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A Correct Definition of Elastic and Plastic Deformation and Its Computational Significance[1]

The plastic part of an elastic-plastic deformation is that remaining when the stress, and hence the elastic strain, is reduced to zero. Elastic deformation is that produced in this purely plastically deformed material by the action of stresses up to yield. The associated exact finite-deformation kinematics shows the almost universal assumption that the total rate of deformation is the sum of elastic and plastic rates to be in error. An incremental elastic-plastic theory is developed using the nonlinear kinematics. The theory is contrasted with that in common use and anomalies in the latter are discussed.

1 Introduction

The kinematics of elastic-plastic deformation at finite strain was incorporated into elastic-plastic constitutive relations by means of the matrix analysis of sequential deformations [1–3]. The configuration of a body in its undisturbed reference state at uniform temperature θ_0 is specified by the Cartesian position coordinates $\mathbf{X} = (X_1, X_2, X_3)$ of the body's material points. After loading beyond the elastic limit the body takes on the configuration $\mathbf{x}$ at time t given by the mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{1}$$

For elastic-plastic analysis the deformation is appropriately expressed in terms of the deformation gradient matrix

$$\mathbf{F}(\mathbf{X}, t) = \partial \mathbf{x}/\partial \mathbf{X}, \quad (F_{iJ} = \partial x_i/\partial X_J) \tag{2}$$

In order to define appropriate variables in which to express elastic-plastic analysis, the body is considered to be destressed and reduced to the initial temperature θ_0 at time t thus releasing the thermoelastic strains. The configuration is then specified by the mapping

$$\alpha = \alpha(\mathbf{X}, t) \tag{3}$$

Since after elastic-plastic deformation a body is commonly left in a state of residual stress when loads are removed and the temperature is reduced to the base value, destressing may require the body to be divided into infinitesimal elements so that the mapping (3) may then be discontinuous and not one-one [3, 4]. However a local "deformation gradient" $\mathbf{F}^p(\mathbf{X}, t)$ can be defined in each element which specifies the deformation after the macroscopic stress, and hence the thermoelastic strain, has been removed. $\mathbf{F}^p$ thus expresses the plastic deformation which has taken place. It corresponds directly to the deformation which would be associated with the migration of dislocations through unstressed crystal lattices according to the physical theory of plasticity. It also corresponds directly to the method of measuring the onset of plasticity through a proof stress test which determines the loss of reversibility to zero strain, and hence of purely elastic behavior, when the stress is cycled back to zero.

A similar local deformation gradient $\mathbf{F}^e$ specifies the mapping from the unstressed plastically deformed configuration α, (3), to the elastically-plastically deformed configuration $\mathbf{x}$, (1), and constitutes the elastic deformation gradient. Since $\boldsymbol{\alpha}$ is not in general a continuous one-one mapping, $\mathbf{F}^e(\mathbf{X}, t)$ is, like $\mathbf{F}^p$, a point function and not the partial derivative $\partial \mathbf{x}/\partial \boldsymbol{\alpha}$ (which usually does not exist).

The configurations of a material element in the neighborhoods of $\mathbf{X}$, $\alpha(\mathbf{X}, t)$ and $\mathbf{x}(\mathbf{X}, t)$ are related since the sequence of transformations $\mathbf{X} \to \alpha(\mathbf{X}, t)$ followed by $\alpha(\mathbf{X}, t) \to \mathbf{x}(\mathbf{X}, t)$ is equivalent to the mapping $\mathbf{X} \to \mathbf{x}(\mathbf{X}, t)$, so that the chain rule determines

$$\mathbf{F}(\mathbf{X}, t) = \mathbf{F}^e(\mathbf{X}, t)\mathbf{F}^p(\mathbf{X}, t) \tag{4}$$

This expresses a simple but generally noncommutative coupling between elastic and plastic deformation.

But plasticity is an incremental or flow-type phenomenon so that increments or rates of deformation must be incorporated into the formulation of the theory. The gradient of the particle velocity

$$\mathbf{v} = \partial \mathbf{x}/\partial t|_{\mathbf{X}} \tag{5}$$

in the current configuration at time t is given by

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{L} \tag{6}$$

where $\dot{\mathbf{F}} = \partial \mathbf{F}/\partial t|_{\mathbf{X}}$. $\mathbf{L}$ can be decomposed into its symmetric part $\mathbf{D}$

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[1] Dedicated to the memory of Suresh Chandra who was studying this problem before his untimely death.

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(the rate of deformation, velocity strain or stretching tensor) and its antisymmetric part $\mathbf{W}$ (the spin tensor):

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \tag{7}$$

Substituting (4) into (6) gives

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}^e\mathbf{F}^{e-1} + \mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p-1}\mathbf{F}^{e-1} = \mathbf{L}^e + \mathbf{F}^e\mathbf{L}^p\mathbf{F}^{e-1} \tag{8}$$

in which $\mathbf{L}^e = \dot{\mathbf{F}}^e\mathbf{F}^{e-1}$ corresponds to the velocity gradient of the purely elastic deformation, and $\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}$ corresponds to the velocity gradient of the purely plastic deformation.

Equation (8) clearly demonstrates, as was pointed out in [3], that in general for strain rates expressed by $\mathbf{D}$, $\mathbf{D}^e$, and $\mathbf{D}^p$ (the symmetric parts of $\mathbf{L}$, $\mathbf{L}^e$, and $\mathbf{L}^p$)

$$\mathbf{D} \neq \mathbf{D}^e + \mathbf{D}^p \tag{9}$$

Relation (9) with an *equality* sign, or equivalently by multiplying by Δt to obtain strain increments,

$$\Delta\epsilon = \Delta\epsilon^e + \Delta\epsilon^p \tag{9a}$$

is the almost universal assumption in finite-element elastic-plastic computer codes and as a basis for incremental measurements of elastic-plastic material characteristics. For a recent reiteration and elaboration of the rate summability concept, see Nemat-Nasser [14]. This anomaly is examined in [4] where it is shown that the adoption of (9a) implies that certain *elastic* strain increments contribute terms then categorized as *plastic*. In the present paper we show that a careful development based on the kinematics expressed in (8) generates a new incremental theory, using the finite-strain elasticity constitutive relation, which exhibits the appropriate structure and symmetry properties to be incorporated into Hill's rate-potential, finite-deformation, variational principle [5] and hence into finite-element computer codes.

We shall assume isotropic elastic and plastic response to stress throughout the deformation and invariant elastic properties since these generate a relatively transparent analysis which is sufficient to contrast with the commonly accepted approach. Such isotropy implies that the plastic strain rate, based on normality with an isotropic yield surface, and the Euler or Almansi elastic strain have the same principal directions as the stress tensor.

It has been pointed out [3] that the component deformation gradients $\mathbf{F}^e$ and $\mathbf{F}^p$ are not uniquely defined because arbitrary local material element rotations in the unstressed state give alternate unstressed configurations. For analytical convenience, and with no basic loss of generality, we take the elastic deformation gradient $\mathbf{F}^e$, associated with destressing, to be rotation free and hence given by $\mathbf{V}^e$, a symmetric matrix.

$$\mathbf{F}^e = \mathbf{V}^e \tag{10}$$

(8) then takes the form

$$\mathbf{L} = \dot{\mathbf{V}}^e\mathbf{V}^{e-1} + \mathbf{V}^e(\mathbf{D}^p + \mathbf{W}^p)\mathbf{V}^{e-1} = \dot{\mathbf{V}}^e\mathbf{V}^{e-1} + \mathbf{V}^e\mathbf{D}^p\mathbf{V}^{e-1} + \mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e-1} \tag{11}$$

and since $\mathbf{V}^e$ has the same principal directions as the elastic Euler strain, and hence also the same principal directions as stress *and* $\mathbf{D}^p$, the multiplications in $\mathbf{V}^e\mathbf{D}^p\mathbf{V}^{e-1}$ are commutative so that, taking symmetric and antisymmetric parts, gives

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p + (\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e-1})_S \tag{12}$$

$$\mathbf{W} = \mathbf{W}^e + (\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e-1})_A \tag{13}$$

where the subscripts S and A indicate the symmetric and antisymmetric parts, respectively.

Thus, in rate form, the elastic-plastic coupling appears to be more involved than the deformation gradient relation (4). An interpretation of the significance of (12) is presented in [4]. The last term in (12) expresses a rate of strain contribution associated with elastic destressing, followed by rotation and elastic restressing, although it appears as a contribution associated with $\mathbf{L}^p$, and constitutes a residual strain increment following application and removal of a stress increment [4]. This may appear to be artificially associated with plastic flow because of the choice (10), but the additional strain-rate term, which contributes an increment of residual strain, arises from rotation of the body relative to the stress tensor and would still appear if the choice (10) had not been made. The theory must be sufficiently flexible to incorporate arbitrary rotation with elastic-plastic deformation if it is to be used as a vehicle for finite-element implementation. As explained in [4], such an elastic residual strain increment has commonly been categorized as contributing to plasticity.

Instead of utilizing the first term on the right-hand-side of (8) to give the elastic strain-rate $\mathbf{D}^e$, the approach developed in this paper is to combine it with the last term in (12) to yield a more cogent separation of elastic and plastic effects. Moreover use of the exact kinematic relation (12) eliminates anomalies associated with *hypothesising* an equality sign in (9), i.e., that the total strain-rate is the sum of elastic and plastic strain-rates.

2 The Elastic-Plastic Constitutive Relation

First, let us consider elastic response to stress. Since, without loss of generality, we define elastic deformation by the strain recovery on destressing without rotation, $\mathbf{F}^e = \mathbf{V}^e$, (10), and using the Truesdell-Noll notation [6 p. 52] the right and left Cauchy-Green tensors, $\mathbf{C}^e$ and $\mathbf{B}^e$ respectively, are equal

$$\mathbf{C}^e = \mathbf{F}^{eT}\mathbf{F}^e = (\mathbf{V}^e)^2 = \mathbf{F}^e\mathbf{F}^{eT} = \mathbf{B}^e \tag{14}$$

Thus the stress deformation relation for an isotropic thermoelastic material (equation (18) of [3] or equation (84.11) of [6]) takes the form

$$\boldsymbol{\tau} = 2\mathbf{C}^e\frac{\partial\psi}{\partial\mathbf{C}^e} \tag{15}$$

where $\boldsymbol{\tau}$ is the Kirchoff stress (det $(\mathbf{F}^e)$ times the Cauchy stress) and ψ is the free energy per unit undeformed volume (equal to $\rho_0\psi$ as defined in [3]). The deduction of (15) depends on the fact that, for an isotropic body, $\mathbf{V}^e$, $\mathbf{C}^e$ and $\partial\psi/\partial\mathbf{C}^e$ all have the same principal axes so that products of the matrices are commutative.

Since the incremental or flow-type structure of the plasticity law demands that the elastic-plastic relation appear in rate form, (15) must be expressed in terms of the velocity gradients developed in the previous section.

$$\dot{\mathbf{C}}^e = \dot{\mathbf{V}}^e\mathbf{V}^e + \mathbf{V}^e\dot{\mathbf{V}}^e = (\mathbf{V}^e\dot{\mathbf{V}}^e)^T + \mathbf{V}^e\dot{\mathbf{V}}^e = 2(\mathbf{V}^e\dot{\mathbf{V}}^e)_S \tag{16}$$

so that

$$\mathbf{D}^e = (\dot{\mathbf{V}}^e\mathbf{V}^{e-1})_S = (\mathbf{V}^{e-1}\mathbf{V}^e\dot{\mathbf{V}}^e\mathbf{V}^{e-1})_S = \mathbf{V}^{e-1}(\mathbf{V}^e\dot{\mathbf{V}}^e)_S\mathbf{V}^{e-1} = \frac{1}{2}\mathbf{V}^{e-1}\dot{\mathbf{C}}^e\mathbf{V}^{e-1} \tag{17}$$

Thus the elastic terms in (12) can be rewritten as

$$\mathbf{D}^e + (\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e-1})_S = \frac{1}{2}\mathbf{V}^{e-1}\dot{\mathbf{C}}^e\mathbf{V}^{e-1} + \mathbf{V}^{e-1}(\mathbf{V}^e\mathbf{V}^e\mathbf{W}^p)_S\mathbf{V}^{e-1} = \frac{1}{2}\mathbf{V}^{e-1}[\dot{\mathbf{C}}^e - \mathbf{W}^p\mathbf{C}^e + \mathbf{C}^e\mathbf{W}^p]\mathbf{V}^{e-1} \tag{18}$$

by making use of the antisymmetry of W^p. The expression in square brackets is the Jaumann or corotational derivative $(\overset{\nabla}{\ })$, [7, p. 402] of $\mathbf{C}^e$ for axes rotating with spin $\mathbf{W}^p$, i.e.,

$$\overset{\nabla}{\mathbf{C}}{}^e = \dot{\mathbf{C}}^e - \mathbf{W}^p\mathbf{C}^e + \mathbf{C}^e\mathbf{W}^p \tag{19}$$

and (12) becomes

$$\mathbf{D} = \mathcal{D}^e + \mathbf{D}^p \tag{20}$$

where

$$\mathcal{D}^e = \frac{1}{2}\mathbf{V}^{e-1}\overset{\nabla}{\mathbf{C}}{}^e\mathbf{V}^{e-1} \tag{21}$$

Thus the rate-of-deformation or stretching tensor $\mathbf{D}$ of the elastic-plastic deformation can be expressed as the sum of an elastic component involving the Jaumann rate of $\mathbf{C}^e$ (effectively a term involving the rate of finite elastic strain) and the plastic rate-of-deformation tensor, $\mathbf{D}^p$. This provides a basis for writing the elastic-plastic constitutive relation in incremental form.

Taking the Jaumann derivative with spin $\mathbf{W}^p$ of the elastic constitutive relation (15) gives

$$\overset{\nabla}{\tau} = 2\overset{\nabla}{\mathbf{C}}^e \frac{\partial\psi}{\partial\mathbf{C}^e} + 2\mathbf{C}^e\left[\left(\frac{\partial^2\psi}{\partial\mathbf{C}^{e2}}:\overset{\nabla}{\mathbf{C}}^e\right) + \frac{\partial^2\psi}{\partial\theta\partial\mathbf{C}^e}\dot{\theta}\right] \qquad (22)$$

where the colon, :, denotes the trace of the matrix product. In component form this becomes, for isothermal response with which this paper is mainly concerned

$$\overset{\nabla}{\tau}_{ij} = \left[2\delta_{i\alpha}\left(\frac{\partial\psi}{\partial C^e}\right)_{\beta j} + 2C^e_{ik}\left(\frac{\partial^2\psi}{\partial C^{e2}}\right)_{kj\alpha\beta}\right]\overset{\nabla}{C}{}^e_{\alpha\beta} \qquad (23)$$

and substituting for $\mathbf{C}^e$ from (21) gives

$$\overset{\nabla}{\tau}_{ij} = 4\left[V^e_{im}V^e_{n\beta}\left(\frac{\partial\psi}{\partial C^e}\right)_{\beta j} + C^e_{ik}V^e_{\alpha m}V^e_{n\beta}\left(\frac{\partial^2\psi}{\partial C^{e2}}\right)_{kj\alpha\beta}\right]\mathcal{D}^e_{mn} \qquad (24)$$

or in concise notation

$$\overset{\nabla}{\tau}_{ij} = \tilde{\Pi}_{ijmn}\mathcal{D}^e_{mn}, \qquad \overset{\nabla}{\tau} = \tilde{\Pi}(\mathcal{D}^e) \qquad (25)$$

Inversion gives

$$\mathcal{D}^e = \tilde{\Lambda}^e(\overset{\nabla}{\tau}) \qquad (26)$$

The plastic rate of deformation or velocity strain, $\mathbf{D}^p$, is given by the plastic-potential, time-independent, hardening law [3, equation (40)]

$$\mathbf{D}^p = \frac{1}{h}\left(\frac{\partial f}{\partial\tau}:\dot{\tau} + \frac{\partial f}{\partial\theta}\dot{\theta}\right)\frac{\partial f}{\partial\tau} \qquad (27)$$

where $f = g(\tau) - c = 0$ is the yield function, g being on isotropic scalar function for isotropic hardening, and c a scalar function of temperature, θ, and history of plastic deformation. h is also a scalar function of plastic deformation history and temperature.

Since f is a scalar function of τ, the first term in the parenthesis in (27), being a part of f, will not be changed by replacing $\dot{\tau}$ by another tensor time derivative, such as $\overset{\nabla}{\tau}$. Thus (27) can be written

$$\mathbf{D}^p = \frac{1}{h}\left(\frac{\partial f}{\partial\tau}:\overset{\nabla}{\tau} + \frac{\partial f}{\partial\theta}\dot{\theta}\right)\frac{\partial f}{\partial\tau} \qquad (28)$$

For the remainder of the development we will restrict ourselves to isothermal deformation ($\dot{\theta} = 0$), and write (28) in the form

$$D^p_{ij} = \left(\frac{1}{h}\frac{\partial f}{\partial\tau_{ij}}\frac{\partial f}{\partial\tau_{mn}}\right)\overset{\nabla}{\tau}_{mn} = \Lambda^p_{ijmn}\overset{\nabla}{\tau}_{mn}, \quad \mathbf{D}^p = \Lambda^p(\overset{\nabla}{\tau}) \qquad (29)$$

Combining (20), (26), and (29) gives

$$\mathbf{D} = (\tilde{\Lambda}^e + \Lambda^p)(\overset{\nabla}{\tau}) = \bar{\Lambda}(\overset{\nabla}{\tau}) \qquad (30)$$

a rate-type law for elastic-plastic material. The operator $\bar{\Lambda}$ is a function of the current state which depends on the stress and the history of deformation. Inversion of (30) gives

$$\overset{\nabla}{\tau} = \bar{\mathcal{L}}(\mathbf{D}) \qquad (31)$$

3 Objectivity and Symmetry Properties

In order to contrast the elastic-plastic kinematic relations (9) (with an equality sign) and (20), we examine the relevant transformation characteristics of quantities involved under rotation of the current configuration by the proper orthogonal transformation $\mathbf{Q}(t)$

$$\mathbf{x}^* = \mathbf{Q}\mathbf{x} \qquad (32)$$

The deformation gradient changes to

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F} \qquad (33)$$

Since elastic destressing is considered to occur without rotation, each element of the unstressed configuration must be subjected to the same rotation, $\mathbf{Q}(t)$, as the current configuration and this constraint must be introduced into the objectivity requirements. Thus

$$\mathbf{F}^{p*} = \mathbf{Q}\mathbf{F}^p \qquad (34)$$

and

$$\mathbf{V}^{e*} = \mathbf{Q}\mathbf{V}^e\mathbf{Q}^T \qquad (35)$$

It is clear that (33)–(35) are consistent with (4) and (10). Using (33)–(35) it can be readily shown that the following transformations arise:

$$\mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \qquad (36)$$

$$\mathbf{D}^{p*} = \mathbf{Q}\mathbf{D}^p\mathbf{Q}^T \qquad (37)$$

$$\mathbf{C}^{e*} = \mathbf{Q}\mathbf{C}^e\mathbf{Q}^T \qquad (38)$$

$$\dot{\mathbf{C}}^{e*} = \mathbf{Q}\dot{\mathbf{C}}^e\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{C}^e\mathbf{Q}^T + \mathbf{Q}\mathbf{C}^e\dot{\mathbf{Q}}^T \qquad (39)$$

$$\overset{\nabla}{\mathbf{C}}{}^{e*} = \mathbf{Q}\overset{\nabla}{\mathbf{C}}{}^e\mathbf{Q}^T \qquad (40)$$

$$\mathbf{D}^{e*} = \mathbf{Q}\mathbf{D}^e\mathbf{Q}^T + \frac{1}{2}\mathbf{Q}(\mathbf{V}^e\dot{\mathbf{Q}}^T\mathbf{Q}\mathbf{V}^{e-1} + \mathbf{V}^{e-1}\mathbf{Q}^T\dot{\mathbf{Q}}\mathbf{V}^e)\mathbf{Q}^T \qquad (41)$$

$$\mathcal{D}^{e*} = \mathbf{Q}\mathcal{D}^e\mathbf{Q}^T \qquad (42)$$

It is clear that (20) is objective since $\mathbf{D}$, $\mathcal{D}^e$, and $\mathbf{D}^p$ all transform in the same way, whereas the common assumption of additive elastic and plastic strain rates; i.e., (9) with an equality sign, cannot be objective since $\mathbf{D}^e$ transforms differently from the other two terms.

It is important to examine the symmetry of the operators $\bar{\Lambda}_{ijmn}$ (30) and $\bar{\mathcal{L}}_{ijmn}$ (31) to check that the structure of the constitutive relation generates a rate-potential function [5] and hence can be incorporated into Hill's variational principle valid for evaluating solutions of problems involving finite deformation.

Equation (30) expresses $\bar{\Lambda}$ as the sum of elastic and plastic operators $\tilde{\Lambda}^e$ and Λ^p, and we examine these parts independently.

For isothermal isotropic elastic response the Helmholtz free energy, ψ, is a function of I_1, I_2, and I_3 the principal invariants of the Cauchy-Green deformation tensor $\mathbf{C}^e$:

$$I_1 = \mathrm{tr}\,(\mathbf{C}^e), \quad I_2 = [(\mathrm{tr}\,\mathbf{C}^e)^2 - \mathrm{tr}\,(\mathbf{C}^{e2})]/2, \quad I_3 = \det\mathbf{C}^e \qquad (43)$$

where tr stands for trace. The derivatives of ψ in (22) are then

$$\frac{\partial\psi}{\partial C^e_{ij}} = \frac{\partial\psi}{\partial I_1}\delta_{ij} + \frac{\partial\psi}{\partial I_2}(I_1\delta_{ij} - C^e_{ij}) + \frac{\partial\psi}{\partial I_3}I_3C^{e-1}_{ij} \qquad (44)$$

and

$$\begin{aligned}\frac{\partial^2\psi}{\partial C^e_{ij}\partial C^e_{\alpha\beta}} &= \left[\frac{\partial^2\psi}{\partial I_1^2}\delta_{\alpha\beta} + \frac{\partial^2\psi}{\partial I_1\partial I_2}(I_1\delta_{\alpha\beta} - C^e_{\alpha\beta}) + \frac{\partial^2\psi}{\partial I_1\partial I_3}I_3C^{e-1}_{\alpha\beta}\right]\delta_{ij} \\ &+ \left[\frac{\partial^2\psi}{\partial I_2\partial I_1}\delta_{\alpha\beta} + \frac{\partial^2\psi}{\partial I_2\partial I_2}(I_1\delta_{\alpha\beta} - C^e_{\alpha\beta}) + \frac{\partial^2\psi}{\partial I_1\partial I_3}I_3C^{e-1}_{\alpha\beta}\right](I_1\delta_{ij} - C^e_{ij}) \\ &+ \left[\frac{\partial^2\psi}{\partial I_3\partial I_1}\delta_{\alpha\beta} + \frac{\partial^2\psi}{\partial I_3\partial I_2}(I_1\delta_{\alpha\beta} - C^e_{\alpha\beta}) + \frac{\partial^2\psi}{\partial I_3\partial I_3}I_3C^{e-1}_{\alpha\beta}\right]I_3C^{e-1}_{ij} \\ &+ \frac{\partial\psi}{\partial I_2}(\delta_{\alpha\beta}\delta_{ij} - \delta_{i\alpha}\delta_{j\beta}) + \frac{\partial\psi}{\partial I_3}(I_3C^{e-1}_{\alpha\beta}C^{e-1}_{ij} - I_3C^{e-1}_{ik}\delta_{k\alpha}\delta_{l\beta}C^{e-1}_{lj})\end{aligned} \qquad (45)$$

Substitution of (44) and (45) into (24) gives the operator $\tilde{\Pi}_{ijmn}$, (25), and term-by-term examination establishes the symmetry

$$\tilde{\Pi}_{ijmn} = \tilde{\Pi}_{mnij} \qquad (46)$$

The variables τ_{ij} and $\mathcal{D}^e_{mn}$ are both symmetric in their suffixes, this property of the former being confirmed by deduction from (21), (24), (44), and (45). This, in combination with (46), permits the constitutive relation (25) to be formulated so that

$$\tilde{\Pi}_{ijmn} = \tilde{\Pi}_{jimn} = \tilde{\Pi}_{ijnm} \qquad (47)$$

In view of the symmetry (46), and regarding (25) as a matrix transformation of a vector with subscript (mn), to one with subscript (ij), inversion to (26) yields a symmetric inverse matrix $\tilde{\Lambda}^e$. This combined with the symmetry of τ_{ij} and $\mathcal{D}^e_{mn}$ establishes the symmetry relations

$$\tilde{\Lambda}^e_{ijkl} = \tilde{\Lambda}^e_{klij} = \tilde{\Lambda}^e_{jikl} = \tilde{\Lambda}^e_{ijlk} \qquad (48)$$

It is immediately clear that the plastic operator, $\mathbf{A}^p$, (29), exhibits the same symmetries, and hence also the elastic-plastic operator $\tilde{\mathbf{A}}$, (30). Again the symmetries are preserved by the matrix inversion of (30) to produce the operator $\tilde{\mathcal{L}}$, (31).

4 An Appropriate Variational Formulation

The symmetries established in the previous section guarantee the existence of a rate-potential function [5, 8] and hence lead to application of Hill's variational principle for velocity fields which is valid for finite deformation. Sequential application and time integration over the period of deformation permits the history of the deformation and stress distributions to be evaluated.

However, substitution of (31) into the variational formulation would involve unnecessary complexity since the Jaumann derivative of stress, $\overset{\nabla}{\tau}$, involves the spin tensor $\mathbf{W}^p$ (equation (19)), which cannot be simply expressed in terms of the velocity field as is evident from (11). In contrast the total spin $\mathbf{W}$ is simply the antisymmetric part of the velocity gradient. Incorporation of the Jaumann derivative based on the total spin is therefore more simply expressed and yields a convenient variational-principal structure for the determination of the velocity field.

To this end we define the Jaumann derivative associated with the total spin $\mathbf{W}$,

$$(\,^{\circ}\,) = (\,\dot{}\,) - \mathbf{W}(\;) + (\;)\mathbf{W} \tag{49}$$

The corresponding differentiation of the elastic constitutive relation (15) yields

$$\overset{\circ}{\tau}_{ij} = 2\left[\delta_{im}\frac{\partial\psi}{\partial C^e_{nj}} + C^e_{i\alpha}\left(\frac{\partial^2\psi}{\partial C^{e2}}\right)_{\alpha jmn}\right]\overset{\circ}{C}{}^e_{mn} \tag{50}$$

where

$$\overset{\circ}{\mathbf{C}}{}^e = \dot{\mathbf{C}}^e - \mathbf{W}\mathbf{C}^e + \mathbf{C}^e\mathbf{W} \tag{51}$$

Manipulation of this relation, by substituting from (13)

$$\mathbf{W} = (\dot{\mathbf{V}}^e\mathbf{V}^{e-1})_A + (\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e-1})_A \tag{52}$$

yields, after some algebraic manipulation

$$\hat{\mathbf{C}}^e = \mathcal{D}^e\mathbf{C}^e + \mathbf{C}^e\mathcal{D}^e \tag{53}$$

$\mathcal{D}^e$ being defined in (21). When this is substituted into (50) we have

$$\overset{\circ}{\tau}_{ij} = 2\left[\delta_{i\alpha}\left(\frac{\partial\psi}{\partial C^e}\right)_{\beta j} + C^e_{ik}\left(\frac{\partial^2\psi}{\partial C^{e2}}\right)_{kj\alpha\beta}\right] \times (\delta_{\alpha m}C_{n\beta} + C_{\alpha m}\delta_{\beta n})\mathcal{D}^e_{mn} \tag{54}$$

and inversion and combination with the plastic strain-rate operator (29) according to (20) gives

$$\mathbf{D} = (\mathbf{\Lambda}^e + \mathbf{\Lambda}^p)\,\overset{\circ}{\boldsymbol{\tau}} = \mathbf{\Lambda}(\overset{\circ}{\boldsymbol{\tau}}) \tag{55}$$

where $\mathbf{\Lambda}^e$ is the inverse operator to that in (54). Inversion of (55) gives

$$\overset{\circ}{\tau} = \mathcal{L}(\mathbf{D}) \tag{56}$$

On the basis of (54), (44), and (45) it can be shown that the operator $\mathcal{L}$ exhibits the same symmetry properties established in the previous section for operator $\tilde{\mathcal{L}}$ in (31). Thus the operator $\mathcal{L}$ can be expressed in rate-potential form and so incorporated into Hill's variational principle [5].

In order to include convection influences associated with finite deformation in a simple and complete manner, Hill's variational principle involves the unsymmetric nominal stress T_{Ji} (Piola-Kirchhoff I) and the gradient of the actual material velocity in the deformed configuration with respect to the reference position coordinates $\mathbf{X}$. The variational principle for the velocity field $\mathbf{v}$, (5), takes the form

$$\int_{V_0}\dot{T}_{Ji}\delta\left(\frac{\partial v_i}{\partial X_J}\right)dV_0 - \int_{V_0}\dot{b}_i\delta v_i dV_0 - \int_{S_0}\dot{f}_i\delta v_i dS_0 = 0 \tag{57}$$

b_i being the body force per unit reference volume, f_i the surface traction per unit reference area, and the superposed dot indicates time derivative. The rate of change of stress is expressed in terms of the rate-potential function E by the relation

$$\dot{T}_{Ji} = \frac{\partial E}{\partial(\partial v_i/\partial X_J)} \tag{58}$$

Thus, to incorporate directly the constitutive relation in the form (56) into (57) and (58), $\mathbf{D}$ being the symmetric part of the velocity gradient in the current configuration, the current configuration must be chosen as the reference configuration, and $\dot{T}_{Ji}$ expressed in terms of the rate of Kirchhoff stress, $\dot{\boldsymbol{\tau}}$, defined with respect to the initial configuration. The velocity distribution is thus evaluated at time t and hence the increments of displacement in the time interval t to $(t + \Delta t)$. The new configuration is then used as the reference state for evaluating the deformation during the next time-step. Iteration is used to improve the accuracy of the sequential procedure, and stresses are determined by integrating the constitutive relation as the deformation proceeds.

Now the Piola-Kirchoff I stress is given in terms of the Kirchoff stress (see, for example, [8] with the notation $\mathbf{s}$ for $\mathbf{T}$ and $J\overset{\circ}{\sigma} = \overset{\circ}{\tau}$ for $\boldsymbol{\tau}$) by

$$\mathbf{T} = \mathbf{F}^{-1}\boldsymbol{\tau} \tag{59}$$

For the reference state coincident with the current state at time t

$$^t\mathbf{F}(t) = \mathbf{I} \tag{60}$$

the unit matrix. Differentiating (59) with respect to time in the interval t to $t + \Delta t$, using (49) and (60), then gives at time t

$$^t\dot{\mathbf{T}} = {}^t\overset{\circ}{\boldsymbol{\tau}} + {}^t\boldsymbol{\tau}\mathbf{D} - \mathbf{D}\,{}^t\boldsymbol{\tau} - {}^t\boldsymbol{\tau}\mathbf{L} \tag{61}$$

where the superscript t indicates that these stresses are defined with the configuration at time t as reference state. But the Kirchhoff stress in (56) is based on the unstressed reference state of density ρ_0, and is defined, in terms of the Cauchy stress $\boldsymbol{\sigma}$ at some arbitrary time, by

$$\boldsymbol{\tau} = \frac{\rho_0}{\rho}\boldsymbol{\sigma} \tag{62}$$

whereas in (60) and (61) the configuration at time t is the reference state, and in the time interval t to $t + \Delta t$

$$^t\boldsymbol{\tau} = \frac{\rho(t)}{\rho}\boldsymbol{\sigma} = \frac{\rho(t)}{\rho_0}\frac{\rho_0}{\rho}\boldsymbol{\sigma} = \frac{\rho(t)}{\rho_0}\boldsymbol{\tau} \tag{63}$$

Substituting into the variational principle (57) with $\mathbf{X} = \mathbf{x}(t)$ and incorporating (56) then yields after some manipulation

$$\int_V \frac{\rho(t)}{\rho_0}[\mathcal{L}_{ijmn}D_{mn}\delta D_{ij} + \tau_{ij}(-2D_{kj}\delta D_{ik} + L_{kj}\delta L_{ki})]dV - \int_V \dot{b}_i\delta v_i dV - \int_S \dot{f}_i\delta v_i dS = 0 \tag{64}$$

where b_i and f_i are per unit volume and area, respectively, in the configuration at time t. The formulation of the variational principle in terms of the Kirchhoff stress is essentially that presented by McMeeking and Rice [9] for the small elastic strain case except for the density ratio term which was not included and may not be important in that case.

The variational principle (64) forms a convenient basis for the generation of finite-element computer codes, many of which are in successful operation for small elastic strains, for example [9–11].

5 Finite Elastic-Plastic Deformation With Small Elastic Strain

Many problems involving the elastic-plastic deformation of metals may fall within the scope of a special case of the theory developed in this paper, that of finite deformation with small elastic strain. Then the squares of elastic strain components can be neglected. The elastic strain tensor can be defined as the Lagrange strain

$$\boldsymbol{\epsilon}^e = (\mathbf{C}^e - \mathbf{I})/2 \tag{65}$$

The Helmholtz free energy for isothermal deformation takes the form of the classical strain energy for infinitesimal strain

$$\psi = (\lambda \epsilon^e_{ii} \epsilon^e_{jj} + 2\mu \epsilon^e_{ij} \epsilon^e_{ji})/2 \tag{66}$$

Substitution of (65) into (66) yields

$$\psi = [\lambda(C^e_{ii}C^e_{jj} - 6C^e_{ii} + 9) + 2\mu(C^e_{ij}C^e_{ji} - 2C^e_{ii} + 3)]/8 \tag{67}$$

Since in (54) $\mathcal{D}^e$, which involves a time derivative of ϵ^e, is considered a first-order small term, analysis to first-order permits zero-order substitution of factors and hence $\mathbf{C}^e$ to be replaced by $\mathbf{I}$. $\mathcal{D}^e$ can then be replaced by $\mathbf{D}^e$, and (54) becomes

$$\overset{\circ}{\tau}_{ij} = 4\left[\delta_{im}\left(\frac{\partial\psi}{\partial C^e}\right)_{ij} + \left(\frac{\partial^2\psi}{\partial C^{e2}}\right)_{ijmn}\right] D^e_{mn} \tag{68}$$

Differentiation of (67) shows that the first derivative of ψ is of order $\boldsymbol{\epsilon}$, and can be neglected compared with the second derivative, so that (68) becomes

$$\overset{\circ}{\tau}_{ij} = (\lambda\delta_{ij}\delta_{mn} + 2\mu\delta_{im}\delta_{nj})D^e_{mn} \tag{69}$$

Inversion gives

$$D^e_{ij} = \frac{1}{2\mu}\left(\delta_{im}\delta_{jn} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}\delta_{mn}\right)\overset{\circ}{\tau}_{mn} \tag{70}$$

Substitution into (20) with the plasticity law (29) then yields

$$D_{ij} = \left[\frac{1}{2\mu}\left(\delta_{im}\delta_{jn} - \frac{\lambda}{3\lambda + 2\mu}\,\delta_{ij}\delta_{mn}\right) + \frac{1}{h}\frac{\partial f}{\partial \tau_{ij}}\frac{\partial f}{\partial \tau_{mn}}\right]\overset{\circ}{\tau}_{mn} \tag{71}$$

which can be inverted to give

$$\overset{\circ}{\tau}_{ij} = 2\mu\left[\delta_{i\alpha}\delta_{j\beta} + \frac{\lambda}{2\mu}\,\delta_{\alpha\beta}\delta_{ij} - \frac{1}{\dfrac{h}{2\mu} + \dfrac{\partial f}{\partial \tau_{mn}}\dfrac{\partial f}{\partial \tau_{mn}}}\frac{\partial f}{\partial \tau_{ij}}\frac{\partial f}{\partial \tau_{\alpha\beta}}\right] D_{\alpha\beta} \tag{72}$$

the operator on the right-hand side being the corresponding $\mathcal{L}$. The development following (69) parallels that given in [9] using different elastic moduli. If in certain problems, for example, stability analyses, a more precise formulation is needed, it may be possible to include higher-order terms in an approximation for the elastic-plastic operator $\mathcal{L}$, (56).

6 Discussion

This paper presents a rate or incremental formulation of the finite-deformation, elastic-plastic theory developed in [3]. The fact that in [3] the elastic constitutive relation was left in total elastic deformation form, rather than in rate form, limited utilization of the theory and application has been restricted to shock-wave analysis in which, by symmetry, principal directions of stress remained fixed in the body. This permits the use of logarithmic or natural strain which generates additivity of both elastic and plastic strain and strain-rates [12, 13]. However, this approach is not possible for arbitrary loading with rotation and the present contribution removes this restriction.

The present theory is based on exact nonlinear kinematics of elastic and plastic deformation. The latter is the deformation the body would exhibit if the macroscopic stress, and hence elastic strain, were maintained at zero while the remaining deformation proceeds. In the case of migration of dislocations, for example, increments of this plastic deformation are directly related to the specifics of the activity of such atomic mechanisms and are not coupled with the elastic strain to which the body is in fact continuously subjected. This is in contrast with an increment of irreversible or residual strain under maintained stress, which depends on the elastic strain because of the elastically deformed material lattice quite apart from the influence of rotation already considered. The elastic deformation is that due to stress acting on the purely plastically deformed material. The exact nonlinear kinematics logically directs the development of the theory including the appropriate stress and strain variables which arise.

The current commonly adopted elastic-plastic theory appears to have grown from a foundation of infinitesimal deformation kinematics which sets rates (or increments) of total strain as the sum of elastic and plastic contributions, using the rate-type law for the plastic strain and the derivative of the classical elasticity law for the elastic contribution. Combining such kinematics and constitutive laws yields a linear stress-rate strain-rate relation with stress and history variables as coefficients. In order to preserve objectivity (or from a more elementary standpoint, to permit rigid-body rotation to correspond to zero stress rate) the Jaumann corotational stress rate is selected for the elastic-plastic rate law. While the final rate law developed is objective, the components from which it was constructed either are not objective or do not correctly express the physical entities they purport to represent. It seems to us that this explains the anomalies which appear in the currently accepted elastic-plastic theory when it is subjected to careful scrutiny.

Perhaps some comments are in order to clarify the concept that the residual strain increment, left after addition and removal of a stress increment $\Delta\boldsymbol{\sigma}$, on a body in which the stress $\boldsymbol{\sigma}$ is maintained, contains an elastic part. Increments of stress and deformation about a state of maintained stress can be analyzed in terms of the rate of deformation $\mathbf{D}$ defined in (11) and (12). During the loading increment, elastic and plastic deformation increments can arise and all the terms in (12) are likely to be nonzero. The simplest unloading assumption for the stress increment $-\Delta\boldsymbol{\sigma}$ is to reverse the sign of $\dot{\mathbf{V}}^e$, the elastic deformation rate, and introduce no additional spin. Then the last term in (12) is zero for unloading, and the value of that term times Δt during the loading $\Delta\boldsymbol{\sigma}$ will be retained as a residual or irreversible strain increment, although it is independent of the plastic *flow* occurring during the loading increment [4]. Of course, the spin tensor W^p associated with the elastic-plastic loading increment could be reversed and incorporated into the elastic unloading increment, in which case the irreversible increment of deformation associated with W^p would become reversible. However this might be difficult to arrange in practice since in the loading increment the plastic strain will normally dominate and superposing the spin W^p on elastic unloading is likely to dominate the deformation process in unloading. In any event, to include spin in the unloading which defines plastic strain, and to have to devise a prescription for such spin, is likely to serve no useful purpose and would complicate the analysis compared with that corresponding to the choice (10), of unloading without rotation. Quite apart from the unloading question, it was already shown in [3] that the power expended by the deformation rate associated with the last term in (12) is zero, in contrast to that due to D^p, so that the former term was shown to be physically associated with elastic deformation even though it does not involve a change in the elastic deformation $\mathbf{V}^e$.

From the standpoint of application, the small elastic strain theory developed in the previous section, which results in essentially the formulation currently adopted in many computer codes valid for finite deformation, can be expected to be satisfactory for the majority of problems of metal deformation. Equation (18) shows that the plastic spin associated Jaumann derivative of the elastic Cauchy-Green tensor, $\mathbf{C}^e$, is a good approximation to the elastic contribution to the strain rate, which includes the effects of both the rate of change of stress and the rotation of the body. Equation (22) shows that this is associated directly with the same derivative of stress, and (13) indicates that the total spin will generally be a close approximation to the spin $\mathbf{W}^p$, so that the Jaumann derivative based on total spin will be a good approximation to that based on W^p. The present new formulation will permit a more accurate version of the theory to be developed if needed for particular problems.

It should perhaps be pointed out that, although the concept of the definition of plastic deformation as that remaining when the macroscopic stress, and hence elastic deformation, is reduced to zero is stressed in this paper, the general structure of the theory presented can encompass the situation when a Bauschinger effect involves reverse plastic flow on unloading before zero stress is reached. In this case there need be no particular difficulty in devising a Helmholtz free-energy function for the elastic range. The application of incremental theory involves only rates or increments of strain and not a prescription of total elastic or plastic strain. Formally one could define the plastic strain to correspond to zero stress evaluated from the elastic law even though this state cannot be reached without additional plastic flow. The utilization of this artificial unstressed state provides no impediment to evaluating stresses according to elastic or elastic-plastic theory where the corresponding physical behavior is occurring.

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