365.

[24] S. P. Timoshenko and J. M. Gere, *Mechanics of Materials*. New York: Van Nostrand Reinhold, 1972.