A Simple Representation of the J-Integral and Some Estimates of Elastic Stress Intensity Factors

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Precise estimates of the elastic stress intensity factors for the Griffith and edge cracks are made by using one convenient representation of the J-integral and asymptotic crack-tip fields, without recourse to the solutions of corresponding boundary value problems.

1 Introduction

We show here that for various two-dimensional elastic crack problems the energy release rate, or J-integral, has a simple representation $J = \sigma A/l$, where σ is a representative measure of applied stress, A is the area defined by the deformed crack faces, and l is the crack length. This directly follows from the M-integral conservation law, although in some cases it can be deduced by more elementary means. Utilizing the crack-opening displacements from the universal, asymptotic crack-tip field, rather than from the specific boundary value problem, we then obtain an estimate of the elastic stress intensity factor, which is for an edge crack only seven percent higher than the true value. By a simple modification, an exact estimate is obtained for the stress intensity of the Griffith crack and an improved estimate, with the error less than three percent, for the edge crack.

2 J Representation

In the case of plane infinitesimal deformation of a homogeneous linearly elastic material, one of the conservation laws introduced by Knowles and Sternberg (1972) states that for any closed contour C within a simply connected region, the following (M) integral vanishes:

$$M(C) = \int_C (Wn_i x_i - T_i u_{i,k} x_k)\, dC. \qquad (1)$$

Here, $W = 1/2\sigma_{ij}\epsilon_{ij}$ is the elastic strain energy density, n_i is the unit normal to C, $T_i = \sigma_{ij}n_j$ the traction, and u_i displacement, all expressed in rectangular Cartesian coordinates, x_1 and x_2. No body forces are considered, so that the equilibrium equations reduce to $\sigma_{ij,j} = 0$. If the region enclosed by C is not simply connected, but contains a cavity, then $M(C) \neq 0$ and Budiansky and Rice (1973) have shown that the value of M is the energy release rate associated with a self-similar expansion of a traction-free cavity. The conservation law $M = 0$ was used by Freund (1978) to determine elastic stress intensity factors for various crack problems, without solving the corresponding elastic boundary value problems. The main characteristics which a problem must have in order for the approach to be useful are discussed in his paper, and they are utilized here.

Consider a two-dimensional infinite medium and a straight crack of length l loaded by uniform compressive stress σ on its two faces (Fig. 1), which is by a superposition equivalent to Griffith crack. Introduce a closed contour C as shown, which consists of two infinitesimal circles Γ_0 around the crack tips, an infinitely large circle Γ_∞, straight segments $\Gamma^\pm$ along the crack faces, and remaining parts $\Gamma_0^\pm$ joining Γ_∞ to one part of Γ_0. The contributions to $M(C)$ from the last two parts cancel each other, $M(\Gamma_0^+ + \Gamma_0^-) = 0$, and the contribution from Γ_∞ is also zero because the applied loads on crack faces are self-equilibrating and stresses decrease faster than $O(r^{-1})$, where r is the radius of large circle (Freund, 1978). Thus, the only

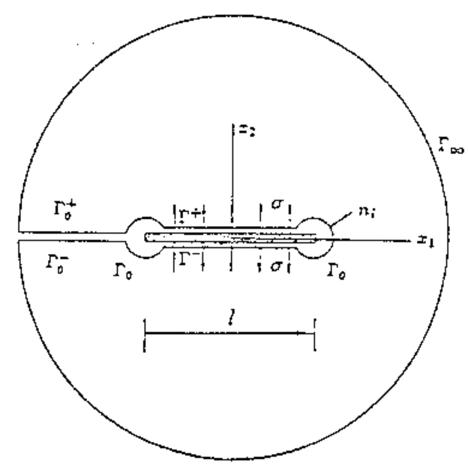

Fig. 1 Closed contour defining simply connected region for which the M integral vanishes. Crack of length l is loaded on its faces by compressive stress σ.

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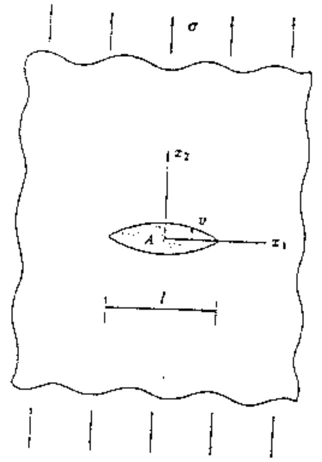

Fig. 2 Deformed configuration of the Griffith crack of length l; v is the crack opening displacement, and A the area within deformed crack faces

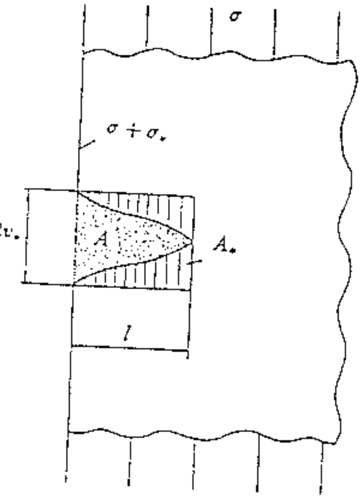

Fig. 3(a) Deformed configuration of edge crack of length l; A is the area within deformed crack faces, while quantities denoted by * appear in formula (6)

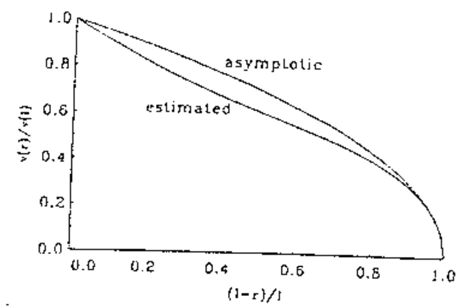

Fig. 3(b) Asymptotic and estimated shapes of the crack-opening displacement $v(r)$

contributions that remain are from Γ_0 and Γ^*. The contribution from Γ_0 is easily seen to be lJ, where

$$J = \int_{\Gamma_0} (W n_1 - T_i u_{i,1}) d\Gamma_0 \tag{2}$$

is Rice's J integral (Rice 1968a, b), while each crack face gives

$$\sigma \int_{-l/2}^{l/2} x_1 dv = -\sigma \int_{-l/2}^{l/2} v dx_1 = -\sigma \frac{A}{2}, \tag{3}$$

where $A/2$ is half of the area enclosed by deformed crack faces (Fig. 2). Partial integration was utilized above, together with the boundary conditions at the crack tips $v(\pm l/2) = 0$, where $v = u_2(x_1, 0)$ denotes lateral displacement (opening) of the crack faces. Hence, from $M = 0$, we have $lJ - \sigma A = 0$, and therefore

$$J = \frac{\sigma A}{l}. \tag{4}$$

Note that since there is no length scale other than l, A is proportional to l^2 and for the Griffith crack, J is a linear function of l.

For example, if we assume the opening profile of the crack to be an ellipse $v = v_0(1 - 4x_1^2/l^2)^{1/2}$, we have $A = \pi l/2 v_0$ and (4) gives $J = \pi/2\, \sigma v_0$. Substituting $v_0 = [(1-\nu)/2\mu]\, \sigma l$ from the exact solution of the Griffith crack (see, for example, Rice 1968a), we recover the well-known relationship between J and l

$$J = kl, \quad k = \frac{1-\nu}{4\mu} \pi \sigma^2 \tag{5}$$

ν being Poisson's ratio and μ the shear modulus.

In retrospect, expression (4) can be directly obtained in the following manner. If Ω is the potential energy, i.e., the strain energy minus the load potential ($\Omega = 1/2\ \sigma A - \sigma A = -1/2\ \sigma A$), the energy release rate or J is, by definition, $J = -\partial\Omega/\partial l = 1/2\ \sigma dA/dl$. Hence, since A is proportional to l^2, (4) follows. Or, equivalently, let us close the crack in Fig. 2, i.e., apply on its deformed faces uniform gradually increasing tension up to amount σ. The faces will come together and the state of stress will be uniform throughout the body. The corresponding work done is $1/2\sigma A$. This must be equal to the strain energy release in the reverse process of cracking the body by Griffith crack of length l, which is $1/2Jl$ (where J plays a role of the crack extension force), hence (4).

The same expression holds for the edge crack (Fig. 3(a)). Due to less material constraint, one expects A in this case to be bigger than half of the area within the opened Griffith crack, and therefore a higher energy release rate is associated with the edge crack. An additional interesting relationship can be obtained from the M integral by locating the coordinate system at the crack tip, so that there is no contribution from the small circle surrounding the tip, but there is from the edge of the semi-infinite medium. It turns out that the stress deviation along this edge, from the uniform value, obeys the relationship

$$\int_{-\infty}^{\infty} \sigma_*^2 dx_2 = \frac{4\mu}{1-\nu} \frac{\sigma A_*}{l}, \tag{6}$$

where $A_* = 2v_* l - A$ is the shaded area in Fig. 3(a).

3 An Estimate of the Elastic Stress Intensity Factor

Since J is equal to the energy release rate associated with co-planar crack growth, it is related to the plane strain mode I stress intensity factor K_I through

$$J = \frac{1-\nu}{2\mu} K_I^2. \tag{7}$$

Combining this with (4) gives

$$K_I^2 = \frac{2\mu}{1-\nu} \frac{\sigma A}{l}. \tag{8}$$

This can be used to make an estimate of K_I without recourse to the solution of the Griffith or edge crack boundary value

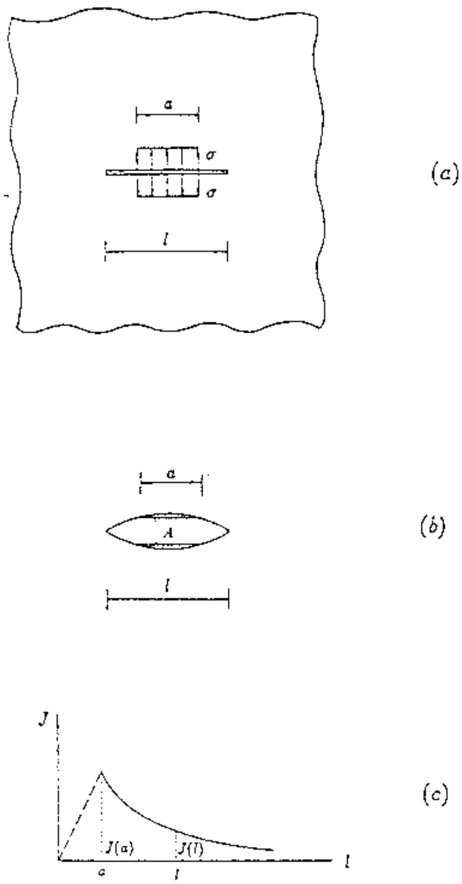

Fig. 4(a) Crack of length l compressed by σ over the part of length a; (b) its deformed configuration indicating the area A which appears in formula (4); (c) qualitative change of the corresponding J integral with crack length

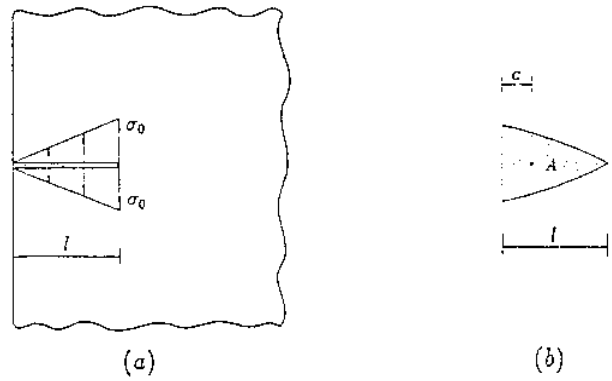

Fig. 5(a) Edge crack under triangular loading; (b) its deformed configuration; c defines the centroid of the crack opening area A

problems, by using the crack opening displacement from the universal, asymptotic elastic crack-tip field, which is (Rice, 1968a)

$$v=\sqrt{\frac{2}{\pi}}\,\frac{1-\nu}{\mu}\,K_I\sqrt{r}. \tag{9}$$

For example, for the edge crack of length l, this gives

$$A=2\int_0^l v\,dr=\frac{8}{3\sqrt{2\pi}}\,\frac{1-\nu}{\mu}\,K_I l\sqrt{\pi l}, \tag{10}$$

and by substitution into (8),

$$K_I=1.2\sigma\sqrt{\pi l}. \tag{11}$$

This is only seven percent higher than the value obtained from the exact solution, giving $1.1215\sigma\sqrt{\pi l}$ (with a possible numerical error of at most one unit in the last digit, Koiter 1965). Applied to the Griffith crack, (11) is an overestimate of 20 percent, since deviation of the asymptotic field displacement from the true one is more pronounced. However, we can improve this estimate by adjusting the asymptotic expression (9) to meet the symmetry requirement and vanishing slope of the deformed Griffith crack faces at $r = l/2$. An obvious modification to satisfy $v'(l/2) = 0$ is obtained by multiplying (9) with $\sqrt{1-r/l}$, i.e.,

$$v(r)=\sqrt{\frac{2}{\pi}}\,\frac{1-\nu}{\mu}\,K_I\sqrt{r\left(1-\frac{r}{l}\right)}. \tag{12}$$

This clearly tends to the asymptotic expression (9) as $r \to 0$. Therefore,

$$A=4\int_0^{l/2} v(r)\,dr=\frac{1-\nu}{2\mu}\,K_I l\sqrt{\pi l/2}, \tag{13}$$

and from (8) we obtain the exact value for the stress intensity factor of the Griffith crack $K_I = \sigma\sqrt{\pi l/2}$. This exact estimate is a consequence of the fact that (12) turns out to be an ellipsoidal shape, which the complete solution of the Griffith crack boundary value problem indeed confirms.

Similarly, an improved estimate can be made for the edge crack, by modifying the asymptotic expression (9) to allow for an inflection point of the deformed faces, which is on physical ground expected to occur somewhere along the edge crack. Denoting a position of this point by $r = a$, we multiply (9) by $\sqrt{1-r/l(1-r/3a)}$, for then

$$v(r)=\sqrt{\frac{2}{\pi}}\,\frac{1-\nu}{\mu}\,K_I\sqrt{r\left[1-\frac{r}{l}\left(1-\frac{r}{3a}\right)\right]} \tag{14}$$

satisfies $v''(a) = 0$. Also, as $r \to 0$, this reduces to (9). By taking $a = l/3$ (so that $v(l)$ given by (14) and (9) coincide (Fig. 3(b))), we obtain $K_I = 1.091\,\sigma\sqrt{\pi l}$, which is 2.7 percent lower than the true value. For $a = 0.3l$ (14) gives $K_I = 1.1216\,\sigma\sqrt{\pi l}$, which is essentially equal to Koiter's value. Nonetheless, (14) is only an approximation to the true crack-opening displacement function $v(r)$, given in Koiter's (1956) paper by its Melin's transform (Eq. (72), with $\kappa_0 = 0$ in (91) and (85)), from which it can be extracted by the inverse transform and numerical procedure. There is also integral equations solution by Sneddon and Das (1971), with a performed numerical evaluation of $v(r)$. The plot in Fig. 1 of their paper clearly indicates the inflection point of the deformed crack faces, although their value for the maximum displacement $2\sigma l(1-\nu^2)/E$ is too small, and would only correspond to maximum opening displacement of the Griffith crack. The value reported in Tada et al. (1973), p. 8.1, is proportional to 2.915, which is about 5.8 percent smaller than the value 3.086, predicted by the estimated shape (14) in the case $a = l/3$.

4 Additional Examples

We give two more crack configurations for which a formula of type (4) holds. The derivation proceeds in a straightforward manner by applying the M integral. For the case in Fig. 4(a), with partially loaded crack faces, the area A appearing in (4) is shown in Fig. 4(b). For the edge crack and triangular loading, as in Fig. 5(a), the representative stress appearing in formula (4) is $\sigma = \sigma_0 2c/l$, where c defines the centroid of the crack opening area A.

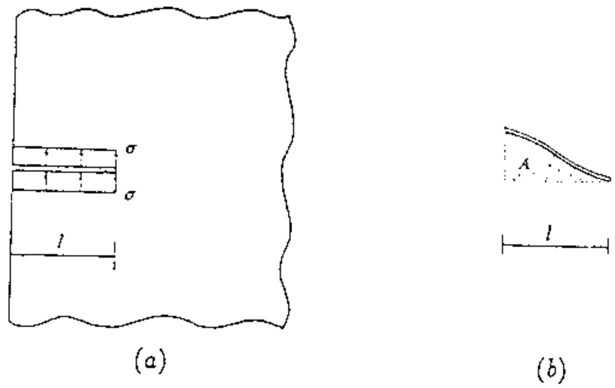

Fig. 6(*a*) Antisymmetric loading of the edge crack; (*b*) deformed faces and the area A appearing in formula (17)

A comment should perhaps be made regarding the case in Fig. 4(a). The associated potential energy is $\Omega = -1/2\sigma A_*$, where $A_* = A + 2av(a, l)$ is the area of the opened crack below the load σ. Hence, from $J = -\partial\Omega/\partial l$ we have by integration

$$\frac{1}{2}(A_* - A_0) = \int_a^l J(l')dl', \tag{15}$$

where A_0 denotes the area within opened Griffith crack of length a. In view of (4), therefore,

$$\frac{1}{2}J(l)l - \frac{1}{2}J(a)a = \int_a^l J(l')dl' - \sigma av(a, l), \tag{16}$$

indicating a deviation of $J(l)$ from linearity in l (due to the second term on the right-hand side). The qualitative plot of $J = J(l)$ is shown in Fig. 4(c). As l becomes large, much bigger than a, $J(l)$ tends to zero, as less and less energy release rate is associated with further crack growth. Indeed, from the complete solution of this problem (Tada et al., 1973, p. 5.12), $J(l) = 4/\pi^2 kl(\sin^{-1} a/l)^2$, with k given in (5).

Lastly, we give a J representation for the case of the antisymmetric crack loading, shown for example for the edge crack in Fig. 6(a). The only difference in the procedure of applying the M integral to this configuration is that there is a contribution to M from the large circle (Γ_∞), due to a concentrated tangential force of amount $2\sigma l$ acting on the edge of the half space. This is (Freund, 1978) $M(\Gamma_\infty) = [(1-\nu)/2\mu](2\sigma l)^2/\pi$, and therefore

$$J = 2\left(\frac{\sigma A}{l} - \frac{1-\nu}{\mu}\frac{\sigma^2 l}{\pi}\right), \tag{17}$$

where A is the area shown in Fig. 6(b). Since the antisymmetric loading in Fig. 6(a) gives rise to mode II stress intensity factor (K_{II}), while symmetric loading and formula (4) corresponds to mode I, (17) can be used in a superposition procedure to decouple two modes of a mixed-mode loading, as in uniform compression on one of the crack faces. If the loading is by a concentrated tangential force P at the left end of the crack face, it follows that

$$K_I = \frac{\pi}{\sqrt{\pi^2-4}}\frac{P}{\sqrt{\pi l}}, \quad K_{II} = \frac{2}{\sqrt{\pi^2-4}}\frac{P}{\sqrt{\pi l}} \tag{18}$$

with the face angle determined from $K_{II}/K_I = 2/\pi$.

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