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A Note on the Effective Lamé Constants of Polycrystalline Aggregates of Cubic Crystals

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It is well known that the Voigt and Reuss estimates of the effective shear modulus of a polycrystalline aggregate of cubic crystals are, respectively, the upper and lower bounds for this constant ($\mu^R \leq \mu \leq \mu^V$). It is pointed out in this note that the opposite is true for the Lamé constant λ ($\lambda^V \leq \lambda \leq \lambda^R$).

1 Introduction

According to the Voigt (1889) assumption, when a polycrystalline aggregate is subjected to the overall uniform strain, the individual crystals are all in the same state of applied strain. From this assumption it follows that the effective elastic moduli of an isotropic aggregate are directional averages of the corresponding moduli of individual crystals. Thus, the effective Lamé and bulk moduli of a polycrystalline aggregate of cubic crystals are

$$\lambda^V = \tfrac{1}{5}(c_{11} + 4c_{12} - 2c_{44}), \quad \mu^V = \tfrac{1}{5}(c_{11} - c_{12} + 3c_{44}),$$

$$\kappa^V = \tfrac{1}{3}(c_{11} + 2c_{12}), \tag{1}$$

where superscript V designates the Voigt assumption. The constants c_{11}, c_{12}, and c_{44} are the elastic moduli of individual crystals.

According to the Reuss (1929) assumption, when a polycrystalline aggregate is subjected to overall uniform stress, the individual crystals are all in the same state of applied stress. From this assumption it follows that the effective elastic compliances of an isotropic aggregate are directional averages of the corresponding compliances of individual crystals. Expressing the result in terms of elastic moduli, this gives

$$\lambda^R = \frac{(c_{11} - c_{12})(c_{11} + 2c_{12}) - 2c_{44}(c_{11} - 3c_{12})}{3(c_{11} - c_{12}) + 4c_{44}},$$

$$\mu^R = \frac{5c_{44}(c_{11} - c_{12})}{3(c_{11} - c_{12}) + 4c_{44}}, \tag{2}$$

and $\kappa^R = (c_{11} + 2c_{12})/3$. Evidently, $\kappa^V = \kappa^R$, since cubic crystals under hydrostatic state of stress behave as isotropic materials. Comparing Eqs. (1) and (2), we further have

$$\lambda^V = \lambda^R - \frac{2}{5}\frac{(c_{11} - c_{12} - 2c_{44})^2}{3(c_{11} - c_{12}) + 4c_{44}},$$

$$\mu^V = \mu^R + \frac{3}{5}\frac{(c_{11} - c_{12} - 2c_{44})^2}{3(c_{11} - c_{12}) + 4c_{44}}. \tag{3}$$

Since from the stability conditions for a single crystal $c_{11} - c_{12} > 0$ and $c_{44} > 0$, from Eq. (3) it follows that

$$\lambda^V \leq \lambda^R, \quad \mu^V \geq \mu^R. \tag{4}$$

2 Bounds

Hill (1952) has shown that the strain energy density in a polycrystalline aggregate with the true effective moduli is not greater than the corresponding energy in the aggregate with the Voigt estimates of the elastic moduli. Likewise, the complementary energy in a polycrystalline aggregate with the true effective compliances is not greater than the corresponding energy in the aggregate with the Reuss estimates of the elastic compliances. From this it follows that the true shear and bulk moduli are related to their Voigt and Reuss estimates by

$$\mu^R \leq \mu \leq \mu^V, \quad \kappa^R = \kappa = \kappa^V. \tag{5}$$

The other well-known consequences are that the effective Young's modulus and Poisson's ratio are bounded by

$$E^R \leq E \leq E^V, \quad \nu^V \leq \nu \leq \nu^R. \tag{6}$$

The bounds on the Lamé constant λ are easily identified. Indeed,

$$\kappa - \tfrac{2}{3}\mu^V \leq \lambda = \kappa - \tfrac{2}{3}\mu \leq \kappa - \tfrac{2}{3}\mu^R, \tag{7}$$

and since $\kappa = \kappa^V = \kappa^R$, we obtain

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$$\lambda^V \leq \lambda \leq \lambda^R. \quad (8)$$

Thus, the upper bound for the Lamé constant λ is the Reuss estimate of this constant, while the lower bound for this constant is given by its Voigt estimate.

This is the message of this brief note. It is brought to the attention of the reader, because a statement appears in the literature that all effective elastic moduli are bounded from above by their Voigt estimates and from below by their Reuss estimates (e.g., Barsch, 1968, p. 3782; Mura, 1987, Eq. (45.19.13) on p. 428).

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Natural Frequencies and Normal Modes for Externally Damped Spinning Timoshenko Beams With General Boundary Conditions

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Vibration analysis of externally damped spinning Timoshenko beams with general boundary conditions is performed analytically. Exact solutions for natural frequencies and normal modes for the six classical boundary conditions are derived for the first time. In the numerical simulations, the trend between the complex frequencies and the damping coefficient is investigated, and complex mode shapes are presented in three-dimensional space.

1 Introduction

Most research in analytical modeling of Timoshenko beams spinning along the longitudinal axis that has been conducted is limited to undamped vibrations. Zu and Han (1992) developed analytical solutions for the natural frequencies and normal modes for a spinning Timoshenko beam with six different boundary conditions. Lee (1995) formulated the equations of motion for a simply supported beam subjected to axial forces and moving loads by Hamilton's principle. Argento (1995) investigated the response and resonance of simply supported and clamped-clamped spinning beams subjected to moving loads using Galerkin's method. Tan and Kuang (1995) obtain closed-form solutions for both Rayleigh and Timoshenko beams by means of a distributed transfer function and a generalized displacement formulation for stepped beams.

For damped vibration of Timoshenko beams, Singh and Abdelnaser (1993) presented a general modal solution to a stationary Timoshenko beam with external damping. They include both external transverse and rotary viscous damping as well as viscoelastic damping. Extensive research on externally damped spinning Timoshenko beams is lacking at the present time. One of the very few papers on external damping is by Medgyesi (1991), in which the external damping was studied in the case of a Jeffcott rotor.

The objective of this paper is to develop analytical solutions for the complex eigenvalues and complex normal modes of an externally damped Timoshenko beam with general boundary conditions. This is a continuation of the work done by Zu and Han (1992) in which undamped natural frequencies and normal modes were solved analytically.

2 Equations of Motion

The equations of motion based on an inertial frame $oxyz$ for a spinning uniform Timoshenko beam are based on the equations presented by Han and Zu (1992) with the inclusion of the external viscous damping. The equations of motion are

$$\frac{\partial^2 u}{\partial t^2} + \frac{c}{\rho A}\frac{\partial u}{\partial t} + \frac{\kappa G}{\rho l^2}\left[l\frac{\partial \psi}{\partial \zeta} - \frac{\partial^2 u}{\partial \zeta^2}\right] = 0 \quad (1)$$

$$\frac{\partial^2 \psi}{\partial t^2} - i\frac{\Omega J_z}{\rho I}\frac{\partial \psi}{\partial t} - \frac{E}{\rho l^2}\frac{\partial^2 \psi}{\partial \zeta^2} + \frac{\kappa A G}{\rho I l}\left[l\psi - \frac{\partial u}{\partial \zeta}\right] = 0 \quad (2)$$

where ρ is the mass density; A is the cross-sectional area; I is the transverse moment of inertia of an axisymmetric cross section; J_z is the polar moment of inertia; c is the viscous damping coefficient; and E, G, and κ are Young's modulus, shear modulus, and shear coefficient, respectively. $\zeta = z/l$ is the nondimensional variable, and $u = u_x + iu_y$, $\psi = \psi_x + i\psi_y$ are the complex transverse deflections the corresponding bending angles. Note that only transverse damping is included while rotary damping is ignored since the effect of rotary damping is much smaller than the transverse damping.

Assume that the solutions to Eqs. (1) and (2) are

$$u = W_u(\zeta)e^{\lambda t}$$
$$\psi = W_\psi(\zeta)e^{\lambda t}. \quad (3)$$

Here $W_u(\zeta)$ and $W_\psi(\zeta)$ are complex normal modes and λ is the complex eigenvalue and they are expressed by

$$W_u(\zeta) = W_{uR}(\zeta) - iW_{uI}(\zeta)$$
$$W_\psi(\zeta) = W_{\psi R}(\zeta) + iW_{\psi I}(\zeta) \quad (4)$$

$$\lambda = \lambda_R + i\lambda_I. \quad (5)$$

Thus the substitution of Eq. (3) into Eqs. (1) and (2) becomes

$$-\lambda\left(\lambda + \frac{c}{\rho A}\right)W_u(\zeta) = \frac{\kappa G}{\rho l}W'_\psi(\zeta) - \frac{\kappa G}{\rho l^2}W''_u(\zeta) \quad (6)$$

$$\left[\lambda^2 - i\frac{\Omega J_z\lambda}{\rho I} + \frac{\kappa A G}{\rho I}\right]W_\psi(\zeta) - \frac{E}{\rho l^2}W''_\psi(\zeta) = \frac{\kappa A G}{\rho I l}W'_u(\zeta). \quad (7)$$

The solutions to Eqs. (6) and (7) can be derived as follows:

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