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On the Stress Field in Sliding Ellipsoidal Inclusions With Shear Eigenstrain

A stress distribution in elliptical inclusions with sliding interface and uniform shear eigenstrain is analyzed. The existence of a nonvanishing stress in nearly circular inclusions is demonstrated, and an approximate method for its determination is suggested. An inherent nonlinear dependence of the stress on applied eigenstrain is indicated and discussed.

1 Introduction

There is an unusual dependence of stress on the semi-axes ratio and applied shear eigenstrain in ellipsoidal or elliptical inclusions with sliding interface. Mura and Furuhashi (1984) found that, under certain conditions, the stresses vanish everywhere in the sliding inclusion and the matrix. The result does not apply to spherical or circular inclusions, for which there is a nonvanishing stress in both the inclusion and the matrix. Since the solution for a spherical shape does not follow as a special case from available solution for an ellipsoidal shape, the situation was referred to as an anomaly of sliding inclusions (Kouris, Tsuchida, and Mura, 1986). In this paper, we explain this unusual situation and show by geometric considerations that, for sufficiently large levels of applied shear eigenstrain, there must be a nonvanishing stress field in nearly circular or spherical inclusions. In particular, we introduce for elliptical inclusions the parameter $s = (a - b)/b\gamma$, where a and b are the principal semi-axes of the ellipse, and γ is the applied shear eigenstrain, and show that the stress in a sliding inclusion is proportional to γ if $s \ll 1$, and proportional to γ^2 (thus, not observable at the level of linear elasticity) if $s \gg 1$. The latter occurs at all levels of infinitesimal strain γ for elliptical inclusions whose semi-axes differ distinctly, and for nearly circular inclusions at very small values of γ. If the parameter s is of the order of one, the stress depends on γ in a more involved, nonlinear manner, rapidly decreasing with the ratio a/b for any given value of γ. The inherent nonlinear dependence of the stress on applied shear eigenstrain is a peculiar feature of ellipsoidal sliding inclusions, because neither large overall strain nor large rotation actually take place in the considered problem. An analogous situation occurs in the case of sliding elliptical or ellipsoidal inclusions under remote uniform shear loading.

2 Geometric Considerations

Consider a two-dimensional situation in which an infinite block of elastic material is subjected to uniform infinitesimal shear strain ϵ^*_{xy}, denoted in sequel by γ, so that the displacement field in rectangular coordinates is given by $u = \gamma y$, $v = \gamma x$. A point with the coordinates (x, y) on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (1)$$

moves to the point (X, Y) with the coordinates $X = x + \gamma y$ and $Y = y + \gamma x$, so that (Fig. 1)

$$x = \frac{1}{1-\gamma^2}(X - \gamma Y), \quad y = \frac{1}{1-\gamma^2}(Y - \gamma X). \qquad (2)$$

Substitution into Eq. (1) thus gives

$$\frac{(X-\gamma Y)^2}{a^2} + \frac{(Y-\gamma X)^2}{b^2} = (1-\gamma^2)^2. \qquad (3)$$

This is an equation of the deformed ellipse, which can be rewritten more conveniently by using the coordinates (ξ, η), parallel to the new principal directions of the ellipse. Thus, denoting by φ the angle between the old and new major axes of the ellipse, we have

$$X = \xi \cos\varphi - \eta \sin\varphi, \quad Y = \xi \sin\varphi + \eta \cos\varphi. \qquad (4)$$

Equation (3) consequently becomes

$$\frac{\xi^2}{A^2} + \frac{\eta^2}{B^2} - C\xi\eta = 1, \qquad (5)$$

where A and B are defined by

$$A = \frac{(1-\gamma^2)a}{[(\cos\varphi - \gamma\sin\varphi)^2 + (a^2/b^2)(\sin\varphi - \gamma\cos\varphi)^2]^{1/2}}, \qquad (6)$$

$$B = \frac{(1-\gamma^2)b}{[(\cos\varphi + \gamma\sin\varphi)^2 + (b^2/a^2)(\sin\varphi + \gamma\cos\varphi)^2]^{1/2}}, \qquad (7)$$

and

$$C = \frac{1}{1-\gamma^2}\left[\frac{1}{a^2}\left(\sin 2\varphi + \frac{2\gamma}{1-\gamma^2}\cos 2\varphi\right) - \frac{1}{b^2}\left(\sin 2\varphi - \frac{2\gamma}{1-\gamma^2}\cos 2\varphi\right)\right]. \qquad (8)$$

The lengths A and B are the lengths of the principal semi-axes of the deformed ellipse, provided that the angle φ is such that $C = 0$, which gives

$$\tan 2\varphi = \frac{a^2 + b^2}{a^2 - b^2}\frac{2\gamma}{1-\gamma^2}. \qquad (9)$$

The variation of A/a versus a/b for two selected values of γ is

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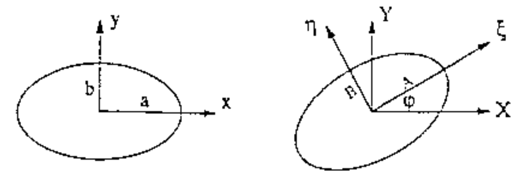

Fig. 1 An ellipse with the principal semi-axes a and b is deformed under uniform shear into an ellipse with the principal semi-axes A and B. The new principal axes of the ellipse are at an angle φ relative to the old axes.

shown in Fig. 2. As the ratio a/b increases from the value of 1, A rapidly approaches a, more so the smaller the value of the applied shear strain. Figure 3 shows plots of A versus γ for two values of the ratio a/b. Except for the circular shape ($a = b$), for which A is a linear function of γ, for all elliptical shapes there is at first a range of γ for which A is essentially equal to a, beyond which A begins to notably increase with γ. This range is more extended the larger the ratio a/b, and for $a/b = 1.02$ it already extends throughout the whole elastic strain range of approximately 10^{-3}. Similar remarks apply to the semi-axis B.

To determine which material direction (semi-axis) of the undeformed ellipse becomes a principal semi-axis of the deformed ellipse, consider an arbitrary point of the undeformed ellipse, with the coordinates $x = a\cos\theta$ and $y = b\sin\theta$, where θ is a parameter of the ellipse. The corresponding point on the deformed ellipse is $X = a\cos\theta + \gamma b\sin\theta$ and $Y = b\sin\theta + \gamma a\cos\theta$. If (X, Y) is to be an apex of the deformed ellipse, the extremum condition $d(X^2 + Y^2)/d\theta = 0$ applies, which defines the angle θ to be

$$\tan 2\theta = \frac{2ab}{a^2 - b^2}\,\frac{2\gamma}{1 + \gamma^2}. \tag{10}$$

The angle ψ, defined as $\tan\psi = (b/a)\tan\theta$, specifies the direction of a semi-axis in the undeformed configuration which becomes the principal semi-axis in the deformed configuration. The angle defining the actual rotation of material lines is $\omega = \varphi - \psi$. As a/b increases, ω rapidly approaches the value of the applied shear strain γ. Thus, although the angle φ may increase up to $\pi/4$, the angle ω remains infinitesimal.

2.1 Limiting Behavior. If the right-hand side of Eq. (9) is sufficiently small for the approximation $\tan 2\varphi \approx 2\varphi$ to accurately apply, we have

$$\varphi = \sin\varphi = \frac{a^2 + b^2}{a^2 - b^2}\gamma,\quad \cos\varphi = 1 - \frac{1}{2}\left(\frac{a^2 + b^2}{a^2 - b^2}\right)^2\gamma^2, \tag{11}$$

neglecting the cubic and higher-order terms in γ. Upon an appropriate expansion, assuming a sufficiently small ratio $b\gamma/(a - b)$, Eqs. (6) and (7) yield

$$A = \left(1 + \frac{1}{2}\,\frac{a^2 + 3b^2}{a^2 - b^2}\gamma^2\right)a, \tag{12}$$

$$B = \left(1 - \frac{1}{2}\,\frac{3a^2 + b^2}{a^2 - b^2}\gamma^2\right)b. \tag{13}$$

Thus, the undeformed and deformed principal semi-axes differ by terms proportional to γ^2. This difference cannot be observed at the level of linear elasticity, and the two ellipses are identical at that level ($A = a$, $B = b$).

In the other limiting case, $(a - b)/b\gamma$ is sufficiently small and the approximation $\cot 2\varphi = (a - b)/2b\gamma$ holds, i.e.,

$$\sin\varphi = \frac{\sqrt{2}}{2}\left(1 - \frac{a - b}{4b\gamma}\right),\quad \cos\varphi = \frac{\sqrt{2}}{2}\left(1 + \frac{a - b}{4b\gamma}\right). \tag{14}$$

The subsequent expansions in Eqs. (6) and (7) yield

$$A = (1 + \gamma)a - \frac{a - b}{2}, \tag{15}$$

$$B = (1 - \gamma)b + \frac{a - b}{2}. \tag{16}$$

Therefore, for a given nonzero γ, if $(a - b) \ll b\gamma$ the deformed ellipse differs in size distinctly from the undeformed ellipse.

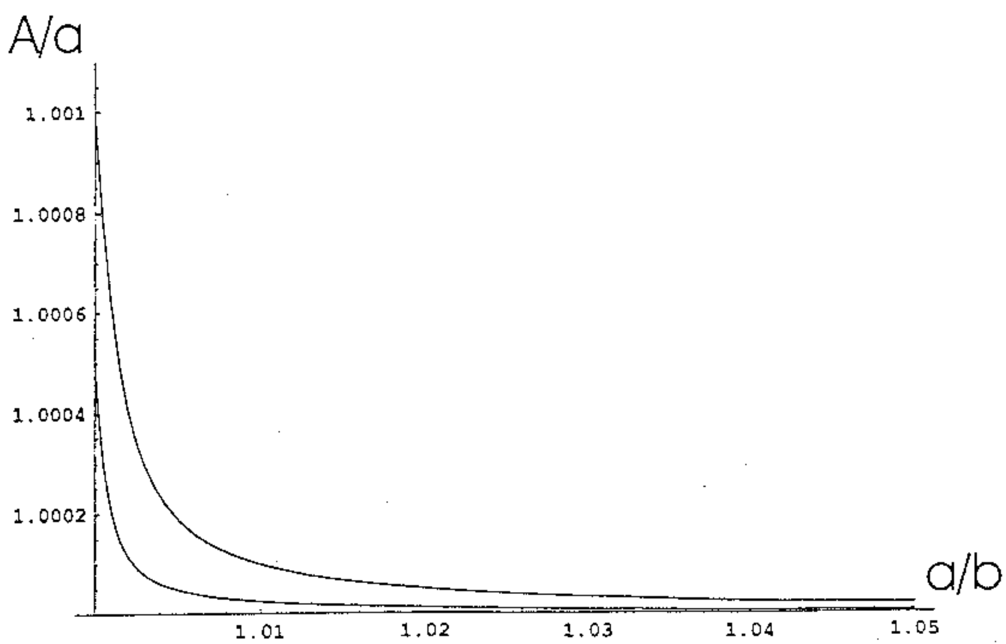

Fig. 2 The ratio A/a of the principal semi-axes of the ellipse in the deformed and undeformed configuration rapidly decreases with the ratio a/b. The upper curve corresponds to strain $\gamma = 10^{-3}$ and the lower to $\gamma = 5\cdot 10^{-4}$.

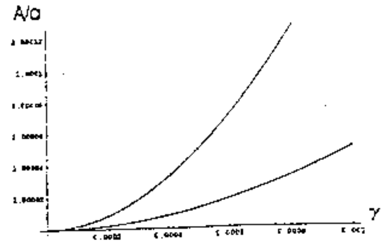

Fig. 3 The variation of A/a with the applied shear strain γ. Except for the circular shape, there is initially a range of γ for which A is essentially equal to a.

Note that for a given level of the shear strain γ, an ellipse with a small ratio $(a-b)/2b\gamma$ has the major semi-axis in the deformed configuration at an angle φ nearly equal to 45 deg, relative to the direction of the major semi-axis in the undeformed configuration. This, of course, does not imply any large rotations taking place, since the major semi-axis in the deformed configuration consists of the material points that were aligned in the undeformed configuration along the direction at an angle ψ, also nearly equal to 45 deg.

2.2 Approximate Expressions. Expressions (6) and (7) have simple but accurate approximations throughout the whole range of γ and a/b, given by

$$A = \left[1 + \frac{\gamma}{1 + k(a-b)/b\gamma}\right]a, \tag{17}$$

$$B = \left[1 - \frac{\gamma}{1 + k(a-b)/b\gamma}\right]b, \tag{18}$$

where $k = \frac{3}{2}$. Evidently, if $(a-b)/b\gamma$ is large, A and B become, within terms proportional to γ^2, equal to a and b. For small values of $(a-b)/b\gamma$, $A/a = 1 + \gamma - k(a-b)/b$, and $B/b = 1 - \gamma + k(a-b)/b$. Numerical evaluations confirm close agreement with exact results. Equations (17) and (18) will be conveniently utilized later in the text.

3 Sliding Inclusion

If the uniform eigenstrain in the elliptical inclusion is pure shear ϵ^*_{xy}, the normal and shear component of the traction at the bonded interface between the inclusion and the surrounding matrix are obtained from the Eshelby (1957) solution as

$$\sigma_n^E = \sigma_{xy}^E \frac{2a^2b^2xy}{b^4x^2 + a^4y^2}, \tag{19}$$

$$\tau_n^E = \sigma_{xy}^E \frac{b^4x^2 - a^4y^2}{b^4x^2 + a^4y^2}. \tag{20}$$

The shear stress $\sigma_{xy}^E = -2k\mu\epsilon^*_{xy}$, with $k = (1-\nu)^{-1}ab/(a+b)^2$, is the Eshelby uniform stress in the inclusion, and μ and ν are the shear modulus and Poisson ratio. Mura and Furuhashi (1984) considered the sliding ellipsoidal inclusion, in which the shear stress at the interface is relaxed to zero by allowing interface sliding or slipping to occur, preserving the normal components of displacement and traction continuous across the interface. The solution can in principle be obtained by superimposing on the Eshelby solution the Volterra solution for a Somigliana dislocation, associated with an appropriate slip variation $\hat{b}_i$ along the interface, i.e.,

$$u_i^V(\mathrm{x}) = \int_S G_{ij,k}(\mathrm{x}-\mathrm{x}')C_{jklm}\hat{b}_l(\mathrm{x}')n_m(\mathrm{x}')dS. \tag{21}$$

The two-dimensional Green's function of an infinite elastic medium is denoted by G_{ij}, a comma denotes differentiation with respect to unprimed coordinates, and C_{ijkl} are the elastic moduli of isotropic material. The slip (or the variable Burgers vector along the interface S) has to be such that $\hat{b}_i n_i = 0$ for the continuity of normal displacement across the interface, and $\tau_n^E + \tau_n^V = 0$ for the vanishing of the shear traction at the interface. It is not easy to analytically determine, or to predict in advance, the structure of $\hat{b}_i$ for an arbitrary ellipsoidal shape of the inclusion, since the problem reduces to integral equations which are difficult to solve (e.g., Wong and Barnett, 1985). Mura and Furuhashi (1984), however, observed that a Somigliana dislocation based on the slip

$$\hat{b}_x = (\epsilon^*_{xy} + \omega_{xy})y, \quad \hat{b}_y = (\epsilon^*_{xy} - \omega_{xy})x, \tag{22}$$

cancels the stress field of the Eshelby inclusion, provided that

$$\omega_{xy} = \frac{a^2 + b^2}{a^2 - b^2}\epsilon^*_{xy}. \tag{23}$$

The right-hand side of Eq. (23) is precisely the angle φ in Eq. (11) of the previous section between the principal semi-axes of the undeformed and deformed ellipse under pure shear, in the case when the deformed ellipse, within the accuracy of linear elasticity, has the same size as the undeformed ellipse ($A = a$, $B = b$). There is no stress left in such an inclusion, because the sheared inclusion can be simply rotated by the angle ω_{xy} and inserted into the matrix without any stress (Mura, 1985). This, of course, cannot be done for the inclusion with bonded interface, where no slip is allowed and where each point on the surface of the inclusion has to be in contact with the point of the surrounding matrix to which it was attached before the eigenstrain took place.

In Section 2 it was found that an ellipse is transformed into an identical ellipse if $b\gamma/(a-b)$ is sufficiently small. For example, $b\gamma/(a-b)$ is less than $5\cdot 10^{-2}$ if $a/b > 1 + 20\gamma$, which for the elastic shear strain of 10^{-3} gives $a/b > 1.02$. For smaller values of the ratio a/b, the principal semi-axes of the deformed ellipse are increasingly more distinct from the undeformed semi-axes, and the inclusion cannot be placed back into the matrix simply by a rotation. The stress must set up to overcome the size difference, which is, for a given eigenstrain, more pronounced the closer the elliptical shape is to the circular shape. Alternatively, in order that terms proportional to ω_{xy} in Eq. (22) represent displacements due to rotation, ω_{xy} in Eq. (23) must be infinitesimal. This is the case if the ratio $\epsilon^*_{xy}b/(a-b)$ is sufficiently small. For larger values of this ratio, significant stresses arise, and no rotation (however large) can by itself place the sliding inclusion back into the matrix.

The slip variation along the boundary of a circular sliding inclusion with uniform shear eigenstrain γ is $\hat{b} = \gamma(x^2 - y^2)/2a$, i.e.,

$$\hat{b}_x = \frac{\gamma}{2}y - \frac{\gamma}{a^2}x^2y, \quad \hat{b}_y = \frac{\gamma}{2}x - \frac{\gamma}{a^2}xy^2, \tag{24}$$

where a is the radius of the inclusion. Note that $\hat{b}$ is proportional to τ_n^E on the boundary of the inclusion. The stress distribution within the inclusion is given by quadratic polynomials in the coordinates x/a and y/a, as

$$\sigma_{xx} = \sigma_{yy} = -\frac{3\mu\gamma}{2(1-\nu)}\frac{xy}{a^2}, \tag{25}$$

$$\sigma_{xy} = -\frac{3\mu\gamma}{4(1-\nu)}\left(1 - \frac{x^2+y^2}{a^2}\right). \tag{26}$$

This was obtained from the Eshelby and Volterra solutions. The average normal stress in the sliding inclusion is zero ($\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = 0$), while the average shear stress is $\bar{\sigma}_{xy} = 3\sigma_{xy}^E/4$, where

$\sigma^E_{xy} = -\mu\gamma/2(1-\nu)$ is the uniform Eshelby stress in the inclusion with bonded interface.

The fact that the stress within the inclusion is given by a polynomial of second degree was expected, since the slip $\hat{b}$ in Eq. (24) gives rise to a quadratic "eigenstrain" $\hat{\epsilon}_{ij} = (\hat{b}_{i,j} + \hat{b}_{j,i})/2$, introduced by formally extending the functions $\hat{b}_i$ from the boundary inside the inclusion. Such a procedure is often utilized in the analysis of Somigliana dislocations (Asaro, 1975).

While in the case of bonded interface, where only uniform shear stress σ^E_{xy} sets up in the inclusion, in the case of sliding inclusion normal and shear stresses arise, both being nonuniformly distributed within the inclusion. That this is so can also be recognized directly from Eqs. (19) and (20). For a circular inclusion, the shear stress τ^E_n equilibrates itself (this, incidentally, is not the case for an elliptical inclusion, for which τ^E_n gives a net moment), and the normal traction given by Eq. (19), applied alone on the boundary, clearly gives a non-uniform stress distribution within the inclusion. Of course, there is also an additional normal traction that must be supplied at the boundary of the inclusion to insure the continuity of normal displacement across the interface, upon the removal of the shear traction τ^E_n there. Indeed, it can be shown (Lubarda and Markenscoff, 1997) that the normal traction at the bonded interface increases by a factor $\frac{3}{2}$, when the shear stress at the interface is relaxed to zero, allowing for the slip to take place.

4 Approximate Stress Expressions

A nonvanishing stress in elliptical inclusions with shear eigenstrain γ can exist only in nearly circular inclusions, whose semi-axes differ by at most a few percent. Such inclusions are normally modeled as circular at the level of small deformation theory. Indeed, although a deformed configuration in infinitesimal theory can differ from the undeformed configuration by up to about two percent, the governing equilibrium equations and boundary conditions are commonly written with respect to the undeformed configuration. Consequently, the stress field in a nearly circular inclusion will be assumed to be as in an average circular inclusion of radius $a_0 = (a+b)/2$, but with an incorporated effect of the length scale $(a-b)$, to which the stress is strongly sensitive when this scale is of the order of $b\gamma$. This will be taken into account by scaling the magnitude of stress by the factor $(A-a)/a$, because it is this size factor that is responsible for the build-up of the stress in sliding elliptical inclusions, when $(a-b) \sim b\gamma$. Thus, from Eqs. (25) and (26),

$$\sigma_{xx} = \sigma_{yy} = -\frac{3\mu\gamma^*}{2(1-\nu)}\frac{xy}{a_0^2}, \tag{27}$$

$$\sigma_{xy} = -\frac{3\mu\gamma^*}{4(1-\nu)}\left(1 - \frac{x^2+y^2}{a_0^2}\right). \tag{28}$$

The apparent strain $\gamma^* = (A-a)/a$ can be calculated from the exact expression (6), or, more conveniently, from the approximate but accurate expression given by Eq. (17). The latter gives

$$\gamma^* = \frac{\gamma}{1 + k(a-b)/b\gamma}. \tag{29}$$

Note that γ^* is also equal to $(b-B)/b$. The average stress in the sliding inclusion can be obtained by integration from Eqs. (27) and (28). The result is

$$\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = 0, \quad \bar{\sigma}_{xy} = -\frac{3\mu\gamma^*}{8(1-\nu)}. \tag{30}$$

For nearly circular inclusions, there is at first a range of γ for which there is no observable stress in the inclusion (the stress being proportional to γ^2). For these very small values of γ, a nearly circular inclusion actually behaves as a very elliptical inclusion, carrying no stress, in accordance with the Mura and Furuhashi result. Beyond this initial range of γ, the stress begins to substantially increase and approach the linear dependence on the eigenstrain γ, faster so smaller the ratio a/b. For the circular inclusion the nonlinear domain shrinks to a point (since $(a-b) \sim b\gamma$ implies $\gamma = 0$ when $a = b$), and the stress becomes a linear function of γ throughout the whole range of γ. Regarding the variation of the stress with the ratio a/b for a given eigenstrain γ, the stress rapidly diminishes to zero as the ratio a/b increases.

5 Discussion

The result that the stress in elliptical sliding inclusions due to shear eigenstrain γ vanishes if the angle φ is small, applies not only to elliptical inclusions with distinctly different semi-axes a and b, but also to nearly circular inclusions at sufficiently small values of γ, which make the parameter $s = (a-b)/b\gamma$ large. Therefore, the initial slope of the stress versus γ curve for nearly circular inclusions is essentially equal to zero. But, as γ increases (still staying within the infinitesimal range), s decreases and the stress eventually begins to increase with a substantial slope relative to γ. Consequently, the stress dependence on γ is inherently nonlinear. When the parameter s becomes small enough, the stress is simply proportional to γ^2, thus not observable at the level of linear elasticity. This explains why the attempts to calculate the stress in nearly circular sliding inclusions by the equations of linear elasticity (e.g., Kouris et al., 1986) were not successful. The point is subtle, because the linear equations of elasticity usually suffice in problems where no large overall strain or rotation take place. However, in the considered problem of sliding elliptical inclusions, the stress strongly depends on the length $(a-b)$, if this is of the same order as $b\gamma$. Therefore, although γ is an infinitesimal strain and $b\gamma$ is an infinitesimal length change relative to the overall geometry and the lengths a and b, the length change $b\gamma$ is large relative to the length scale $(a-b)$. From a combined kinematic-kinetic point of view, there is an unusual interplay of the contributions from the rotation and stress in fitting the sliding inclusion into the matrix. The stress is taken to be proportional to $(A-a)/a$, because the rotation cannot overcome this size barrier and insert by itself the inclusion into the matrix. For bonded inclusions, on the other hand, the stress is not proportional to $(A-a)/a$, because sliding is not available to help the accommodation of the inclusion, and the stress does exist even when $A = a$.

The unusual behavior, caused by an interplay of the rotation and stress in fitting the sliding inclusion into the matrix, is present only when the shear eigenstrain is "parallel" to the principal directions (x and y) of the ellipse (i.e., $\gamma = \epsilon^*_{xy}$). The effect is absent in the case of biaxial eigenstrain, parallel to the principal directions of the ellipse, since the ellipse deformed in this way always differs in size from the undeformed ellipse. There is, however, a particular situation in which the stress in the inclusion and the matrix can vanish. This happens for a nearly circular inclusion with the biaxial eigenstrain $\epsilon_x = b/a - 1$ and $\epsilon_y = a/b - 1$, which deforms a into b, and b into a. A rotation by 90 deg of an inclusion so strained would place it back into the matrix, without any stress. Such a configuration would correspond to the minimum (zero) total energy of the inclusion and the matrix.

Although the derivation presented in this paper was mainly devoted to elliptical inclusions in plane strain, the conclusion regarding the existence of nonvanishing stress in nearly circular inclusions can be extended to nearly spherical inclusions, as well. Thus, for a given level of applied shear eigenstrain, the stress in nearly spherical inclusions rapidly diminishes as the inclusion becomes less spherical, for the same reasons as in the case of nearly circular inclusions. This rapid decrease of stress, characteristic for a sliding inclusion, is not a feature of the Eshelby inclusion at all, since the stress there decreases far more gradually with the ratio a/b, being practically equal in spherical and nearly spherical inclusions.

Finally, an analogous situation to the one considered in this paper in the case of shear eigenstrain, also occurs in the case of sliding elliptical or ellipsoidal inclusions under remote uniform shear loading. Here, the stress in distinctly elliptical inclusions is of the order τ^2/μ, where τ is the remote shear stress and μ the shear modulus of the material. Such inclusions behave as voids from the standpoint of linear elasticity, since they accommodate deformation of the surrounding matrix simply by an infinitesimal rotation. A nonlinear analysis is required to determine the stress in nearly circular or nearly spherical sliding inclusions. Only at very small values of the remote shear stress do such inclusions (or inhomogeneities) act as voids. The overall stiffness and the resistance of the system to applied shear stress become quite different for larger values of τ. The inclusion is unable to further adjust to the deforming matrix by a rigid-body rotation, and the normal traction builds up at the interface, causing a stress field in the inclusion, and a change in the overall stiffness of the considered matrix-inclusion composite.

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