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Complementary energy release rates and dual conservation integrals in micropolar elasticity

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Abstract

The complementary energy momentum tensor, expressed in terms of the spatial gradients of stress and couple-stress, is used to construct the $\hat{J}_k$ and $\hat{L}_k$ conservation integrals of infinitesimal micropolar elasticity. The derived integrals are related to the release rates of the complementary potential energy associated with a defect translation or rotation. A nonconserved $\hat{M}$ integral is also derived and related to the energy release rate that is associated with a self-similar cavity expansion. The results are compared to those obtained on the basis of the classical energy momentum tensor, expressed in terms of the spatial gradients of displacement and rotation, and the release rates of the potential energy. It is shown that the evaluation of the complementary conservation integrals is of similar complexity to that of the classical conservation integrals, so that either can be effectively used in the energetic analysis of the mechanics of defects. The two-dimensional versions of the dual conservation integrals are then derived and applied to an out-of-plane shearing of a long cracked slab.

Keywords: Complementary energy; Configurational force; Conservation integrals; Dual integrals; Micropolar elasticity

1. Introduction

There has been a great amount of research during the past several decades devoted to conservation integrals in classical, micropolar, and nonlocal elasticities, thermoelasticity,

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piezoelectricity, finite-strain elasticity, and related branches of continuum mechanics. Most of this research has been inspired by Eshelby's (1951, 1956) pioneering work on the energy momentum tensor and configurational forces on moving material defects such as inclusions, voids, cracks, dislocations, and phase boundaries. Additional impetus was provided by the work of Knowles and Sternberg (1972), Budiansky and Rice (1973), and Eshelby (1975), who related the conservation integrals to Noether's theorem on invariant variational principles and established their physical interpretations. These results were of practical significance for fracture and damage mechanics, dislocations studies, mechanics of moving interfaces, biomechanics of tissue growth and remodeling, and other problems concerning the micromechanics of heterogeneous materials (Rice, 1985; Maugin, 1995; Gurtin, 2000; Kienzler and Herrmann, 2001).

The classical conservation integrals are expressed in terms of the spatial gradients of displacements and rotations, and are related to the release rates of the potential energy associated with a defect motion within the material. The complementary conservation integrals are related to the release rates of the complementary potential energy, and are expressed in terms of the spatial gradients of stresses and couple stresses. The consideration of the complementary or dual conservation integrals was initiated by Bui's (1973, 1974) introduction of the *I*-integral, a dual to Rice's (1968) *J*-integral of two-dimensional fracture mechanics. An independent study of the complementary conservation integrals was presented by Carlsson (1974). The subsequent research includes, among others, the work by Sun (1985), Moran and Shih (1987), Li (1988), Bui (1994), Trimarco and Maugin (1995), and Li and Gupta (2006). In recent paper on dual conservation integrals in classical elasticity, Lubarda and Markenscoff (2007) pointed out and corrected some conceptual errors made by others in the analysis and derivation of the relationship between the dual integrals and the release rates of the complementary potential energy in nonpolar elasticity. In the present paper, devoted to micropolar elasticity, we derive the complementary energy momentum tensor and the complementary or dual $\hat{J}_k$, $\hat{L}_k$, and $\hat{M}$ integrals, which were neither studied nor reported in the literature before. We then relate them to the release rates of the potential and complementary potential energy associated with particular types of the defect motion within the material. The conservation laws $\hat{J}_k = 0$ and $\hat{L}_k = 0$ are proved for any closed surface that does not embrace a singularity or a defect. If there is a defect inside the surface, the values of $\hat{J}_k$ and$\hat{L}_k$ are related to the release rates of the complementary potential energy that is associated with a defect translation or rotation. A nonconserved $\hat{M}$ integral is also derived and related to the energetic force due to a self-similar expansion of the cavity. A complete duality between the two formulations is established. The two-dimensional versions of the dual integrals are then deduced and applied to an out-of-plane shearing of a long cracked slab. The calculations illustrate that the evaluation of the complementary conservation integrals is of similar complexity to that of the classical conservation integrals, so that either can be used in the energetic analysis of the mechanics of defects.

2. Basic equations of micropolar elasticity

In a micropolar continuum the deformation is described by the displacement vector and an independent rotation vector, so that an infinitesimal material element can experience a microrotation without undergoing a macrodisplacement. An infinitesimal surface element transmits a force and a couple vector, which give rise to nonsymmetric stress and couple-stress tensors. The nonsymmetric stress is related to nonsymmetric strain tensor, and the couple stress is related to the gradient of the rotation vector. This type of the continuum mechanics was introduced by Voigt (1887) and the brothers Cosserat (1909), and later further developed by numerous investigators in the second half of the last century. The books by Brulin and Hsieh (1982), Nowacki (1986), and Eringen (1999) offer extensive list of pertinent references. A finite strain and rotation theories of polar elasticity and thermoelasticity are presented by Maugin (1998).

The physical rationale for the extension of the classical to micropolar and couple-stress theory was that the classical theory was not able to predict the size effect experimentally observed in problems which had a geometric length scale comparable to material's microstructural length, such as the grain size in a polycrystalline or granular aggregate. The classical theory was also in disagreement with experiments for high-frequency ultrashort wave propagation problems, in which the wavelength is comparable to the material's microstructural length. Furthermore, couple stresses can affect the singular nature of the crack tip fields, and may be of interest in explaining the deformation mechanisms of micro and nanostructured materials, inelastic localization and instability phenomena (Asaro and Lubarda, 2006).

A brief summary of the governing equations of micropolar elasticity is as follows. An infinitesimal deformation of a micropolar elastic material is described by the displacement vector u_i and an independent rotation vector φ_i. The surface forces are in equilibrium with the nonsymmetric Cauchy stress t_{ij}, and the surface couples are in equilibrium with the nonsymmetric couple-stress m_{ij}, such that

$$T_i = n_j t_{ji}, \quad M_i = n_j m_{ji}, \tag{1}$$

where n_j are the components of the unit vector orthogonal to the surface element under consideration. In the absence of body forces and body couples, the conservation laws for vanishing linear and angular momenta are the integral conditions of equilibrium

$$\int_S T_i \,\mathrm{d}S = 0, \quad \int_S (M_i + e_{ijk} x_j T_k)\,\mathrm{d}S = 0, \tag{2}$$

where e_{ijk} are the components of the permutation tensor. The corresponding differential equations of equilibrium are (e.g., Mindlin, 1964)

$$t_{ji,j} = 0, \quad m_{ji,j} + e_{ijk} t_{jk} = 0. \tag{3}$$

Note that the second equation in (3) is equivalent to $(m_{ji} + e_{ilk} t_{jk} x_l)_{,j} = 0$. For elastic deformations of micropolar continuum, the strain energy is $W = W(\gamma_{ij}, \kappa_{ij})$, with the complementary strain energy, as its counterpart, defined by

$$\Phi(t_{ij}, m_{ij}) = t_{ij}\gamma_{ij} + m_{ij}\kappa_{ij} - W(\gamma_{ij}, \kappa_{ij}). \tag{4}$$

Their rates are

$$\dot{W} = t_{ij}\dot{\gamma}_{ij} + m_{ij}\dot{\kappa}_{ij}, \quad \dot{\Phi} = \gamma_{ij}\dot{t}_{ij} + \kappa_{ij}\dot{m}_{ij}, \tag{5}$$

where

$$\gamma_{ij} = u_{j,i} - e_{ijk}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i} \tag{6}$$

are the nonsymmetric strain and curvature tensors, respectively. The constitutive relations of micropolar elasticity are

$$t_{ij} = \frac{\partial W}{\partial \gamma_{ij}}, \quad m_{ij} = \frac{\partial W}{\partial \kappa_{ij}},$$
$$\gamma_{ij} = \frac{\partial \Phi}{\partial t_{ij}}, \quad \kappa_{ij} = \frac{\partial \Phi}{\partial m_{ij}}. \tag{7}$$

If the material is linearly elastic, the strain energy W and the complementary strain energy Φ are the quadratic functions of their arguments,

$$W = \tfrac{1}{2} C_{ijkl}\, \gamma_{ij}\gamma_{kl} + \tfrac{1}{2} K_{ijkl}\, \kappa_{ij}\kappa_{kl},$$
$$\Phi = \tfrac{1}{2} C^{-1}_{ijkl}\, t_{ij}t_{kl} + \tfrac{1}{2} K^{-1}_{ijkl}\, m_{ij}m_{kl}. \tag{8}$$

The components of the fourth-order tensors of micropolar elastic moduli are C_{ijkl} and K_{ijkl}, while the components of their inverse tensors are the elastic compliances C^{-1}_{ijkl} and K^{-1}_{ijkl}. Since the strain and curvature tensors are not symmetric, only the reciprocal symmetries $C_{ijkl} = C_{klij}$ and $K_{ijkl} = K_{klij}$ hold, and likewise for the compliances. The inverse tensors are thus defined such that $C_{ijmn}C^{-1}_{mnkl} = \delta_{ik}\delta_{jl}$ and $K_{ijmn}K^{-1}_{mnkl} = \delta_{ik}\delta_{jl}$. The constitutive expressions (7), associated with the strain energies (8), are

$$t_{ij} = C_{ijkl}\,\gamma_{kl}, \quad m_{ij} = K_{ijkl}\kappa_{kl},$$
$$\gamma_{ij} = C^{-1}_{ijkl}\, t_{kl}, \quad \kappa_{ij} = K^{-1}_{ijkl}\, m_{kl}. \tag{9}$$

In the material is isotropic, and using the notation of Nowacki (1986), the moduli are

$$C_{ijkl} = (\mu + \bar{\mu})\delta_{ik}\delta_{jl} + (\mu - \bar{\mu})\delta_{il}\delta_{jk} + \lambda\delta_{ij}\,\delta_{kl},$$
$$K_{ijkl} = (\alpha + \bar{\alpha})\,\delta_{ik}\delta_{jl} + (\alpha - \bar{\alpha})\delta_{il}\delta_{jk} + \beta\delta_{ij}\,\delta_{kl}, \tag{10}$$

where $\mu, \bar{\mu}, \lambda$ and $\alpha, \bar{\alpha}, \beta$ are the Lamé-type elastic constants. The corresponding elastic compliances are

$$C^{-1}_{ijkl} = \frac{1}{4}\left(\frac{1}{\mu} + \frac{1}{\bar{\mu}}\right)\delta_{ik}\delta_{jl} + \frac{1}{4}\left(\frac{1}{\mu} - \frac{1}{\bar{\mu}}\right)\delta_{il}\,\delta_{jk} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}\delta_{kl},$$
$$K^{-1}_{ijkl} = \frac{1}{4}\left(\frac{1}{\alpha} + \frac{1}{\bar{\alpha}}\right)\delta_{ik}\delta_{jl} + \frac{1}{4}\left(\frac{1}{\alpha} - \frac{1}{\bar{\alpha}}\right)\delta_{il}\delta_{jk} - \frac{\beta}{2\alpha(3\beta + 2\alpha)}\delta_{ij}\delta_{kl}, \tag{11}$$

where both $\hat{\mu}$ and $\hat{\alpha}$ are assumed to be different from zero.

3. Dual J integrals in micropolar elasticity

A spatial gradient of the strain energy function $W = W(\gamma_{ij}, \kappa_{ij})$ is

$$W_{,k} = \frac{\partial W}{\partial \gamma_{ij}}\gamma_{ij,k} + \frac{\partial W}{\partial \kappa_{ij}}\kappa_{ij,k} = t_{ij}\gamma_{ij,k} + m_{ij}\kappa_{ij,k}, \tag{12}$$

which can be rewritten, by using (6), as

$$W_{,j}\delta_{jk} - t_{ji}u_{i,jk} - m_{ji}\varphi_{i,jk} + t_{ji}e_{jil}\varphi_{l,k} = 0. \tag{13}$$

In view of the equilibrium equations (3), this reduces to

$$(W\delta_{jk} - t_{ji}u_{i,k} - m_{ji}\varphi_{i,k})_{,j} = 0, \tag{14}$$

which defines a divergence-free energy momentum tensor of micropolar elasticity, given by

$$P_{jk} = W\delta_{jk} - t_{ji}u_{i,k} - m_{ji}\varphi_{i,k}, \quad P_{jk,j} = 0. \tag{15}$$

Consequently, there is a conservation law

$$J_k = \int_S P_{jk}n_j \, \mathrm{d}S = 0, \tag{16}$$

for any closed surface S which does not enclose a singularity or a defect.

An alternative derivation to the above simple derivation of the J_k conservation law was earlier presented by Dai (1986) and Jaric (1986) in the case of elastostatics, and by Vukobrat (1989) in the case of elastodynamics. A derivation based on Noether's theorem on invariant variational principles, was given by Pucci and Saccomandi (1990) and, in a more general context, by Lubarda and Markenscoff (2003). An extension of the analysis to account for the material nonhomogeneity and anisotropy effects, as well as for body forces and body couples, was recently presented by Lazar and Kirchner (2007).

3.1. Dual $\hat{J}_k$ integral

Consider next a spatial gradient of the complementary strain energy function $\Phi = \Phi(t_{ij}, m_{ij})$,

$$\Phi_{,k} = \frac{\partial \Phi}{\partial t_{ij}} t_{ij,k} + \frac{\partial \Phi}{\partial m_{ij}} m_{ij,k} = \gamma_{ij}t_{ij,k} + \kappa_{ij}m_{ij,k}. \tag{17}$$

In view of the kinematic expressions (6), this becomes

$$\Phi_{,j}\delta_{jk} - u_{i,j}t_{ji,k} + e_{jir}\varphi_r t_{ji,k} - \varphi_{i,j}m_{ji,k} = 0. \tag{18}$$

Incorporating the equilibrium equations (3), the above reduces to

$$(\Phi\delta_{jk} - u_i t_{ji,k} - \varphi_i m_{ji,k})_{,j} = 0. \tag{19}$$

From this we recognize a divergence-free complementary energy momentum tensor, defined by

$$\hat{P}_{jk} = \Phi\delta_{jk} - u_i t_{ji,k} - \varphi_i m_{ji,k}, \quad \hat{P}_{jk,j} = 0. \tag{20}$$

Consequently, there is a dual conservation law[1]

$$\hat{J}_k = \int_S \hat{P}_{jk}n_j \, \mathrm{d}S = 0, \tag{21}$$

for any closed surface S that does not embrace a singularity or a defect.

The J_k integral in (16) is expressed in terms of the spatial gradients of the displacement and rotation, while $\hat{J}_k$ in (21) is expressed in terms of the stress and couple-stress gradients.

[1] In the context of nonpolar elasticity, a dual conservation law of this type was originally introduced by Bui (1973, 1974).

It readily follows that

$$P_{jk} + \hat{P}_{jk} = (W + \Phi)\delta_{jk} - (t_{ji}u_i + m_{ji}\varphi_i)_{,k},$$
$$P_{kk} = 3W - t_{ij}\gamma_{ij} - m_{ij}\kappa_{ij} - e_{ijk}t_{ij}\varphi_k, \quad \hat{P}_{kk} = 3\Phi - e_{ijk}t_{ij}\varphi_k. \tag{22}$$

When specialized to classical nonpolar elasticity, the first of these expressions reduces to the result noted by Li and Gupta (2006) in their study of the relationship between the dual conservation laws and the invariance of an appropriate variational principle.

In a particular case, when the strain energy W is a homogeneous function of degree r in both the strain and curvature components, the complementary strain energy Φ is a homogeneous function of degree $s = r/(r-1)$ in the stress and couple-stress components, and $\Phi = rW/s$. For this case it can be shown that

$$rJ_k - s\hat{J}_k = \int_S [s(u_i t_{ji,k} + \varphi_i m_{ji,k}) - r(t_{ji}u_{i,k} + m_{ji}\varphi_{i,k})]n_j \, \mathrm{d}S. \tag{23}$$

If S encloses a defect then $\hat{J}_k = -J_k \neq 0$ (shown in Section 6), and (23) yields

$$J_k = \int_S \left[\frac{1}{r}(u_i t_{ji,k} + \varphi_i m_{ji,k}) - \frac{1}{s}(t_{ji}u_{i,k} + m_{ji}\varphi_{i,k})\right] n_j \, \mathrm{d}S. \tag{24}$$

This result is later used in Section 7 to evaluate the dual integrals around the crack tip in a long rectangular slab weakened by a semi-infinite crack.

4. Dual M integrals in micropolar elasticity

If the strain energy $W = W(\gamma_{ij}, \kappa_{ij})$ is a homogeneous function of degree r in both the strain and curvature components, it can be written as

$$W = \frac{1}{r}(t_{jk}\gamma_{jk} + m_{jk}\kappa_{jk}). \tag{25}$$

Being divergence-free, the energy momentum tensor (15) satisfies the equation

$$(P_{jk}x_k)_{,j} - P_{kk} = 0. \tag{26}$$

From the expression for P_{kk} in (22), and by using (25), we have

$$P_{kk} = \frac{3-r}{r}(t_{jk}u_{k,j} + m_{jk}\varphi_{k,j}) - \frac{3}{r}e_{ijk}t_{ij}\varphi_k. \tag{27}$$

The substitution into (26) then yields

$$\left[P_{jk}x_k - \frac{3-r}{r}(t_{jk}u_k + m_{jk}\varphi_k)\right]_{,j} = -e_{ijk}t_{ij}\varphi_k. \tag{28}$$

When this is subjected to the Gauss divergence theorem, we deduce an integral of the form

$$N = \int_S \left[P_{jk}x_k - \frac{3-r}{r}(t_{jk}u_k + m_{jk}\varphi_k)\right] n_j \, \mathrm{d}S = -e_{ijk}\int_V t_{ij}\varphi_k \, \mathrm{d}V. \tag{29}$$

The surface integral in the expression for N is not equal to zero but to the volume integral on the right-hand side of (29), so that the N integral is not a conserved integral. Since

$$m_{jk}\kappa_{jk} = (m_{jk}\varphi_k)_{,j} + e_{ijk}t_{ij}\varphi_k, \tag{30}$$

the expression (29) yields another nonconserved surface integral, defined by

$$M = \int_S \left(P_{jk}x_k - \frac{3-r}{r}\, t_{jk}u_k - \frac{3}{r}\, m_{jk}\varphi_k \right) n_j \, \mathrm{d}S = -\int_V m_{jk}\kappa_{jk} \, \mathrm{d}V. \tag{31}$$

For the quadratic strain energy, $r = 2$ and (31) reduces to the expression derived by Lubarda and Markenscoff (2003). In the absence of micropolar effects, there is a conservation law

$$M = \int_S \left(P_{jk}x_k - \frac{3-r}{r}\, t_{jk}u_k \right) n_j \, \mathrm{d}S = 0, \tag{32}$$

for any closed surface that does not embrace a singularity or a defect (Günther, 1962; Knowles and Sternberg, 1972; Budiansky and Rice, 1973).

4.1. Dual $\hat{M}$ and $\hat{N}$ integrals

The complementary energy momentum tensor (20) is a divergence-free tensor and thus it satisfies the equation

$$(\hat{P}_{jk}x_k)_{,j} - \hat{P}_{kk} = 0. \tag{33}$$

The complementary strain energy corresponding to the homogeneous strain energy function of degree r, given by (25), is

$$\Phi = \frac{1}{s}(t_{jk}\gamma_{jk} + m_{jk}\kappa_{jk}), \quad s = \frac{r}{r-1}. \tag{34}$$

Thus, from the second expression in (22), we obtain

$$\hat{P}_{kk} = \frac{3}{s}\left(u_k t_{jk} + \varphi_k m_{jk}\right)_{,j} + e_{ijk}t_{ij}\varphi_k. \tag{35}$$

When this is substituted into (33), there follows

$$\left[\hat{P}_{jk}x_k - \frac{3}{s}(u_k t_{jk} + \varphi_k m_{jk}) \right]_{,j} = e_{ijk}t_{ij}\varphi_k. \tag{36}$$

Therefore, upon the application of the Gauss divergence theorem, we identify a dual $\hat{N}$ integral,

$$\hat{N} = \int_S \left[\hat{P}_{jk}x_k - \frac{3}{s}(u_k t_{jk} + \varphi_k m_{jk}) \right] n_j \, \mathrm{d}S = e_{ijk}\int_V t_{ij}\varphi_k \, \mathrm{d}V, \tag{37}$$

and a dual $\hat{M}$ integral,

$$\hat{M} = \int_S \left(\hat{P}_{jk}x_k - \frac{3}{s}u_k t_{jk} - \frac{3-s}{s}\varphi_k m_{jk} \right) n_j \, \mathrm{d}S = \int_V m_{jk}\kappa_{jk} \, \mathrm{d}V. \tag{38}$$

The duality is such that $N + \hat{N} = 0$ and $M + \hat{M} = 0$, where the M and N integrals are expressed in terms of the spatial gradients of the displacement and rotation, while $\hat{N}$ and $\hat{M}$ integrals are in terms of the stress and couple-stress gradients.

In the absence of micropolar effects, there is a dual conservation law (Sun, 1985)

$$\hat{M} = \hat{N} = \int_S \left(\hat{P}_{jk}x_k - \frac{3}{s}u_k t_{jk} \right) n_j \, \mathrm{d}S = 0. \tag{39}$$

5. Dual L integrals in micropolar elasticity

To derive the L integral of isotropic micropolar elasticity, consider the identity

$$c_k = e_{kij}(t_{il}\gamma_{jl} + t_{li}\gamma_{lj} + m_{il}\kappa_{jl} + m_{li}\kappa_{lj}) = 0. \tag{40}$$

This holds because the tensors $(t_{il}\gamma_{jl} + t_{li}\gamma_{lj})$ and $(m_{il}\kappa_{jl} + m_{li}\kappa_{lj})$ are both symmetric in ij (for isotropic elasticity), as can be verified by the substitution of the constitutive expressions for stresses and couple stresses. In view of (6), the expression for c_k in (40) can be rewritten as

$$c_k = e_{kij}(t_{il}u_{l,j} + t_{li}u_{j,l} + m_{il}\varphi_{l,j} + m_{li}\varphi_{j,l} - e_{irs}t_{rs}\varphi_j). \tag{41}$$

By using the energy momentum tensor (15), the above becomes

$$c_k = e_{kij}(P_{ji} + t_{li}u_{j,l} + m_{li}\varphi_{j,l} - e_{irs}t_{rs}\varphi_j). \tag{42}$$

The energy momentum and stress tensors, in the absence of body forces, are divergence-free tensors ($P_{li,l} = 0$, $t_{li,l} = 0$). Thus, recalling that $m_{li,l} = -e_{irs}t_{rs}$, we can express c_k as

$$c_k = d_{kl,l}, \quad d_{kl} = e_{kij}(P_{li}x_j + t_{li}u_j + m_{li}\varphi_j). \tag{43}$$

Since $c_k = 0$, the application of the Gauss divergence theorem yields the conservation law

$$L_k = e_{kij}\int_S (P_{li}x_j + t_{li}u_j + m_{li}\varphi_j)n_l \, \mathrm{d}S = 0, \tag{44}$$

for any closed surface S that does not embrace a singularity or a defect.[2]

5.1. Dual $\hat{L}_k$ integral

In a dual analysis, we first introduce a dual vector $\hat{c}_k$, such that $\hat{c}_k + c_k = 0$. From (41) it follows that

$$\hat{c}_k = e_{kij}(u_{i,l}t_{lj} + u_{l,i}t_{jl} + \varphi_{i,l}m_{lj} + \varphi_{l,i}m_{jl} - \varphi_i e_{jrs}t_{rs}), \tag{45}$$

or, by using the expression for the complementary energy momentum tensor (20),

$$\hat{c}_k = e_{kij}(\hat{P}_{ji} + u_{i,l}t_{lj} + \varphi_{i,l}m_{lj} - \varphi_i e_{jrs}t_{rs} + u_{l,i}t_{jl} + u_l t_{jl,i} + \varphi_{l,i}m_{jl} + \varphi_l m_{jl,i}). \tag{46}$$

The complementary energy momentum tensor is divergence-free, and (46) can be recast in terms of the spatial gradient of $\hat{d}_{kl}$, such that

$$\hat{c}_k = \hat{d}_{kl,l}, \quad \hat{d}_{kl} = e_{kij}[\hat{P}_{li}x_j + u_i t_{lj} + \varphi_i m_{lj} + \delta_{il}(u_r t_{jr} + \varphi_r m_{jr})]. \tag{47}$$

Since $\hat{c}_k = 0$, (47) yields a dual conservation law

$$\hat{L}_k = e_{kij}\int_S [\hat{P}_{li}x_j + u_i t_{lj} + \varphi_i m_{lj} + \delta_{il}(u_r t_{jr} + \varphi_r m_{jr})]n_l \, \mathrm{d}S = 0, \tag{48}$$

for any closed surface S that does not embrace a singularity or a defect.

[2] A derivation of (44) based on Noether's theorem on invariant variational principles, for both couple-stress and micropolar elasticity, was given by Lubarda and Markenscoff (2000, 2003).

6. Dual conservation integrals and energy release rates

The conservation integrals of micropolar elasticity introduced in the previous sections can be given a physical interpretation based on the consideration of the potential and complementary potential energies. For classical nonpolar elasticity this was recently presented by Lubarda and Markenscoff (2007), who pointed out and corrected the errors in the earlier derivations of the relationship between the dual integrals and the release rates of the complementary potential energy.

6.1. Release rates of potential energy

Let the body of volume V be loaded by the surface tractions $T_i = \bar{T}_i$ over the portion S_T of its external bounding surface S, and by the surface couples $M_i = \bar{M}_i$ applied over S_M. The displacements $u_i = \bar{u}_i$ are prescribed over S_u and the rotations $\varphi_i = \bar{\varphi}_i$ over S_φ. Within a body there is an unloaded cavity (or crack) of the bounding surface S_0. The potential energy of this body is

$$\Pi = \int_V W \, \mathrm{d}V - \int_{S_T} \bar{T}_i u_i \, \mathrm{d}S - \int_{S_M} \bar{M}_i \varphi_i \, \mathrm{d}S. \tag{49}$$

Extending the nonpolar analysis of Budiansky and Rice (1973), without changing the boundary conditions on S, the rate of change of the potential energy associated with the spatial variation of the cavity surface S_0, described by its velocity field $\dot{u}_i^0$, is

$$\dot{\Pi} = \int_V \dot{W} \, \mathrm{d}V - \int_{S_0} W \dot{u}_i^0 n_i \, \mathrm{d}S - \int_{S_T} \bar{T}_i \dot{u}_i \, \mathrm{d}S - \int_{S_M} \bar{M}_i \dot{\varphi}_i \, \mathrm{d}S, \tag{50}$$

where $\dot{u}_i$ and $\dot{\varphi}_i$ are the associated kinematic fields within $V(t)$ due to the imposed velocity $\dot{u}_i^0$. The surface integral over S_0 comes from the Reynolds transport theorem, where n_i is the unit normal to S_0 directed into the material. Assuming that $\dot{u}_i$ and $\dot{\varphi}_i$ are kinematically admissible fields within $V(t)$, the rate of the strain energy is

$$\dot{W} = t_{ij}\dot{\gamma}_{ij} + m_{ij}\dot{\kappa}_{ij}, \quad \dot{\gamma}_{ij} = \dot{u}_{j,i} - e_{ijk}\dot{\varphi}_k, \quad \dot{\kappa}_{ij} = \dot{\varphi}_{j,i}. \tag{51}$$

By using the equilibrium conditions $t_{ji,j} = 0$ and $m_{ij,i} = -e_{jkl}t_{kl}$, (51) can be rewritten as

$$\dot{W} = t_{ij}\dot{\gamma}_{ij} + m_{ij}\dot{\kappa}_{ij} = (t_{ij}\dot{u}_j + m_{ij}\dot{\varphi}_j)_{,i}. \tag{52}$$

Since the surface of the cavity is unloaded, the application of the Gauss divergence theorem gives[3]

$$\int_V \dot{W} \, \mathrm{d}V = \int_{S_T} \bar{T}_j \dot{u}_j \, \mathrm{d}S + \int_{S_M} \bar{M}_i \dot{\varphi}_i \, \mathrm{d}S. \tag{53}$$

The substitution of (53) into (50) yields

$$\dot{\Pi} = -\int_{S_0} W \dot{u}_i^0 n_i \, \mathrm{d}S. \tag{54}$$

The rate of the energy release due to spatial variation of S_0, specified by a prescribed velocity field $\dot{u}_i^0$, is $f = -\dot{\Pi}$, which represents a configurational force on the cavity (defect).

[3] Eq. (53) can also be viewed as a direct consequence of the principle of virtual work.

Since $Wn_i = P_{ji}n_j$ over the unloaded surface of the cavity, we obtain from (54),

$$f = -\dot{\Pi} = \int_{S_0} P_{ji}\dot{u}_i^0 n_j \, \mathrm{d}S. \tag{55}$$

If the cavity translates with a unit velocity in the k-direction, then $\dot{u}_i^0$ can be replaced by δ_{ik}, and (55) gives the rate of the energy release per unit cavity translation in the k-direction, i.e.,

$$f_k = \int_{S_0} P_{jk}n_j \, \mathrm{d}S = J_k(S_0). \tag{56}$$

Since the cavity is assumed to be unloaded, f_k is equal to J_k, evaluated over S_0. By the conservation law $J_k = 0$, applied to the surface $S_0 + S$ bounding a region between S_0 and any closed surface S around the cavity, the configurational force f_k is also equal to J_k evaluated over S, so that $f_k = J_k(S_0) = J_k(S)$.

If the cavity is given a unit angular velocity around the k-axis, then $\dot{u}_i^0$ in (55) can be replaced by $-e_{kil}x_l$, and

$$f_k = -e_{kil}\int_{S_0} P_{ji}x_l n_j \, \mathrm{d}S = -L_k(S_0). \tag{57}$$

Again, by the conservation law $L_k = 0$ applied over $S_0 + S$, the configurational force is also equal to $-L_k$ evaluated over S, so that $f_k = -L_k(S_0) = -L_k(S)$.

Finally, if the cavity transforms such that $\dot{u}_i^0 = x_i$, the corresponding configurational force is

$$f = \int_{S_0} P_{ji}x_i n_j \, \mathrm{d}S = M(S_0). \tag{58}$$

In this case the energy release rate is with respect to $t = l(t)/l - 1$, where l is any characteristic length of the original cavity, scaling according to $l(t) = (1+t)l$. The corresponding configurational force f is equal to M evaluated over S_0, but not over any other surface enclosing the cavity. In fact, in view of (31), we have

$$f = M(S_0) = M(S) + \int_V m_{ij}\kappa_{ij} \, \mathrm{d}V, \tag{59}$$

where V is the volume between S_0 and S. In the absence of polar effects, $f = M$ evaluated over any closed surface surrounding the cavity (Budiansky and Rice, 1973).

6.2. Release rates of complementary potential energy

The complementary potential energy is defined by

$$\Omega = \int_V \Phi \, \mathrm{d}V - \int_{S_u} \bar{u}_i T_i \, \mathrm{d}S - \int_{S_\varphi} \bar{\varphi}_i M_i \, \mathrm{d}S. \tag{60}$$

Since the surface of the cavity is unloaded, we have

$$\Pi + \Omega = \int_V (W + \Phi) \, \mathrm{d}V - \int_S (T_i u_i + M_i \varphi_i) \, \mathrm{d}S = 0, \tag{61}$$

which follows from $W + \Phi = t_{ij}\gamma_{ij} + m_{ij}\kappa_{ij}$ using the equilibrium conditions (3), geometric relationships (6), and the Gauss divergence theorem. The rate of Ω, associated with a spatial variation of the cavity due to its imposed velocity field $\dot{u}_i^0$, is

$$\dot{\Omega} = \int_V \dot{\Phi}\,\mathrm{d}V - \int_{S_0} \Phi\dot{u}_i^0 n_i\,\mathrm{d}S - \int_{S_u} \bar{u}_i\dot{T}_i\,\mathrm{d}S - \int_{S_\varphi} \bar{\varphi}_i\dot{M}_i\,\mathrm{d}S, \tag{62}$$

where $\dot{T}_i$ and $\dot{M}_i$ are the loading rates on S_u and S_φ due to infinitesimal motion of S_0. Assuming that the stress and couple-stress rate fields within $V(t)$ are statically admissible ($\dot{t}_{ji,j} = 0$ and $\dot{m}_{ij,i} = -e_{jkl}\dot{t}_{kl}$), we can write

$$\dot{\Phi} = \gamma_{ij}\dot{t}_{ij} + \kappa_{ij}\dot{m}_{ij} = (u_j\dot{t}_{ij} + \varphi_j\dot{m}_{ij})_{,i}. \tag{63}$$

The rates $\dot{\sigma}_{ij}$ and $\dot{m}_{ij}$ are the rates at fixed points in space (local or nonconvected rates). Thus, by integrating (63) over the entire volume of the body,

$$\int_V \dot{\Phi}\,\mathrm{d}V = \int_S (u_j\dot{t}_{ij} + \varphi_j\dot{m}_{ij})n_i\,\mathrm{d}S + \int_{S_0} (u_j\dot{t}_{ij} + \varphi_j\dot{m}_{ij})n_i\,\mathrm{d}S. \tag{64}$$

Within a geometrically linear theory, we can write $\dot{\sigma}_{ij}n_i = \dot{T}_j$ and $\dot{m}_{ij}n_i = \dot{M}_j$ on S ($\dot{T}_j$ and $\dot{M}_j$ being equal to zero on S_T and S_M, respectively). Consequently, (64) can be rewritten as

$$\int_V \dot{\Phi}\,\mathrm{d}V = \int_{S_u} \bar{u}_j\dot{T}_j\,\mathrm{d}S + \int_{S_\varphi} \bar{\varphi}_j\dot{M}_j\,\mathrm{d}S + \int_{S_0} (u_j\dot{t}_{ij} + \varphi_j\dot{m}_{ij})n_i\,\mathrm{d}S. \tag{65}$$

The substitution into (62) then yields

$$\dot{\Omega} = -\int_{S_0} (\Phi\dot{u}_i^0 + u_j\dot{t}_{ij} + \varphi_j\dot{m}_{ij})n_i\,\mathrm{d}S. \tag{66}$$

The surface of the cavity is unloaded, so that its tractions $T_j = n_i t_{ij}$ and couples $M_j = n_i m_{ij}$ remain zero throughout the motion. This means that

$$\frac{\mathrm{d}T_j}{\mathrm{d}\tau} = \frac{\mathrm{d}n_i}{\mathrm{d}\tau}\,t_{ij} + n_i\frac{\mathrm{d}t_{ij}}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}M_j}{\mathrm{d}\tau} = \frac{\mathrm{d}n_i}{\mathrm{d}\tau}\,m_{ij} + n_i\frac{\mathrm{d}m_{ij}}{\mathrm{d}\tau} = 0, \tag{67}$$

where $\mathrm{d}/\mathrm{d}\tau$ designates the material time derivative, following the particle. Expressing the material derivative of stress as the sum of its local ($\dot{t}_{ij}$) and convected ($t_{ij,l}\dot{u}_l^0$) parts, and similarly for the material derivative of the couple-stress, we obtain from (67) the following expressions:

$$n_i\dot{t}_{ij} = -\frac{\mathrm{d}n_i}{\mathrm{d}\tau}\,t_{ij} - n_i t_{ij,l}\dot{u}_l^0, \quad n_i\dot{m}_{ij} = -\frac{\mathrm{d}n_i}{\mathrm{d}\tau}\,m_{ij} - n_i m_{ij,l}\dot{u}_l^0. \tag{68}$$

In the case when the cavity translates and/or expands in a self-similar manner, we can write $\mathrm{d}n_i/\mathrm{d}\tau = 0$, and (68) reduce to

$$n_i\dot{t}_{ij} = -n_i t_{ij,l}\dot{u}_l^0, \quad n_i\dot{m}_{ij} = -n_i m_{ij,l}\dot{u}_l^0. \tag{69}$$

When this is introduced in (66), there follows

$$\dot{\Omega} = \int_{S_0} (-\Phi\delta_{il} + u_j t_{ij,l} + \varphi_j m_{ij,l})n_i\dot{u}_l^0\,\mathrm{d}S = -\int_{S_0} \hat{P}_{il}n_i\dot{u}_l^0\,\mathrm{d}S. \tag{70}$$

Recalling that $\Pi + \Omega = 0$, and in view of (55), the release rate of the complementary potential energy due to spatial variation of the cavity is

$$f = -\dot{\Pi} = \dot{\Omega} = -\int_{S_0} \hat{P}_{il} n_i \dot{u}_l^0 \, \mathrm{d}S. \tag{71}$$

If the cavity translates with a unit velocity in the k-direction, then $\dot{u}_l^0$ can be replaced by δ_{kl}, and (71) gives the release rate of the complementary potential energy per unit cavity translation in the k-direction,

$$f_k = -\int_{S_0} \hat{P}_{ik} n_i \, \mathrm{d}S = -\hat{J}_k(S_0). \tag{72}$$

By the conservation law $\hat{J}_k = 0$ applied over $S_0 + S$, the configurational force f_k is also equal to $-\hat{J}_k$ evaluated over S. Furthermore, by comparing with (56), we conclude that $\hat{J}_k = -J_k$ over S_0, or any other closed surface surrounding the cavity.

If the cavity transforms such that $\dot{u}_l^0 = x_l$, the energy release rate is

$$f = -\int_{S_0} \hat{P}_{il} n_i x_l \, \mathrm{d}S = -\hat{M}(S_0). \tag{73}$$

Since $\hat{M}$ is not conserved, but given by (38), we have

$$f = -\hat{M}(S_0) = -\hat{M}(S) + \int_V m_{ij} \kappa_{ij} \, \mathrm{d}V, \tag{74}$$

where V is the volume between S_0 and S. When the polar effects are absent, any closed surface surrounding the cavity can be used to evaluate $f = -\hat{M}$.

If the cavity rotates within the material, then

$$\frac{\mathrm{d}n_i}{\mathrm{d}\tau} = -n_j Q_{ji}, \tag{75}$$

where Q_{ji} are the components of anti-symmetric spin matrix, and $\dot{u}_i^0 = Q_{ij} x_j$. When (75) is introduced into (68), we obtain

$$n_i \dot{t}_{ij} = (\delta_{ik} t_{lj} - t_{ij,k} x_l) n_i Q_{kl}, \quad n_i \dot{m}_{ij} = (\delta_{ik} m_{lj} - m_{ij,k} x_l) n_i Q_{kl}. \tag{76}$$

In this case, (66) gives

$$f = \dot{\Omega} = -\int_{S_0} [\hat{P}_{ik} x_l + \delta_{ik}(u_j t_{lj} + \varphi_j m_{lj})] n_i Q_{kl} \, \mathrm{d}S. \tag{77}$$

Suppose that a spin of unit magnitude is about the k-axis, then $Q_{ij} = -e_{ijk}$. The corresponding configurational force, from (77), is

$$f_k = e_{ijk} \int_{S_0} [\hat{P}_{li} x_j + \delta_{li}(u_r t_{jr} + \varphi_r m_{jr})] n_l \, \mathrm{d}S. \tag{78}$$

By comparing with $\hat{L}_k(S_0)$ in (48), we conclude that

$$f_k = \hat{L}_k, \tag{79}$$

where $\hat{L}_k$ is evaluated over S_0 or any other closed surface surrounding the cavity. Furthermore, by comparing (79) with (57), we identify the relationship between the dual L integrals, which is $\hat{L}_k = -L_k$.

7. Dual conservation integrals in two-dimensional micropolar elasticity

7.1. Dual conservation integrals for plane strain

In the case of plane strain parallel to (x_1, x_2) plane, the components φ_3, m_{13} and m_{23} are generally different from zero, while other rotation and couple-stress components are equal to zero. The corresponding energy momentum tensor and its dual can be written as

$$P_{\alpha\beta} = W\delta_{\alpha\beta} - t_{\alpha\gamma}u_{\gamma,\beta} - m_{\alpha3}\varphi_{3,\beta}, \quad P_{33} = W, \tag{80}$$

$$\hat{P}_{\alpha\beta} = \Phi\delta_{\alpha\beta} - u_\gamma t_{\alpha\gamma,\beta} - \varphi_3 m_{\alpha3,\beta}, \quad \hat{P}_{33} = \Phi, \tag{81}$$

where the Greek subscripts range from 1 to 2. The dual J integrals are

$$J_\beta = \int_C P_{\alpha\beta}n_\alpha \, \mathrm{d}C, \quad \hat{J}_\beta = \int_C \hat{P}_{\alpha\beta}n_\alpha \, \mathrm{d}C. \tag{82}$$

If a closed contour C does not surround a defect, the above integrals vanish. The dual L integrals are

$$L_3 = e_{\alpha\beta3}\int_C (P_{\gamma\alpha}x_\beta + t_{\gamma\alpha}u_\beta)n_\gamma \, \mathrm{d}C, \tag{83}$$

$$\hat{L}_3 = e_{\alpha\beta3}\int_C [\hat{P}_{\gamma\alpha}x_\beta + u_\alpha t_{\gamma\beta} + \delta_{\alpha\gamma}(u_\delta t_{\beta\delta} + \varphi_3 m_{\beta3})]n_\gamma \, \mathrm{d}C. \tag{84}$$

The nonconserved dual M integrals are

$$M = \int_C \left(P_{\alpha\beta}x_\beta - \frac{2-r}{r}\, t_{\alpha\beta}u_\beta - \frac{2}{r}\, m_{\alpha3}\varphi_3 \right) n_\alpha \, \mathrm{d}C,$$
$$\hat{M} = \int_C \left(\hat{P}_{\alpha\beta}x_\beta - \frac{2}{s}\, u_\beta t_{\alpha\beta} - \frac{2-s}{s}\varphi_3 m_{\alpha3} \right) n_\alpha \, \mathrm{d}C. \tag{85}$$

7.2. Dual conservation integrals for anti-plane strain

In the case of anti-plane strain, $u_1 = u_2 = 0$ and $u_3 = u_3(x_1, x_2)$, so that the dual energy momentum tensors are

$$P_{\alpha\beta} = W\delta_{\alpha\beta} - t_{\alpha3}u_{3,\beta} - m_{\alpha\gamma}\varphi_{\gamma,\beta}, \quad \hat{P}_{\alpha\beta} = \Phi\delta_{\alpha\beta} - u_3 t_{\alpha3,\beta} - \varphi_\gamma m_{\alpha\gamma,\beta}. \tag{86}$$

The corresponding dual integrals are given by

$$J_\beta = \int_C P_{\alpha\beta}n_\alpha \, \mathrm{d}C, \quad \hat{J}_\beta = \int_C \hat{P}_{\alpha\beta}n_\alpha \, \mathrm{d}C, \tag{87}$$

$$L_3 = e_{\alpha\beta3}\int_C (P_{\gamma\alpha}x_\beta + m_{\gamma\alpha}\varphi_\beta)n_\gamma \, \mathrm{d}C, \tag{88}$$

$$\hat{L}_3 = e_{\alpha\beta3}\int_C [\hat{P}_{\gamma\alpha}x_\beta + \varphi_\alpha m_{\gamma\beta} + \delta_{\alpha\gamma}(u_3 t_{\beta3} + \varphi_\delta m_{\beta\delta})]n_\gamma \, \mathrm{d}C. \tag{89}$$

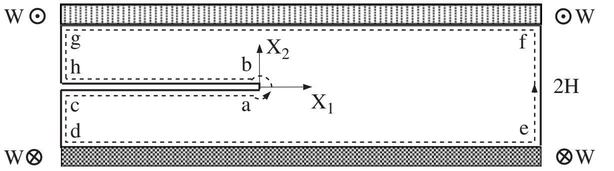

Fig. 1. An infinitely long slab of thickness $2H$ with a semi-infinite crack. The upper and lower sides of the slab are given uniform out-of-plane displacements $\pm w$.

The nonconserved dual M integrals are

$$M = \int_C \left(P_{\alpha\beta} x_\beta - \frac{2-r}{r} t_{\alpha 3} u_3 - \frac{2}{r} m_{\alpha\beta} \varphi_\beta \right) n_\alpha \, \mathrm{d}C, \tag{90}$$

$$\hat{M} = \int_C \left(\hat{P}_{\alpha\beta} x_\beta - \frac{2}{s} u_3 t_{\alpha 3} - \frac{2-s}{s} \varphi_\beta m_{\alpha\beta} \right) n_\alpha \, \mathrm{d}C. \tag{91}$$

7.3. Dual J_1 integrals around the crack tip in a long slab

An infinitely long rectangular slab is weakened by a semi-infinite crack as shown in Fig. 1. The top and bottom side of the slab are given uniform out-of-plane displacements $\pm w$. The objective is to evaluate dual integrals J_1 and $\hat{J}_1$ along a small contour ab around the crack tip. The J_1 integral is

$$J_1 = \int_C [W n_1 - (t_{13} u_{3,1} + m_{11} \varphi_{1,1} + m_{12} \varphi_{2,1}) n_1 - (t_{23} u_{3,1} + m_{21} \varphi_{1,1} + m_{22} \varphi_{2,1}) n_2] \, \mathrm{d}C.$$

Since $J_1 = 0$ along the unloaded crack faces bh and ca, there is path-independent property

$$J_1^{ab} = J_1^{cdefghb}. \tag{92}$$

As in the nonpolar elasticity, the only nonvanishing contribution to J_1 along the path $cdefghb$ is from the segment ef. Since the field variations with respect to x_1 along the segment ef, sufficiently far from the crack tip, can be ignored, the results from the Appendix can be used to obtain

$$J_1^{ef} = \int_e^f W \, \mathrm{d}x_2 = w t_{23}(f) = K \frac{\mu w^2}{H}, \tag{93}$$

where

$$K = \left[1 - \frac{\bar{\mu}}{\mu + \bar{\mu}} \frac{\tanh(kH)}{kH} \right]^{-1}, \quad k^2 = \frac{4\mu\bar{\mu}}{(\mu + \bar{\mu})(\alpha + \bar{\alpha})}. \tag{94}$$

An analogous expression was derived by Atkinson and Leppington (1974) for the Mode II loading of a cracked slab.

The $\hat{J}_1$ integral is defined by

$$\hat{J}_1 = \int_C [\Phi n_1 - (u_3 t_{13,1} + \varphi_1 m_{11,1} + \varphi_2 m_{12,1}) n_1 - (u_3 t_{23,1} + \varphi_1 m_{21,1} + \varphi_2 m_{22,1}) n_2] \, \mathrm{d}C.$$

Since $\hat{J}_1 = 0$ along the unloaded crack faces bh and ca, there is a path-independent property of the dual integral,

$$\hat{J}_1^{ab} = \hat{J}_1^{cdefghb}. \tag{95}$$

The only nonvanishing contributions to $\hat{J}_1$ along $cdefghb$ are from the segments de, fg, and ef. These contributions are, respectively,

$$\hat{J}_1^{de} = -w \int_d^e t_{23,1}\, \mathrm{d}x_1 = -wt_{23}(e), \tag{96}$$

$$\hat{J}_1^{fg} = w \int_f^g t_{23,1}\, \mathrm{d}x_1 = -wt_{23}(f), \tag{97}$$

$$\hat{J}_1^{ef} = \int_e^f \Phi\, \mathrm{d}x_2. \tag{98}$$

As shown in the Appendix, the stress t_{23} at the points e and f is the same, so that

$$\hat{J}_1^{ab} = \int_e^f \Phi\, \mathrm{d}x_2 - 2wt_{23}(f). \tag{99}$$

Since for linear elasticity $W = \Phi$, and since $t_{23}(f) = \mu K w/H$, the above gives

$$\hat{J}_1^{ab} = -wt_{23}(f) = -K\,\frac{\mu w^2}{H}. \tag{100}$$

In retrospect, we note that J_1 and $\hat{J}_1$ can be calculated without actually evaluating the integral of the strain energy along the segment ef, by combining the expressions for the two dual integrals. Indeed, from (93) and (99), we have

$$\hat{J}_1^{ab} = J_1^{ab} - 2wt_{23}(f). \tag{101}$$

Since $\hat{J}_1 = -J_1$, there follows

$$J_1^{ab} = wt_{23}(f) = K\,\frac{\mu w^2}{H}. \tag{102}$$

This result can also be obtained from (24), with $r = s = 2$, which gives

$$J_1 = \frac{1}{2}\int_C [(u_3 t_{13,1} + \varphi_1 m_{11,1} + \varphi_2 m_{12,1} - t_{13} u_{3,1} - m_{11}\varphi_{1,1} + m_{12}\varphi_{2,1})n_1 + (u_3 t_{23,1} + \varphi_1 m_{21,1} + \varphi_2 m_{22,1} - t_{23} u_{3,1} - m_{21}\varphi_{1,1} - m_{22}\varphi_{2,1})n_2]\, \mathrm{d}C.$$

The only nonvanishing contributions to this integral along the path $hgfedcb$ are the two equal contributions from the horizontal segments de and fg. Since rotations vanish along these segments, we obtain

$$J_1 = \int_f^g wt_{23,1}\,(-\mathrm{d}x) = wt_{23}(f) = \frac{K\mu w^2}{H}.$$

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Appendix A. Out-of-plane shearing of a long rectangular slab

The upper and lower sides of an infinitely long rectangular slab of height $2H$ are given uniform out-of-plane displacements $\pm w$, while keeping the rotation components constrained, i.e., $u_3 = \pm w$ and $\varphi_1 = \varphi_2 = 0$ at $x_2 = \pm H$.[4] The slab being infinitely long, with uniform boundary conditions, suggests that in the interior of the slab

$$u_3 = u_3(x_2), \quad \varphi_1 = \varphi_1(x_2), \quad \varphi_2 = 0.$$

The nonvanishing strain and curvature components are

$$\gamma_{32} = \varphi_1, \quad \gamma_{23} = u_{3,2} - \varphi_1, \quad \kappa_{21} = \varphi_{1,2},$$

with the corresponding stress and couple-stress components

$$t_{23} = (\mu + \bar{\mu})u_{3,2} - 2\bar{\mu}\varphi_1, \quad t_{32} = (\mu - \bar{\mu})u_{3,2} + 2\bar{\mu}\varphi_1,$$

$$m_{21} = (\alpha + \bar{\alpha})\varphi_{1,2}, \quad m_{12} = (\alpha - \bar{\alpha})\varphi_{1,2}.$$

The two equilibrium equations are

$$(\mu + \bar{\mu})u_{3,22} = 2\bar{\mu}\varphi_{1,2},$$

$$(\alpha + \bar{\alpha})\varphi_{1,22} - 4\bar{\mu}\varphi_1 = -2\bar{\mu}u_{3,2}.$$

Their solution is found to be

$$u_3 = Kw\left[\frac{x_2}{H} - \frac{\bar{\mu}}{\mu + \bar{\mu}}\frac{\sinh(kx_2)}{kH\cosh(kH)}\right],$$

$$\varphi_1 = \frac{Kw}{2H}\left[1 - \frac{\cosh(kx_2)}{\cosh(kH)}\right],$$

where

$$K = \left[1 - \frac{\bar{\mu}}{\mu + \bar{\mu}}\frac{\tanh(kH)}{kH}\right]^{-1}, \quad k^2 = \frac{4\mu\bar{\mu}}{(\mu + \bar{\mu})(\alpha + \bar{\alpha})}.$$

The corresponding strain and curvature components are

$$\gamma_{32} = \varphi_1, \quad \gamma_{23} = \frac{Kw}{2H}\left[1 + \frac{\mu - \bar{\mu}}{\mu + \bar{\mu}}\frac{\cosh(kx_2)}{\cosh(kH)}\right],$$

$$\kappa_{21} = -\frac{Kkw}{2H}\frac{\sinh(kx_2)}{\cosh(kH)}.$$

[4] The effects of the boundary and interface conditions in couple-stress elasticity problems were discussed by Lubarda (2003).

The stress and couple-stress components in the slab follow as

$$t_{32} = \frac{Kw}{H}\,\mu\left[1 - \frac{2\bar{\mu}}{\mu+\bar{\mu}}\frac{\cosh(kx_2)}{\cosh(kH)}\right], \quad t_{23} = \frac{Kw}{H}\,\mu,$$

$$m_{12} = -(\alpha - \bar{\alpha})\frac{Kkw}{2H}\frac{\sinh(kx_2)}{\cosh(kH)}, \quad m_{21} = -(\alpha + \bar{\alpha})\frac{Kkw}{2H}\frac{\sinh(kx_2)}{\cosh(kH)}.$$

The strain energy density at an arbitrary point of the slab is

$$W = \tfrac{1}{2}[(\mu+\bar{\mu})(\gamma_{23}^2 + \gamma_{32}^2) + 2(\mu - \bar{\mu})\gamma_{23}\gamma_{32} + (\alpha + \bar{\alpha})\kappa_{21}^2],$$

while the strain energy per unit length of the slab in the x_1 direction is

$$\int_{-H}^{H} W\,\mathrm{d}x_2.$$

The evaluation of W, by using the derived expressions for the strain and curvature components and a tedious integration, can be circumvented by observing that the above strain energy integral must be equal to the external work done per unit length of the slab, i.e.,

$$\int_{-H}^{H} W\,\mathrm{d}x_2 = t_{23}(H)w = \frac{K\mu w^2}{H}\,\mu.$$

This result is conveniently used in Section 7 to derive the expressions for the dual J_1 integrals around the crack tip in a long cracked slab.

References

Asaro, R.J., Lubarda, V.A., 2006. Mechanics of Solids and Materials. Cambridge University Press, Cambridge.

Atkinson, C., Leppington, F.G., 1974. Some calculations of the energy-release rate G for cracks in micropolar and couple-stress elastic media. Int. J. Fract. 10, 599–602.

Brulin, O., Hsieh, R.K.T. (Eds.), 1982. Mechanics of Micropolar Media. World Scientific, Singapore.

Budiansky, B., Rice, J.R., 1973. Conservation laws and energy-release rates. J. Appl. Mech. 40, 201–203.

Bui, H.D., 1973. Dualité entre les intégrals indépendentes du contour dans la théorie des solides fissurés. C. R. Acad. Sci. Paris 276, 1425–1428.

Bui, H.D., 1974. Dual path independent integrals in the boundary-value problems of cracks. Eng. Fract. Mech. 6, 287–296.

Bui, H.D., 1994. Inverse Problems in the Mechanics of Materials: An Introduction. CRC Press, Boca Raton, FL.

Carlsson, A.J., 1974. Path independent integrals in fracture mechanics and their relation to variational principles. In: Sih, G.C., van Elst, H.C., Broek, D. (Eds.), Prospects of Fracture Mechanics. Noordhoff, Leyden, The Netherlands, pp. 139–158.

Cosserat, E., Cosserat, F., 1909. Theorie des Corps Deformables. Hermann, Paris.

Dai, T.A., 1986. Some path-independent integrals for micropolar media. Int. J. Solids Struct. 22, 729–735.

Eringen, A.C., 1999. Microcontinuum Field Theories. Springer, New York.

Eshelby, J.D., 1951. The force on an elastic singularity. Philos. Trans. R. Soc. A 244, 87–112.

Eshelby, J.D., 1956. The continuum theory of lattice defects. Solid State Phys. 3, 79–144.

Eshelby, J.D., 1975. The elastic energy-momentum tensor. J. Elasticity 5, 321–335.

Günther, W., 1962. Über einige Randintegrale der Elastomechanik. Abh. Braunschw. Wiss. Ges. 14, 53–72.

Gurtin, M.E., 2000. Configurational Forces as Basic Concepts of Continuum Mechanics. Springer, New York.

Jaric, J.P., 1986. The energy-release rate in quasi-static crack propagation and J-integral. Int. J. Solids Struct. 22, 767–778.

Kienzler, R., Herrmann, G., 2001. Mechanics in Material Space. Springer, Berlin.

Knowles, J.K., Sternberg, E., 1972. On a class of conservation laws in linearized and finite elastostatics. Arch. Ration. Mech. Anal. 44, 187–211.

Lazar, M., Kirchner, H.O.K., 2007. The Eshelby stress tensor, angular momentum tensor and scaling flux in micropolar elasticity. Int. J. Solids Struct. 44, 4613–4620.

Li, S., Gupta, A., 2006. On dual configurational forces. J. Elasticity 84, 13–31.

Li, X., 1988. Dual conservation laws in elastostatics. Eng. Fract. Mech. 29, 233–241.

Lubarda, V.A., 2003. Circular inclusions in anti-plane strain couple stress elasticity. Int. J. Solids Struct. 40, 3827–3851.

Lubarda, V.A., Markenscoff, X., 2000. Conservation integrals in couple stress elasticity. J. Mech. Phys. Solids 48, 553–564.

Lubarda, V.A., Markenscoff, X., 2003. On conservation integrals in micropolar elasticity. Philos. Mag. A 83, 1365–1377.

Lubarda, V.A., Markenscoff, X., 2007. Dual conservation integrals and energy release rates. Int. J. Solids Struct. 44, 4079–4091.

Maugin, G.A., 1995. Material forces: concepts and applications. Appl. Mech. Rev. 48, 247–285.

Maugin, G.A., 1998. On the structure of the theory of polar elasticity. Philos. Trans. R. Soc. London A 356, 1367–1395.

Mindlin, R.D., 1964. Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–78.

Moran, B., Shih, C.F., 1987. A general treatment of crack tip contour integrals. Int. J. Fract. 35, 295–310.

Nowacki, W., 1986. Theory of Asymmetric Elasticity. Pergamon, Oxford, UK.

Pucci, E., Saccomandi, G., 1990. Symmetries and conservation laws in micropolar elasticity. Int. J. Eng. Sci. 28, 557–562.

Rice, J.R., 1968. A path independent integral and approximate analysis of strain concentration by notches and cracks. J. Appl. Mech. 38, 379–386.

Rice, J.R., 1985. Conserved integrals and energetic forces. In: Bilby, B.A., Miller, K.J., Willis, J.R. (Eds.), Fundamentals of Deformation and Fracture. Cambridge University Press, Cambridge, pp. 33–56.

Sun, S.-X., 1985. Dual conservation laws in elastostatics. Int. J. Eng. Sci. 23, 1179–1186.

Trimarco, C., Maugin, G.A., 1995. Bui's path-independent integral in finite elasticity. Meccanica 30, 139–145.

Voigt, W., 1887. Theoritiscke Studienuber die Elastizitatsverhaltnisse der Krystalle. Abhandl. Ges. Wiss. Gottingen 34, 3–51.

Vukobrat, M.D., 1989. Conservations laws in micropolar elastodynamics and path-independent integrals. Int. J. Eng. Sci. 27, 1093–1106.