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CLASSROOM NOTE

On the proofs of orthogonality of eigenfunctions for heat conduction, wave propagation, and advection-diffusion problems

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ABSTRACT

The orthogonality of eigenfunctions in problems of unsteady heat conduction in an infinite slab with symmetric and nonsymmetric convective boundary conditions are demonstrated by performing actual integration of the products of the derived forms of eigenfunctions and by implementing the corresponding eigenvalue conditions. The analysis also applies to longitudinal vibrations of an elastic rod attached at its ends to linear elastic springs, and to advection-diffusion problems under appropriate boundary conditions. The same form of eigenfunctions and the same type of eigenvalue condition apply in all three considered cases. The proofs are compared with the well-known general proof of orthogonality of eigenfunctions, which circumvents the actual integration. The presented analysis is pedagogically appealing for use in engineering and applied physics education.

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1. Introduction

The well-known proof of orthogonality of eigenfunctions of the Sturm–Liouville problem circumvents the actual evaluation of the integral of the product of specific forms of eigenfunctions by showing instead that this integral is equal to a certain combination of the boundary terms which vanishes by the application of the prescribed homogeneous boundary conditions (Boyce et al., 2021; Greenberg, 1998; Kreyszig, 2011; O'Neil, 2018; Zill, 2018). While this elegant and powerful proof applies to all kinds of prescribed homogeneous boundary conditions, and the corresponding forms of eigenfunctions, one is often curious or tempted to verify the orthogonality of eigenfunctions in each considered boundary-value problem by performing actual integration of the product of the derived forms of eigenfunctions and by implementing the corresponding eigenvalue conditions. Such integration is for some boundary conditions simple, but for some it is less so. This is illustrated and discussed in this paper on the example of unsteady heat conduction in an infinitely long slab with symmetric and nonsymmetric convective boundary conditions (Cengel, 2002; Lienhard IV & Lienhard V, 2017; Mills & Coimbra, 2015),

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the analysis of longitudinal vibrations of an elastic rod attached at its ends to linear elastic springs (Rao, 2016; Weaver Jr. et al., 1991), and the analysis of one-dimensional convection-diffusion problems under appropriate boundary conditions (Bennett, 2012; Stynes & Stynes, 2018).

Figure 1 shows an infinite slab of thickness $2L$, heat conduction coefficient k, and initial (uniform or symmetric in x) temperature T_0, emersed in a fluid whose remote temperature is T_f. An infinite slab is a suitable model for slabs whose height and width are much greater than their thickness, so that possible gradients along y and z directions can be neglected sufficiently away from the ends. Convective boundary conditions with the same convection coefficient h apply on both sides of the slab, $-k(\partial T/\partial x)_{x=\pm L} = \pm h[T(\pm L, t) - T_\mathrm{f}]$, where $T = T(x, t)$ is the temperature field in the slab, dependent on the spatial coordinate x and time t. The governing parabolic-type partial differential equation for the transient temperature field $\tau(x, t) = T(x, t) - T_\mathrm{f}$ is $\partial\tau/\partial t = \alpha(\partial^2\tau/\partial x^2)$, where $\alpha = k/c_p$ is the thermal diffusivity of the slab, c_p being its specific heat capacity per unit volume. Upon using the method of separation of variables to write $\tau(x, t) = f(x)g(t)$, the x-dependence of the temperature is found as the solution to the following eigenvalue problem

$$f'' + (\mu/L)^2 f = 0, \tag{1}$$

$$f'(0) = 0, \quad f'(L) + (h/k)f(L) = 0. \tag{2}$$

The prime denotes the derivative with respect to x, and the symmetry condition $f'(0) = 0$ is conveniently imposed. The corresponding eigenfunctions are

$$f_n = \cos(\mu_n x/L), \tag{3}$$

where μ_n are the numerically determined roots to the transcendental equation

$$\tan\mu = \frac{\mathrm{Bi}}{\mu}, \quad \mathrm{Bi} = \frac{hL}{k} \quad \text{(Biot number)}. \tag{4}$$

Physically, the Biot number represents the ratio of the conduction resistance inside a body to the convection resistance at the external surface of a body. Equation (4) follows from the second boundary condition in (2), while the symmetry condition $f'(0) = 0$ eliminates the sine functions. The functions f_n are orthogonal to each other on the interval $[0, L]$,

$$\int_0^L f_n(x) f_m(x)\,\mathrm{d}x = 0, \quad n \neq m, \tag{5}$$

which enables the analytical determination of the coefficients c_n appearing in the general expression for the transient temperature field,

$$\tau = \sum_{n=1}^{\infty} c_n \cos(\mu_n x/L)\,\mathrm{e}^{-\alpha(\mu_n/L)^2 t}. \tag{6}$$

Indeed, from the initial condition $\tau(x, 0) = T_0 - T_\mathrm{f}$, by multiplying both sides with $\cos(\mu_m x/L)$ and by integrating from 0 to L, we obtain

$$\sum_{m=1}^{\infty} c_m \cos(\mu_m x/L) = T_0 - T_\mathrm{f} \quad \Rightarrow \quad c_n = \frac{\int_0^L (T_0 - T_\mathrm{f})\cos(\mu_n x/L)\,\mathrm{d}x}{\int_0^L \cos^2(\mu_n x/L)\,\mathrm{d}x}, \tag{7}$$

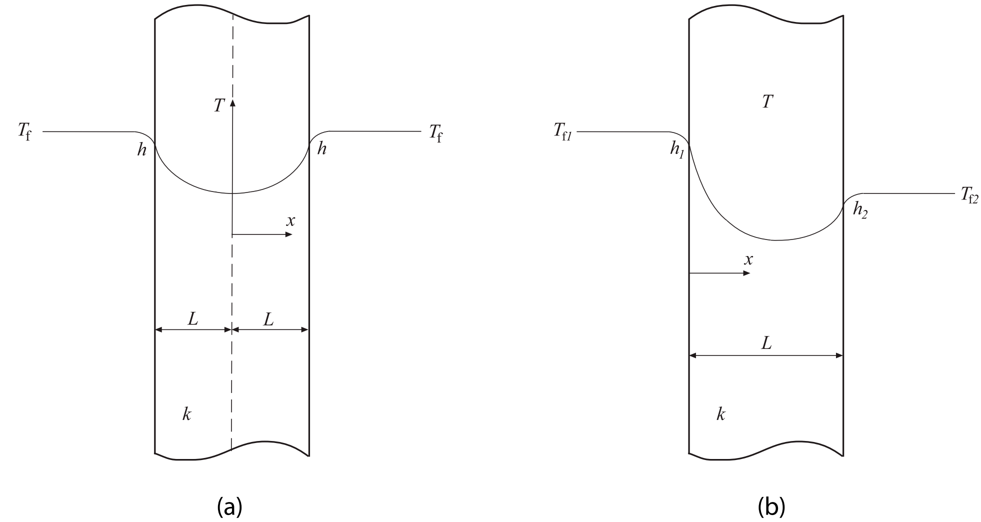

Figure 1. (a) A sketch of the temperature profile $T = T(x, t)$ in an infinitely long slab of thickness $2L$ and heat conduction coefficient k at an arbitrary instant of time t. The remote temperature of the surrounding fluid is T_f. Convective boundary conditions with the same convection coefficient h apply on both sides of the slab. The initial uniform temperature of the slab is T_0. (b) A sketch of the temperature profile under nonsymmetric convective boundary conditions. The remote temperatures of the surrounding fluid at the two sides of the slab are T_{f1} and T_{f2}. Convective boundary conditions apply with convection coefficient h_1 on the left, and h_2 on the right side of the slab.

which gives, in the case of uniform initial temperature, the well-known result (Cengel, 2002; Lienhard IV & Lienhard V, 2017; Mills & Coimbra, 2015)

$$c_n = 4(T_0 - T_f)\,\frac{\sin\mu_n}{2\mu_n + \sin 2\mu_n}, \qquad n = 1, 2, 3, \ldots. \tag{8}$$

The coefficients c_n rapidly diminish with the increase of n. For example, if $\mathrm{Bi} = 1$, the three lowest eigenvalues are $\mu_1 = 0.8603$, $\mu_2 = 3.4256$, and $\mu_3 = 6.4373$, and (8) gives $c_1 = 1.1191$, $c_2 = -0.1517$, and $c_3 = 0.0466$, all multiplied by $(T_0 - T_f)$. Thus, the third coefficient is only 4.2% of the first coefficient, and therefore only several leading eigenfunctions dominate the spatial variation of τ. For smaller values of Biot's number, the convergence of the series (6) is even faster. For example, for $\mathrm{Bi} = 0.1$, one obtains $\mu_1 = 0.3111$, $\mu_2 = 3.1731$, and $\mu_3 = 6.2991$, and (8) gives $c_1 = 1.0161$, $c_2 = -0.0197$, and $c_3 = 0.0050$, multiplied by $(T_0 - T_f)$. Thus, the third coefficient is in this case only 0.5% of the first coefficient, which is not surprising, because for small values of Biot's number (e.g. $\mathrm{Bi} <= 0.1$), the temperature variation within a body is closer to uniform temperature.

The classical proof of orthogonality of eigenfunctions (5) is an indirect proof. If the differential Equation (1), written for the eigenfunction f_n, is multiplied by f_m $(m \neq n)$, one obtains $(\mu_n/L)^2 f_n f_m = -f_m f_n''$. An analogous expression can be written by interchanging the indices n and m, $(\mu_m/L)^2 f_n f_m = -f_n f_m''$. By subtracting the last two expressions,

$$\frac{\mu_n^2 - \mu_m^2}{L^2} f_n f_m = f_n f_m'' - f_m f_n'' \equiv \left(f_n f_m' - f_m f_n'\right)', \tag{9}$$

and, upon integration,

$$\frac{\mu_n^2 - \mu_m^2}{L^2} \int_0^L f_n f_m \, \mathrm{d}x = \left(f_n f_m' - f_m f_n'\right)_0^L . \tag{10}$$

The right-hand side of (10) is identically equal to zero by the boundary conditions (2), and (10) reduces to

$$(\mu_n^2 - \mu_m^2) \int_0^L f_n f_m \, \mathrm{d}x = 0, \quad n \neq m. \tag{11}$$

Thus, since $\mu_n \neq \mu_m$ for $n \neq m$, (11) implies the orthogonality of functions f_n, as stated in (5).

2. Verification of orthogonality by explicit integration

The verification of orthogonality of eigenfunctions by direct integration in (5) in the case of negligible surface resistance and the boundary conditions $f'(0) = 0$ and $f(L) = 0$ is simple. The verification in the case of the convective boundary conditions (2) is somewhat more involved. First, by using the relationship

$$2\cos(\mu_n x/L)\cos(\mu_m x/L) = \cos[(\mu_n + \mu_m)x/L] + \cos[(\mu_n - \mu_m)x/L)], \tag{12}$$

one obtains

$$\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x = \frac{L}{2}\left[\frac{\sin(\mu_n + \mu_m)}{\mu_n + \mu_m} + \frac{\sin(\mu_n - \mu_m)}{\mu_n - \mu_m}\right]. \tag{13}$$

Second, from the additive theorems of trigonometry $\sin(\mu_n \pm \mu_m) = \sin\mu_n \cos\mu_m \pm \cos\mu_n \sin\mu_m$, and by incorporating the eigenvalue condition (4), it follows that

$$\frac{\sin(\mu_n + \mu_m)}{\mu_n + \mu_m} = \mathrm{Bi}\,\frac{\cos\mu_n \cos\mu_m}{\mu_n \mu_m}, \qquad \frac{\sin(\mu_n - \mu_m)}{\mu_n - \mu_m} = -\mathrm{Bi}\,\frac{\cos\mu_n \cos\mu_m}{\mu_n \mu_m}. \tag{14}$$

The substitution of (14) into (13) establishes the orthogonality

$$\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x = 0, \quad n \neq m. \tag{15}$$

2.1. Nonorthogonality of sine functions

In contrast to negligible surface resistance and the boundary conditions $f'(0) = 0$ and $f(L) = 0$, when both the sine and cosine functions are orthogonal on $[0, L]$, in the case of convective boundary conditions (2), the sine functions are not orthogonal on $[0, L]$.

Instead,

$$\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x = -L\mathrm{Bi}\,\frac{\cos\mu_n\cos\mu_m}{\mu_n\mu_m}. \tag{16}$$

This follows by an analogous derivation as used to prove (15), beginning from the trigonometric relationship

$$-2\sin(\mu_n x/L)\sin(\mu_m x/L) = \cos[(\mu_n+\mu_m)x/L] - \cos[(\mu_n-\mu_m)x/L)]. \tag{17}$$

Furthermore, the integral of the product of the derivatives of the eigenfunctions is

$$\int_0^L f_n' f_m'\,\mathrm{d}x = \frac{\mu_n\mu_m}{L^2}\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x = -\frac{\mathrm{Bi}}{L}\cos\mu_n\cos\mu_m. \tag{18}$$

2.2. Alternative proof of orthogonality of cosine functions

An alternative proof of orthogonality of eigenfunctions $\cos(\mu_n x/L)$ in the considered boundary value problem can be constructed by applying integration by parts to the integral of the product of sine functions,

$$\begin{aligned}\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x &= -\frac{L}{\mu_m}\int_0^L \sin(\mu_n x/L)\mathrm{d}[\cos(\mu_m x/L)]\\ &= -\frac{L}{\mu_m}(\sin\mu_n\cos\mu_m) + \frac{\mu_n}{\mu_m}\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x.\end{aligned} \tag{19}$$

By substituting into (19) the eigenvalue condition (4), it follows that

$$\begin{aligned}&\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x\\ &= -L\mathrm{Bi}\,\frac{\cos\mu_n\cos\mu_m}{\mu_n\mu_m} + \frac{\mu_n}{\mu_m}\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x.\end{aligned} \tag{20}$$

The left-hand side of (20) is symmetric in n and m, and so is the first term on the right-hand side. Consequently, the second term on the right-hand side of (20), with a nonsymmetric pre-factor μ_n/μ_m, must vanish. This is possible if only if

$$\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x = 0, \quad n \neq m, \tag{21}$$

which establishes the orthogonality of eigenfunctions $\cos(\mu_n x/L)$.

2.3. Simultaneous proofs of orthogonality of cosine and nonorthogonality of sine functions

By applying integration by parts to the integral of the product of cosine functions, it follows that

$$\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x = \frac{L}{\mu_m}\int_0^L \cos(\mu_n x/L)\mathrm{d}[\sin(\mu_m x/L)]$$
$$= \frac{L}{\mu_m}(\sin\mu_m\cos\mu_n) + \frac{\mu_n}{\mu_m}\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x. \tag{22}$$

After substituting the eigenfunction condition (4), the integral in (22) becomes

$$\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x$$
$$= L\mathrm{Bi}\,\frac{\cos\mu_n\cos\mu_m}{\mu_m^2} + \frac{\mu_n}{\mu_m}\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x. \tag{23}$$

An analogous expression can be written by interchanging the indices n and m in (23), i.e.

$$\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x$$
$$= L\mathrm{Bi}\,\frac{\cos\mu_n\cos\mu_m}{\mu_n^2} + \frac{\mu_m}{\mu_n}\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x. \tag{24}$$

Subtracting (24) from (23) gives

$$\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x = -L\mathrm{Bi}\,\frac{\cos\mu_n\cos\mu_m}{\mu_n\mu_m}, \tag{25}$$

which demonstrates the nonorthogonality of $\sin(\mu_n x/L)$ functions on the interval $[0, L]$. When (25) is substituted in either (23) or (24), it follows that

$$\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x = 0, \quad n \neq m, \tag{26}$$

which establishes the orthogonality of eigenfunctions $\cos(\mu_n x/L)$ on the interval $[0, L]$.

3. Nonsymmetric convective boundary conditions

The presented direct proofs of orthogonality of eigenfunctions in the case of symmetric convective boundary conditions are rather straightforward, but in the case of nonsymmetric convective boundary conditions, they become more involved. This is illustrated and discussed in this section. Figure 1(b) shows an infinite slab of thickness L, having heat conduction coefficient k, under convective boundary conditions $-k(\partial T/\partial x)_{x=0} = h_1[T_{\mathrm{f}1} - T(0,t)]$ and $-k(\partial T/\partial x)_{x=L} = h_2[T(L,t) - T_{\mathrm{f}2}]$, where h_1 is the convection coefficient on the side $x = 0$, and h_2 on the side $x = L$. The remote fluid temperatures on the two sides of the slab are $T_{\mathrm{f}1}$ and $T_{\mathrm{f}2}$, and the initial temperature of the slab is T_0. These boundary conditions follow from Newton's law of cooling, by which the rate of heat loss is proportional to the difference between the surface temperature of the body and the remote temperature of the environment. The temperature field in the slab $T = T(x,t)$ can be expressed as the sum $T = \tau(x,t) + T_{ss}(x)$ of the transient (τ) and steady-state temperature (T_{ss}) fields. The latter is given by

$$T_{ss} = T_1 - \frac{T_1 - T_2}{L}\,x, \tag{27}$$

where

$$T_1 = \frac{\mathrm{Bi}_1(1+\mathrm{Bi}_2)T_{f1} + \mathrm{Bi}_2 T_{f2}}{\mathrm{Bi}_1(1+\mathrm{Bi}_2) + \mathrm{Bi}_2}, \quad T_2 = \frac{\mathrm{Bi}_2(1+\mathrm{Bi}_1)T_{f2} + \mathrm{Bi}_2 T_{f1}}{\mathrm{Bi}_2(1+\mathrm{Bi}_1) + \mathrm{Bi}_1} \tag{28}$$

are the final temperatures of the two sides of the slab. The two Biot numbers in (28) are $\mathrm{Bi}_1 = h_1 L/k$ and $\mathrm{Bi}_2 = h_2 L/k$.

The transient temperature field is governed by the differential equation $\partial\tau/\partial t = \alpha(\partial^2\tau/\partial x^2)$, with the initial condition $\tau(x,0) = T_0 - T_{ss}$ and the homogeneous boundary conditions

$$\left(\frac{\partial\tau}{\partial x} - \frac{\mathrm{Bi}_1}{L}\,\tau\right)_{x=0} = 0, \quad \left(\frac{\partial\tau}{\partial x} + \frac{\mathrm{Bi}_2}{L}\,\tau\right)_{x=L} = 0. \tag{29}$$

Upon using the method of separation of variables to write $\tau(x,t) = f(x)g(t)$, it follows that the differential equation for $f(x)$ is

$$f'' + (\mu/L)^2 f = 0, \tag{30}$$

accompanied by the Robin-type boundary conditions

$$f'(0) - \frac{\mathrm{Bi}_1}{L} f(0) = 0, \quad f'(L) + \frac{\mathrm{Bi}_2}{L} f(L) = 0. \tag{31}$$

The corresponding eigenfunctions are

$$f_n = \sin(\mu_n x/L) + \frac{\mu_n}{\mathrm{Bi}_1}\,\cos(\mu_n x/L), \tag{32}$$

where μ_n are the numerically determined roots to the transcendental equation

$$(\mu^2 - \mathrm{Bi}_1\mathrm{Bi}_2)\sin\mu - (\mathrm{Bi}_1 + \mathrm{Bi}_2)\mu\cos\mu = 0. \tag{33}$$

The functions $f_n(x)$ could also be defined by $f_n = \cos(\mu_n x/L) + (\mathrm{Bi}_1/\mu_n)\sin(\mu_n x/L)$, but we shall proceed with their representation (32).

The eigenfunctions $f_n(x)$ are orthogonal to each other on the interval $[0, L]$, in the sense that

$$\int_0^L f_n(x) f_m(x)\,\mathrm{d}x = 0, \quad n \neq m. \tag{34}$$

The classical proof of this orthogonality is identical to that presented in the introductory Section 1 because the boundary conditions (31) ensure that

$$f_n(L)f_m'(L) - f_m(L)f_n'(L) = 0, \quad f_n(0)f_m'(0) - f_m(0)f_n'(0) = 0. \tag{35}$$

3.1. Verification of orthogonality

In contrast to symmetric convective boundary conditions from Section 2, the verification of orthogonality of eigenfunctions by direct integration in the case of nonsymmetric convective boundary conditions becomes quite tedious, albeit still straightforward. The product of the eigenfunctions f_n and f_m is, from (32),

$$\begin{aligned} f_n f_m &= \sin(\mu_n x/L)\sin(\mu_m x/L) + \frac{\mu_n \mu_m}{\mathrm{Bi}_1^2}\cos(\mu_n x/L)\cos(\mu_m x/L) \\ &\quad + \frac{\mu_n}{\mathrm{Bi}_1}\cos(\mu_n x/L)\sin(\mu_m x/L) + \frac{\mu_m}{\mathrm{Bi}_1}\sin(\mu_n x/L)\cos(\mu_m x/L). \end{aligned} \tag{36}$$

Upon integration, and after using in the resulting expressions, the eigenvalue condition (33) to eliminate $\cos\mu_n$ and $\cos\mu_m$ in terms of $\sin\mu_n$ and $\sin\mu_m$, it follows that

$$\int_0^L \sin(\mu_n x/L)\sin(\mu_m x/L)\,\mathrm{d}x = -\frac{L}{\mathrm{Bi}_1 + \mathrm{Bi}_2}\sin\mu_n \sin\mu_m, \tag{37}$$

$$\frac{\mu_n \mu_m}{\mathrm{Bi}_1^2}\int_0^L \cos(\mu_n x/L)\cos(\mu_m x/L)\,\mathrm{d}x = -\frac{\mathrm{Bi}_2}{\mathrm{Bi}_1}\frac{L}{\mathrm{Bi}_1 + \mathrm{Bi}_2}\sin\mu_n \sin\mu_m, \tag{38}$$

$$\begin{aligned} &\int_0^L \left[\frac{\mu_n}{\mathrm{Bi}_1}\cos(\mu_n x/L)\sin(\mu_m x/L) + \frac{\mu_m}{\mathrm{Bi}_1}\sin(\mu_n x/L)\cos(\mu_m x/L)\right]\mathrm{d}x \\ &= \frac{L}{\mathrm{Bi}_1}\sin\mu_n \sin\mu_m. \end{aligned} \tag{39}$$

The addition of (37)–(39) establishes the orthogonality

$$\int_{-L}^{L} f_n f_m\,\mathrm{d}x = 0, \quad f_n = \sin(\mu_n x/L) + \frac{\mu_n}{\mathrm{Bi}_1}\cos(\mu_n x/L), \; n \neq m. \tag{40}$$

4. Longitudinal vibrations of an elastic rod

Although we have discussed in previous sections the verification of orthogonality in the context of heat conduction, the presented analysis can also be applied to other engineering problems. For example, the same Robin-type boundary conditions, as in (31), appear in the problem of longitudinal vibrations of an elastic rod of length L and cross-sectional area A, whose two ends are attached to linear elastic springs with spring constants k_1 and k_2 as shown in Figure 2(a). The governing wave-type partial differential equation for the longitudinal displacement $u = u(x,t)$ is $\partial^2 u/\partial t^2 = v^2(\partial^2 u/\partial x^2)$, where $v = (E/\rho)^{1/2}$ is the wave speed, expressed in terms of the elastic modulus E of the rod and its mass density ρ (Weaver Jr. et al., 1991; Rao, 2011). The accompanying boundary conditions are

$$\left(\frac{\partial u}{\partial x} - \frac{k_1}{EA}u\right)_{x=0} = 0, \quad \left(\frac{\partial u}{\partial x} + \frac{k_2}{EA}u\right)_{x=L} = 0, \tag{41}$$

which are analogous to (29). By writing $u(x,t) = f(x)g(t)$, (41) reduces to homogeneous Robin-type boundary conditions for $f(x)$,

$$f'(0) - \frac{k_1}{EA}f(0) = 0, \quad f'(L) + \frac{k_2}{EA}f(L) = 0. \tag{42}$$

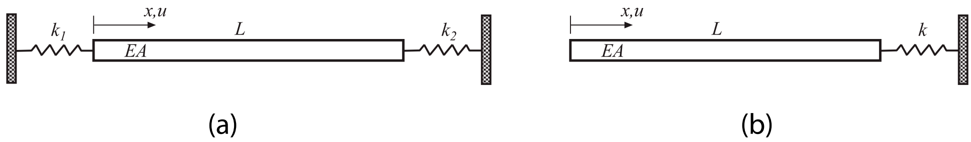

Figure 2. (a) A vibrating elastic rod of elastic stiffness $c = EA/L$, attached at its two ends to elastic springs with spring constants k_1 and k_2. (b) A vibrating elastic rod attached at its right end to elastic spring with spring constant k, while its left end is traction-free. The longitudinal displacement is $u = u(x,t)$.

The corresponding eigenfunctions are

$$f_n = \sin(\mu_n x/L) + \frac{c}{k_1}\,\mu_n \cos(\mu_n x/L), \tag{43}$$

with μ_n determined from the eigenvalue condition

$$\left(\mu^2 - \frac{k_1 k_2}{c^2}\right)\sin\mu - \frac{k_1 + k_2}{c}\mu\cos\mu = 0, \tag{44}$$

where $c = EA/L$ is the overall elastic stiffness of the rod. Equation (44) is an analogue to (33); the transition between the two is made by the replacement $\mathrm{Bi}_j \leftrightarrow k_j/c$ $(j = 1, 2)$. The vibrating rod shown in Figure 2(b) gives rise to eigenfunction boundary conditions $f'(0) = 0$ and $f'(L) + kf(L)/(EA) = 0$, and is thus an analogue to the heat conduction problem from Figure 1(a). The corresponding eigenfunctions are $f_n = \cos(\mu_n x/L)$, while the eigenvalue condition is $\mu \sin\mu = (k/c)\cos\mu$, in duality with (4). All the orthogonality considerations for the heat conduction problems from Sections 1–3 thus apply to the case of vibrating rods shown in Figure 2. If the initial conditions are $u(x,0) = u_0 = \text{const.}$ and $(\partial u/\partial t)_{t=0} = 0$, the longitudinal displacement is

$$u = \sum_{n=1}^{\infty} c_n f_n(x)\cos(\omega_n t), \quad c_n = \frac{u_0 \int_0^L f_n(x)\,\mathrm{d}x}{\int_0^L f_n^2(x)\,\mathrm{d}x}, \tag{45}$$

where $\omega_n = v(\mu_n/L)$ are the frequencies of the eigenmodes of longitudinal vibrations.

5. Advection-diffusion problems

The proofs from previous sections can be extended to advection-diffusion problems with appropriate boundary conditions of interest in various branches of chemical, nuclear, geological, environmental, and bioengineering (Grindrod, 1996; Masters & Ela, 2008). The concentration flux is one-dimensional advection-diffusion problem is defined by $q_x = UC - D\partial C/\partial x$, where U is the average velocity of fluid that convects the solute whose concentration is $C = C(x,t)$, and D is the diffusion coefficient. Thus, from the conservation of solute $\partial C/\partial t = -\partial q_x/\partial x$, it follows that the partial differential equation for the concentration C is

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - U\frac{\partial C}{\partial x}. \tag{46}$$

The initial condition is $C(x,0) = C_0 = \text{const.}$ If the entry flux of solute is constant and equal to UC_e, and if the exit concentration is assumed to be $C(L^+,t) \equiv C(L^-,t)$, where

L is the length of the advection domain, the boundary conditions are (Danckwerts, 1953; Van Genuchten & Parker, 1984)

$$\left(UC - D\,\frac{\partial C}{\partial x}\right)_{x=0} = UC_{\mathrm{e}}, \quad \left(\frac{\partial C}{\partial x}\right)_{x=L} = 0. \tag{47}$$

By introducing the auxiliary function ϕ, such that (Brenner, 1962)

$$C(x,t) = C_{\mathrm{e}} + (C_0 - C_{\mathrm{e}})\,\mathrm{e}^{\frac{Ux}{2D} - \frac{U^2 t}{4D}}\,\phi(x,t), \tag{48}$$

the advection-diffusion problem (46) is mathematically reduced to the diffusion problem

$$\frac{\partial \phi}{\partial t} = D\,\frac{\partial^2 \phi}{\partial x^2}, \tag{49}$$

with homogeneous boundary conditions

$$\left(U\phi - 2D\,\frac{\partial \phi}{\partial x}\right)_{x=0} = 0, \quad \left(U\phi + 2D\,\frac{\partial \phi}{\partial x}\right)_{x=L} = 0, \tag{50}$$

and the initial condition $\phi(x,0) = \mathrm{e}^{-\frac{Ux}{2D}}$. If ϕ is expressed as $\phi(x,t) = f(x)g(t)$, it readily follows that the eigenfunctions are

$$f_n = \sin(\mu_n x/L) + \frac{2\mu_n}{\mathrm{Pe}}\cos(\mu_n x/L), \quad \mathrm{Pe} = \frac{UL}{D} \quad \text{(Péclet number)}. \tag{51}$$

Physically, the Péclet number represents the ratio of the advective transport rate to the diffusive transport rate. The eigenvalues μ_n are the numerically determined roots to the transcendental equation

$$(4\mu^2 - \mathrm{Pe}^2)\sin\mu - 4\mathrm{Pe}\,\mu\cos\mu = 0, \quad \text{i.e.}\ \tan\mu = \frac{4\mathrm{Pe}\,\mu}{4\mu^2 - \mathrm{Pe}^2}. \tag{52}$$

The corresponding time-dependent functions are $g_n(t) = \mathrm{e}^{-D\mu_n^2 t/L^2}$. Because (51) and (52) are of the same type as expressions (32) and (33), the eigenfunctions f_n are clearly orthogonal on $[0,L]$.

The considered advection-diffusion problem can also be solved by using, instead of (48), the transformation

$$C(x,t) = C_{\mathrm{e}} + (C_0 - C_{\mathrm{e}})\varphi(x,t), \tag{53}$$

which gives rise to homogeneous boundary conditions for the auxiliary function φ. Upon using the separation of variables $\varphi(x,t) = F(x)G(t)$, the governing differential equation for the function $F = F(x)$ is $F'' - (U/D)F' + \lambda F = 0$. The corresponding eigenfunctions are $F_n(x) = \mathrm{e}^{\frac{Ux}{2D}} f_n(x)$ with $f_n(x)$ given by (51), while $G_n(t) = \mathrm{e}^{-\frac{U^2 t}{4D}} g_n(t)$ with $g_n(t) = \mathrm{e}^{-D\mu_n^2 t/L^2}$. The eigenfunctions $F_n(x)$ are orthogonal on $[0,L]$ with respect to the weight function $w(x) = \mathrm{e}^{-\frac{Ux}{D}}$, because the Sturm–Liouville form of the differential equation for F is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{Ux}{D}}\,\frac{\mathrm{d}F}{\mathrm{d}x}\right) + \lambda w(x)\,F = 0, \quad w(x) = \mathrm{e}^{-\frac{Ux}{D}}. \tag{54}$$

6. Conclusion

Direct proofs of orthogonality of eigenfunctions in problems of unsteady heat conduction in an infinite slab with convective boundary conditions are presented by performing the actual integration of the product of the derived eigenfunctions and by implementing the corresponding eigenvalue conditions. In the case of a symmetric initial condition and symmetric boundary conditions at the two sides of the slab, the proofs are simple and involve only elementary trigonometric transformations and elementary evaluation of integrals. In the case of nonsymmetric boundary conditions, the proofs are more lengthy, although still straightforward. The same proofs apply to problems of longitudinal vibrations of an elastic rod attached at its ends to linear elastic springs and to one-dimensional advection-diffusion problems under appropriate boundary conditions. The same form of eigenfunctions and the same type of eigenvalue condition apply in all three considered cases. The presented analysis is particularly appealing from the pedagogical point of view because students of engineering mathematics and applied physics are often tempted to verify the orthogonality independently of the general proof, by performing the actual integration of the product of the derived eigenfunctions and by incorporating the corresponding eigenvalue conditions. From our experience, such exercise is instructive because it provides to students an opportunity to apply their basic trigonometry and integral calculus skills. In addition, the numerical determination of the first ten or so roots of nonlinear Equations (4), (33), (44), and (52) by writing independent codes, or by using built-in MATLAB or Python functions, in conjunction with the analysis of the rate of convergence of the derived series solutions such as (6) and (31), provides an opportunity to apply numerical and computational skills which complement students' analytical skills.

Disclosure statement

No potential conflict of interest was reported by the authors.

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