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On the recoverable and dissipative parts of higher order stresses in strain gradient plasticity

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ABSTRACT

The expressions for the free energy in two recent formulations of strain gradient plasticity are extended to include the locked-in strain energy around statistically stored dislocations. This is accomplished by using the strain dependent factor $\eta(e_{\rm p})$, which represents the fraction of the rate of plastic work converted into heat in accordance with the latent heat measurements from classical metal plasticity. The expressions for the plastic work in the two formulations differ by different representations of the portion of plastic work associated with the existence of plastic strain gradients and the corresponding network of geometrically necessary dislocations, while the dissipative parts of plastic work are assumed to be the same in both formulations. The expressions for the recoverable and dissipative parts of the higher order stresses, defined as the work-conjugates to plastic strain and its gradient, are then derived. It is shown that the stress and strain fields of isothermal boundary-value problems of strain gradient plasticity are independent of η, but that this factor may be of importance for non-isothermal analysis in which the dissipated plastic work acts as an internal heat source. The effects of plastic strain gradient on the plastic response of twisted hollow circular tubes made of a rigid-plastic material with different hardening properties are then evaluated and discussed.

1. Introduction

In classical plasticity there is no material length scale in the framework of the constitutive theory, so that this theory cannot predict the size effects experimentally observed in plastic deformation problems at the micron scale, as in the bending and torsion testing of very thin beams and wires, inelastic response of nanograined materials, dispersion strengthening by small particles, measurements of micro-indentation hardness, thin film applications, micro-imprinting processes, etc. (Fleck et al., 1994; Nix and Gao, 1998; Stölken and Evans, 1998; Qiu et al., 2003; Keller et al., 2011; Ma et al., 2012; Liu et al., 2013; Nielsen et al., 2014). In general, the observed trend is that smaller is stronger. This size-dependent strengthening has been attributed to the effects of strain gradients on plastic deformation. The theory which includes these effects has been put forward by Aifantis (1984), Mühlhaus and Aifantis (1991), Fleck and Hutchinson (1993, 1997,2001), and Gao et al. (1999), with subsequent developments by many investigators, including, inter alia, Huang et al. (2000, 2004), Hutchinson (2000, 2012), Gurtin (2002, 2003,2004), Gudmundson (2004), Anand et al. (2005), Gurtin and Anand (2005a,2005b,2009), Bardella (2006, 2007), Fleck and Willis (2009a,2009b), Polizzotto (2009), Voyiadjis et al. (2010), Voyiadjis and Faghihi (2012),

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Dahlberg et al. (2013), Nielsen and Niordson (2014), Mayeur and McDowell (2014), Fleck et al. (2014, 2015), Bardella and Panteghini (2015), and Anand et al. (2015). From the dislocation point of view, the gradient of plastic strain is associated with the storage of geometrically necessary dislocations, while the uniform strain is associated with random trapping and storage of statistically stored dislocations (Ashby, 1970; Fleck et al., 1994; Nix and Gao, 1998; Kysar et al., 2010; Öztop et al., 2013).

In the present paper we extend the strain gradient plasticity analysis of Hutchinson (2012), and Fleck et al. (2014) to include in their expressions for the free energy the locked-in strain energy around statistically stored dislocations. The strain dependent factor $\eta(e_{\rm p})$ is used to represent the fraction of the rate of plastic work converted into heat in accordance with the latent heat measurements from classical metal plasticity. The utilized expressions for the plastic work differ by the different representations of the portion of plastic work associated with the existence of plastic strain gradients and the corresponding network of geometrically necessary dislocations, while the dissipative parts of plastic work are assumed to be the same in both formulations. The expressions for the recoverable and dissipative parts of the work-conjugates to plastic strain and its gradient are derived in each case. It is shown that the stress and strain fields of isothermal boundary-value problems of strain gradient plasticity are independent of η, but that this factor may be of importance for non-isothermal analysis in which the dissipated plastic work acts as an internal heat source. The effects of plastic strain gradient on the plastic response of twisted hollow circular tubes made of a rigid-plastic material are then evaluated and discussed. The shear stress, the edge line forces, and the applied torque are determined for various values of the material length parameter. Results for solid rods, hollow and thin-walled tubes are given at the onset and beyond plastic yield for linear and nonlinear hardening.

As in Hutchinson (2012) and Fleck et al. (2014), the presented analysis is phenomenological, without explicit referral to specific dislocation mechanisms and interactions among individual dislocations. The latter are considered in the discrete dislocation dynamics and dislocation based plasticity theory at submicron scales, e.g., Devincre and Kubin (1997), Tadmor et al. (1999), Needleman (2000), Zbib et al. (2002), Bittencourt et al. (2003), Senger et al. (2011), Taheri-Nassaj and Zbib (2015), and Wulfinghoff and Böhlke (2015).

2. Gradient-enhanced effective plastic strain

It is assumed that the elastoplastic rate of strain is the sum of elastic and plastic contributions, such that $\dot\varepsilon_{ij} = \dot\varepsilon^{\rm e}_{ij} + \dot\varepsilon^{\rm p}_{ij}$. The elastic part of the strain rate depends on the rate of the Cauchy stress (σ_{ij}) according to the generalized Hooke's law. The plastic part of the strain rate is assumed to be codirectional with the deviatoric part of stress (σ'_{ij}), as in the classical J_2 flow theory of plasticity,

$$\dot\varepsilon^{\rm p}_{ij} = \dot e_{\rm p} m_{ij}\,, \quad m_{ij} = \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_{\rm eq}}. \tag{1}$$

The equivalent stress is $\sigma_{\rm eq} = [(3/2)\sigma'_{ij}\sigma'_{ij}]^{1/2}$, while the loading index satisfies

$$\dot e_{\rm p} = \left(\frac{2}{3}\dot\varepsilon^{\rm p}_{ij}\dot\varepsilon^{\rm p}_{ij}\right)^{1/2}. \tag{2}$$

Its path-dependent integral over the history of deformation gives the effective plastic strain $e_{\rm p}$. The spatial gradient of $e_{\rm p}$ will be used as a cumulative measure of plastic strain gradients, so that

$$e_{\rm p} = \int_0^t \dot e_{\rm p}\,{\rm d}t\,, \quad e_{{\rm p},k} = \int_0^t \dot e_{{\rm p},k}\,{\rm d}t. \tag{3}$$

In the strain gradient plasticity, a gradient-enhanced effective plastic strain can be defined by Hutchinson (2012)

$$E_{\rm p} = \left(e_{\rm p}^2 + l^2 e_{{\rm p},k}e_{{\rm p},k}\right)^{1/2}, \tag{4}$$

where l is the material length scale of the specific problem at hand, introduced in (4) on the dimensional ground. While $e_{\rm p}$ is a monotonically increasing measure of plastic strain during the course of plastic deformation, $e_{{\rm p},k}$ is not necessarily increasing because $\dot e_{{\rm p},k}$ can be negative for certain non-proportional strainings, so that the gradient-enhanced plastic strain $E_{\rm p}$ is not necessarily an increasing measure of strain either (i.e., $\dot E_{\rm p}$ could be negative).

In the classical J_2 flow theory of plasticity, the rate of plastic work (per unit volume) is $\dot w_{\rm p} = \sigma'_{ij}\dot\varepsilon^{\rm p}_{ij} \equiv \sigma_0(e_{\rm p})\dot e_{\rm p}$, where $\sigma_0 = \sigma_0(e_{\rm p})$ is the stress-plastic strain curve in uniaxial simple tension test, and $\sigma_{\rm eq}=\sigma_0(e_{\rm p})$ is the yield condition. In the strain gradient plasticity it has been proposed (Hutchinson, 2012) that the specific plastic work is

$$w_{\rm p}(E_{\rm p}) = \int_0^{E_{\rm p}} \sigma_0(\epsilon_{\rm p})\,{\rm d}\epsilon_{\rm p}, \tag{5}$$

where the gradient-enhanced cumulative plastic strain $E_{\rm p}$ is defined by (4). The symbol ϵ is used in (5) and elsewhere in this paper as a dummy variable of integration. The expression (5) implies that the plastic work to deform a material element to the strain level $e_{\rm p}$ in the presence of plastic strain gradients is greater than in the absence of these gradients (because $E_{\rm p} > e_{\rm p}$). Indeed, if (5) is rewritten as

$$w_{\rm p}(E_{\rm p}) = \int_0^{e_{\rm p}} \sigma_0(\epsilon_{\rm p})\,{\rm d}\epsilon_{\rm p} + \int_{e_{\rm p}}^{E_{\rm p}} \sigma_0(\epsilon_{\rm p})\,{\rm d}\epsilon_{\rm p}, \tag{6}$$

the second integral on the right-hand side represents the strain-gradient contribution to plastic work. It will be assumed that this portion of plastic work is, in principle, entirely recoverable, being associated with the current network of geometrically necessary dislocations and the elastic strain energy around them. The rate of plastic work associated with (5) is $\dot{w}_{\rm p} = \sigma_0(E_{\rm p})\dot{E}_{\rm p}$, which is positive or negative depending on the sign of $\dot{E}_{\rm p}$.

The elastic portion of the work, associated with the infinitesimal elastic strain $\varepsilon_{ij}^{\rm e}$, is

$$w_{\rm e}\left(\varepsilon_{ij}^{\rm e}\right) = \mu\varepsilon_{ij}^{\rm e\prime}\varepsilon_{ij}^{\rm e\prime} + \frac{1}{2}\kappa\varepsilon_{kk}^{\rm e\;2}\,, \quad \varepsilon_{ij}^{\rm e\prime} = \varepsilon_{ij}^{\rm e} - \frac{1}{3}\varepsilon_{kk}^{\rm e}\delta_{ij}\,, \tag{7}$$

where μ and κ are the elastic shear and bulk moduli of an isotropic material, respectively. The total work per unit volume is $w = w_{\rm e} + w_{\rm p}$. If the plastic part of strain $\varepsilon_{ij}^{\rm p}$ is obtained from $\dot{\varepsilon}_{ij}^{\rm p}$ by integration along a specified history of deformation, the elastic part of the total strain ε_{ij} follows from $\varepsilon_{ij}^{\rm e} = \varepsilon_{ij} - \varepsilon_{ij}^{\rm p}$.

2.1. An alternative definition of the gradient-enhanced effective plastic strain

The path-dependent integral in (3) over the history of deformation gives the effective plastic strain $e_{\rm p}$. Another measure of the effective plastic strain, defined in terms of the current components of plastic part of strain $\varepsilon_{ij}^{\rm p} = \varepsilon_{ij} - \varepsilon_{ij}^{\rm e}$, regardless of the history of deformation by which they are produced, thus reflecting upon the current geometric structure only, is

$$\varepsilon_{\rm p} = \left(\frac{2}{3}\varepsilon_{ij}^{\rm p}\varepsilon_{ij}^{\rm p}\right)^{1/2}. \tag{8}$$

While $e_{\rm p}$ is non-decreasing, $\varepsilon_{\rm p}$ can increase or decrease during the course of nonproportional plastic straining. Guided by the anticipated physical relationship between the plastic strain gradients and the network of geometrically necessary dislocations which accompany them, Fleck et al. (2014) introduced the gradient-enhanced effective plastic strain, which also reflects only upon the current geometric structure, by

$$\mathbb{E}_{\rm p} = \left(\frac{2}{3}\varepsilon_{ij}^{\rm p}\varepsilon_{ij}^{\rm p} + \frac{2}{3}l^2\varepsilon_{ij,k}^{\rm p}\varepsilon_{ij,k}^{\rm p}\right)^{1/2}. \tag{9}$$

With the cumulative path-dependent plastic strain $e_{\rm p}$, as in (3), and the path-independent strain and strain gradient measures $\varepsilon_{\rm p}$ and $\mathbb{E}_{\rm p}$, as in (8) and (9), Fleck et al. (2014) proposed the following expression for the specific plastic work

$$w_{\rm p}(e_{\rm p},\varepsilon_{\rm p},\mathbb{E}_{\rm p}) = \int_0^{e_{\rm p}} \sigma_0(\epsilon_{\rm p})\,{\rm d}\epsilon_{\rm p} + \int_0^{\mathbb{E}_{\rm p}} \sigma_0(\epsilon_{\rm p})\,{\rm d}\epsilon_{\rm p} - \int_0^{\varepsilon_{\rm p}} \sigma_0(\epsilon_{\rm p})\,{\rm d}\epsilon_{\rm p}, \tag{10}$$

where $\sigma_0 = \sigma_0(e_{\rm p})$ is again obtained from the stress-plastic strain curve in simple tension test. The last two terms together are intended to represent the plastic work contribution associated with the plastic strain gradients, incorporated in the definition of $\mathbb{E}_{\rm p}$ through the length scale l, while the first term on the right-hand side of (10) represents the path-dependent portion of plastic work, corresponding to the current effective plastic strain $e_{\rm p}$ and a specified path of deformation by which it is reached. In the absence of plastic strain gradient effects ($l = 0$), the last two integrals cancel each other and (10) reduces to the plastic work expression of classical plasticity. In the case of proportional plastic straining, $\varepsilon_{\rm p} = e_{\rm p}$ and the first and third integral on the right-hand side of (10) cancel each other.

If expression (10) is rewritten as

$$w_{\mathrm{p}}(e_{\mathrm{p}}, \varepsilon_{\mathrm{p}}, \mathbb{E}_{\mathrm{p}}) = \int_0^{e_{\mathrm{p}}} \sigma_0(\epsilon_{\mathrm{p}})\, \mathrm{d}\epsilon_{\mathrm{p}} + \int_{\varepsilon_{\mathrm{p}}}^{\mathbb{E}_{\mathrm{p}}} \sigma_0(\epsilon_{\mathrm{p}})\, \mathrm{d}\epsilon_{\mathrm{p}}, \tag{11}$$

the first integral on the right-hand side can be interpreted as the work spent on the evolution of statistically stored dislocations from the initial to the current state, while the second integral represents the work associated with the creation of the network of geometrically necessary dislocations giving rise to plastic strain gradients in the current state. By comparing (6) and (11), it is seen that the two plastic work expressions differ by the different integration limits used in the second integral on the right-hand sides accounting for the plastic work contribution from the geometrically necessary dislocations.

2.2. Other strain measures

Other measures of the gradient-enhanced effective plastic strain could be considered, such as $\Xi_{\mathrm{p}} = e_{\mathrm{p}} + l(e_{\mathrm{p},k}e_{\mathrm{p},k})^{1/2}$ (Evans and Hutchinson, 2009; Niordson and Hutchinson, 2011). The strain gradient theories with up to three length scales have also been used (Fleck and Hutchinson, 1997, 2001; Wei and Hutchinson, 2003; Lele and Anand, 2009; Fleck and Willis, 2009a,b; Danas et al., 2012), as well as the strain-dependent length scales (Evans and Hutchinson, 2009). Fleck and Hutchinson (2001) suggested that one length parameter is needed to characterize problems in which shearing gradients are dominant, and the other with dominant stretch gradients. Gurtin (2004) introduced an effective distortion-rate which involves the plastic spin, in addition to the plastic strain rate and its gradient. See also the distortion gradient plasticity analysis of the torsion of thin metal wires by Bardella and Panteghini (2015). For the geometrical and physical foundations of classical and higher-order plasticity models, with a historical perspective, recent books by Gurtin et al. (2010), Clayton (2011), and Steinmann (2015) can also be consulted.

3. Work-conjugates to plastic strain and its gradient

The rate of work is the sum of elastic and plastic parts, $\dot{w} = \dot{w}_{\mathrm{e}} + \dot{w}_{\mathrm{p}}$. The elastic part is obtained from (7) as

$$\dot{w}_{\mathrm{e}} = \sigma_{ij}\dot{\varepsilon}^{\mathrm{e}}_{ij}\,, \quad \sigma_{ij} = 2\mu\varepsilon^{\mathrm{e}\prime}_{ij} + \kappa\varepsilon^{\mathrm{e}}_{kk}\delta_{ij}. \tag{12}$$

The expression for the plastic part is derived by differentiation of (5), which gives

$$\dot{w}_{\mathrm{p}} = \sigma_0(E_{\mathrm{p}})\dot{E}_{\mathrm{p}}\,, \quad \dot{E}_{\mathrm{p}} = \frac{e_{\mathrm{p}}}{E_{\mathrm{p}}}\dot{e}_{\mathrm{p}} + l^2\frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}}\dot{e}_{\mathrm{p},k}. \tag{13}$$

This can be expressed as

$$\dot{w}_{\mathrm{p}} = \overline{q}\dot{e}_{\mathrm{p}} + \overline{\tau}_k\dot{e}_{\mathrm{p},k}\,, \tag{14}$$

where the quantities $\overline{q}$ and $\overline{\tau}_k$ are the work-conjugates to plastic strain and strain gradient measures e_{p} and $e_{\mathrm{p},k}$. By comparing (13) and (14), these are (Hutchinson, 2012)

$$\overline{q} = \sigma_0(E_{\mathrm{p}})\frac{e_{\mathrm{p}}}{E_{\mathrm{p}}}\,, \quad \overline{\tau}_k = l^2\sigma_0(E_{\mathrm{p}})\frac{e_{\mathrm{p},k}}{E_{\mathrm{p}}}. \tag{15}$$

The two quantities are related by $\overline{q}^2 + l^{-2}\overline{\tau}_k\overline{\tau}_k = \sigma_0^2(E_{\mathrm{p}})$.

The work-conjugates to plastic strain $\varepsilon^{\mathrm{p}}_{ij}$ and its gradient $\varepsilon^{\mathrm{p}}_{ij,k}$, denoted by q_{ij} and τ_{ijk}, are defined such that

$$\dot{w}_{\mathrm{p}} = q_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij} + \tau_{ijk}\dot{\varepsilon}^{\mathrm{p}}_{ij,k}. \tag{16}$$

They can be identified by comparing (16) with the rate of plastic work expression (13), in which $\dot{e}_{\mathrm{p}} = (2/3)m_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij}$ and $\dot{e}_{\mathrm{p},k} = (2/3)m_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij,k}$ have been used in the expression for $\dot{E}_{\mathrm{p}}$. This gives

$$\dot{w}_{\mathrm{p}} = \frac{2}{3}\frac{\sigma_0(E_{\mathrm{p}})}{E_{\mathrm{p}}}\left(e_{\mathrm{p}}m_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij} + l^2e_{\mathrm{p},k}m_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij,k}\right). \tag{17}$$

Thus, the comparison of (16) and (17) identifies, up to their workless parts, the work-conjugates

$$q_{ij} = \frac{2}{3}\frac{\sigma_0(E_{\mathrm{p}})}{E_{\mathrm{p}}}e_{\mathrm{p}}m_{ij}, \quad \tau_{ijk} = \frac{2}{3}l^2\frac{\sigma_0(E_{\mathrm{p}})}{E_{\mathrm{p}}}e_{\mathrm{p},k}m_{ij}. \tag{18}$$

The comparison of (15) and (18) establishes the relationships

$$\overline{q} = q_{ij}m_{ij}, \quad \overline{\tau}_k = \tau_{ijk}m_{ij}. \tag{19}$$

In the case of classical plasticity, $q_{ij} = \sigma'_{ij}$ and $\overline{q} = \sigma_{\rm eq}$, while τ_{ijk} and $\overline{\tau}_k$ both vanish.

3.1. Work-conjugates in the case of alternative measures of effective plastic strain

In the case of effective plastic strain measures $e_{\rm p}$, $\varepsilon_{\rm p}$, and $\mathbb{E}_{\rm p}$ from Section 2.1, the rate of plastic work is obtained by differentiation of (10), which gives

$$\dot{w}_{\rm p} = \sigma_0(e_{\rm p})\dot{e}_{\rm p} + \sigma_0(\mathbb{E}_{\rm p})\dot{\mathbb{E}}_{\rm p} - \sigma_0(\varepsilon_{\rm p})\dot{\varepsilon}_{\rm p}. \tag{20}$$

While $\dot{e}_{\rm p}$ is necessarily positive during plastic deformation, $\dot{\varepsilon}_{\rm p}$ and $\dot{\mathbb{E}}_{\rm p}$ can be, in general, either positive or negative. It can be readily shown by differentiating (8) and (9) that

$$\varepsilon_{\rm p}\dot{\varepsilon}_{\rm p} = \left(\frac{2}{3}m_{ij}\varepsilon^{\rm p}_{ij}\right)\dot{e}_{\rm p}, \quad \mathbb{E}_{\rm p}\dot{\mathbb{E}}_{\rm p} = \varepsilon_{\rm p}\dot{\varepsilon}_{\rm p} + \frac{2}{3}l^2\varepsilon^{\rm p}_{ij,k}\dot{\varepsilon}^{\rm p}_{ij,k}. \tag{21}$$

Since the gradient of the rate of plastic strain can be evaluated from (1) as $\dot{\varepsilon}^{\rm p}_{ij,k} = m_{ij}\dot{e}_{{\rm p},k} + m_{ij,k}\dot{e}_{\rm p}$, the expression for the rate $\dot{\mathbb{E}}_{\rm p}$ in (21) becomes

$$\dot{\mathbb{E}}_{\rm p} = \frac{2}{3E_{\rm p}}\left[\left(m_{ij}\varepsilon^{\rm p}_{ij} + l^2 m_{ij,k}\varepsilon^{\rm p}_{ij,k}\right)\dot{e}_{\rm p} + \left(l^2 m_{ij}\varepsilon^{\rm p}_{ij,k}\right)\dot{e}_{{\rm p},k}\right]. \tag{22}$$

Consequently, by substituting (22) into (20), the rate of plastic work can be expressed as

$$\dot{w}_{\rm p} = \overline{q}\dot{e}_{\rm p} + \overline{\tau}_k\dot{e}_{{\rm p},k}, \tag{23}$$

where the work-conjugates to plastic strain and strain gradient measures $e_{\rm p}$ and $e_{{\rm p},k}$ are

$$\overline{q} = \sigma_0(e_{\rm p}) + \frac{2}{3}\left[\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}} - \frac{\sigma_0(\varepsilon_{\rm p})}{\varepsilon_{\rm p}}\right]m_{ij}\varepsilon^{\rm p}_{ij} + \frac{2}{3}l^2\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}m_{ij,k}\varepsilon^{\rm p}_{ij,k}, \tag{24}$$

$$\overline{\tau}_k = \frac{2}{3}l^2\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}m_{ij}\varepsilon^{\rm p}_{ij,k}. \tag{25}$$

The work-conjugates to plastic strain $\varepsilon^{\rm p}_{ij}$ and its gradient $\varepsilon^{\rm p}_{ij,k}$, appearing in the representation (16), can be identified as well. Since, from (2), (8), and (9),

$$\dot{e}_{\rm p} = \frac{2}{3}m_{ij}\dot{\varepsilon}^{\rm p}_{ij}, \quad \dot{\varepsilon}_{\rm p} = \frac{2}{3\varepsilon_{\rm p}}\varepsilon_{ij}\dot{\varepsilon}^{\rm p}_{ij}, \quad \dot{\mathbb{E}}_{\rm p} = \frac{2}{3\mathbb{E}_{\rm p}}\left(\varepsilon_{ij}\dot{\varepsilon}^{\rm p}_{ij} + l^2\varepsilon_{ij,k}\dot{\varepsilon}^{\rm p}_{ij,k}\right), \tag{26}$$

the substitution of (26) into (20) specifies, up to their workless parts, the work conjugates (Fleck et al., 2014)

$$q_{ij} = \frac{2}{3}\sigma_0(e_{\rm p})m_{ij} + \frac{2}{3}\left[\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}} - \frac{\sigma_0(\varepsilon_{\rm p})}{\varepsilon_{\rm p}}\right]\varepsilon^{\rm p}_{ij}, \quad \tau_{ijk} = \frac{2}{3}l^2\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}\varepsilon^{\rm p}_{ij,k}. \tag{27}$$

The comparison of (27) with (24) and (25) establishes the connections

$$\overline{q} = q_{ij}m_{ij} + \tau_{ijk}m_{ij,k}, \quad \overline{\tau}_k = \tau_{ijk}m_{ij}. \tag{28}$$

4. Principle of virtual work

In the absence of body forces, the principal of virtual work for the quasi-static strain gradient plasticity reads (Gudmundson, 2004; Gurtin and Anand, 2005a,b; Fleck et al., 2014)

$$\int_V \left(\sigma_{ij}\delta\varepsilon^{\rm e}_{ij} + q_{ij}\delta\varepsilon^{\rm p}_{ij} + \tau_{ijk}\delta\varepsilon^{\rm p}_{ij,k}\right){\rm d}V = \int_S \left(T_i\delta u_i + t_{ij}\delta\varepsilon^{\rm p}_{ij}\right){\rm d}S. \tag{29}$$

The Gauss divergence theorem applied to (29) yields the equations of equilibrium

$$\sigma_{ij,j} = 0, \quad \tau_{ijk,k} + \sigma'_{ij} - q_{ij} = 0, \tag{30}$$

and the relations $T_i = \sigma_{ij} n_j$ and $t_{ij} = \tau_{ijk} n_k$ between the traction vector T_i and the Cauchy stress tensor σ_{ij}, and between the (deviatoric) moment traction tensor t_{ij} are the moment stress tensor τ_{ijk}. The components of the outward unit vector, orthogonal to the considered surface element, are denoted by n_i. The displacement components are u_i.

The principle of virtual work can also be expressed in terms of $\delta e_{\rm p}$ and $\delta e_{{\rm p},k}$. Since $\delta\epsilon_{ij}^{\rm p} = \delta e_{\rm p} m_{ij}$ and $\sigma_{\rm eq} = \sigma'_{ij} m_{ij}$, (29) can be recast as (Fleck and Hutchinson, 2001; Hutchinson, 2012)

$$\int\limits_V \left(\sigma_{ij}\delta\epsilon_{ij}^{\rm e} + \bar{q}\delta e_{\rm p} + \bar{\tau}_k \delta e_{{\rm p},k}\right) {\rm d}V = \int\limits_S (T_i \delta u_i + \bar{t}\delta e_{\rm p}) {\rm d}S, \tag{31}$$

where

$$\bar{t} = t_{ij} m_{ij}, \quad \bar{\tau}_k = \tau_{ijk} m_{ij}, \quad \bar{q} = q_{ij} m_{ij} + \tau_{ijk} m_{ij,k}, \tag{32}$$

provided that the equilibrium conditions hold

$$\sigma_{ij,j} = 0, \quad \bar{\tau}_{k,k} + \sigma_{\rm eq} - \bar{q} = 0, \tag{33}$$

together with $T_i = \sigma_{ij} n_j$ and $\bar{t} = \bar{\tau}_k n_k$. The last two expressions in (32) confirm (28). For the formulation of the principle of virtual work in the case of a rigid-plastic material model in strain gradient plasticity, the paper by Fleck and Willis (2009b) can be consulted.

5. Free energy

By considering $\dot{e}_{\rm p}$ and $\dot{e}_{{\rm p},k}$ to be the fluxes whose conjugate thermodynamic forces (affinities) are denoted by f and g_k, the rate of internal energy dissipation due to inelastic deformation processes can be expressed as

$$\mathscr{D} = \dot{w}_{\rm p}^{\rm dis} = f\dot{e}_{\rm p} + g_k \dot{e}_{{\rm p},k} > 0. \tag{34}$$

The rate of the Helmholtz free energy under isothermal conditions is then $\dot{\psi} = \dot{w} - \mathscr{D}$. It will be assumed that the dissipative part of the rate of plastic work is

$$\dot{w}_{\rm p}^{\rm dis} = \eta(e_{\rm p})\sigma_0(e_{\rm p})\dot{e}_{\rm p}, \quad 0.9 \le \eta(e_{\rm p}) \le 1. \tag{35}$$

The factor $\eta(e_{\rm p})$ specifies the fraction of the rate of plastic work converted into heat. This factor is assumed to be in the indicated range from 0.9 to 1, based on the early experimental work by Farren and Taylor (1925) and Taylor and Quinney (1934) from classical metal plasticity (without strain gradient effects). The functional dependence $\eta = \eta(e_{\rm p})$ has been subsequently studied by many investigators, some of which work has been reviewed by Bever et al. (1973). More recently, Rosakis et al. (2000) examined, analytically and experimentally, the dependence of η on plastic strain, strain-rate, and the history of plastic deformation for various materials, which is particularly important for determining thermomechanical response under high-rate of deformation.

By comparing (34) and (35), the affinities are

$$f(e_{\rm p}) = \eta(e_{\rm p})\sigma_0(e_{\rm p}), \quad g_k = 0. \tag{36}$$

In the early stage of plastic deformation for which the quasistatic, small strain analysis of this paper is intended to apply, η may be approximately taken to be 0.9. The dissipated plastic work from the initial to the current state is

$$w_{\rm p}^{\rm dis} = \int\limits_0^{e_{\rm p}} \eta(\epsilon_{\rm p})\sigma_0(\epsilon_{\rm p}){\rm d}\epsilon_{\rm p}. \tag{37}$$

The non-dissipative (conceptually recoverable) part of the rate of plastic work, associated with the locked-in energy around both statistically stored and geometrically necessary dislocations, is the plastic contribution to the rate of free energy ($\dot{\psi}_{\rm p}$). In view of (13) and (35), this is

$$\dot{\psi}_{\rm p} = \dot{w}_{\rm p}^{\rm rec} = \dot{w}_{\rm p} - \dot{w}_{\rm p}^{\rm dis} = \sigma_0(E_{\rm p})\dot{E}_{\rm p} - \eta(e_{\rm p})\sigma_0(e_{\rm p})\dot{e}_{\rm p}, \tag{38}$$

which, in general, can be either positive or negative (certainly negative if $\dot{E}_{\rm p} < 0$). The negative value of the increment ${\rm d}\psi_{\rm p}$ means that in an increment of plastic deformation the released strain energy associated with plastic gradients from geometrically necessary dislocations exceeds the stored strain energy around newly created or displaced statistically stored dislocations. In the case of classical plasticity $\dot{\psi}_{\rm p} = [1 - \eta(e_{\rm p})]\sigma_0(e_{\rm p})\dot{e}_{\rm p} \ge 0$.

The overall plastic contribution to the free energy is

$$\psi_{\mathrm{p}}(e_{\mathrm{p}},E_{\mathrm{p}})=w_{\mathrm{p}}^{\mathrm{rec}}(e_{\mathrm{p}},E_{\mathrm{p}})=\int_0^{E_{\mathrm{p}}}\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}}-\int_0^{e_{\mathrm{p}}}\eta(\epsilon_{\mathrm{p}})\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}},\tag{39}$$

i.e.,

$$\psi_{\mathrm{p}}(e_{\mathrm{p}},E_{\mathrm{p}})=w_{\mathrm{p}}^{\mathrm{rec}}(e_{\mathrm{p}},E_{\mathrm{p}})=\int_0^{e_{\mathrm{p}}}[1-\eta(\epsilon_{\mathrm{p}})]\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}}+\int_{e_{\mathrm{p}}}^{E_{\mathrm{p}}}\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}},\tag{40}$$

such that $w_{\mathrm{p}}=w_{\mathrm{p}}^{\mathrm{rec}}+w_{\mathrm{p}}^{\mathrm{dis}}$. If it is assumed that all plastic work in the model of classical plasticity is dissipated (η=1), (39) reduces to the expression originally proposed by Hutchinson (2012). In that case, the entire free energy due to plastic deformation is associated with the plastic strain gradient effects.

In an isothermal boundary-value problem, η does not affect the stress and strain fields, because it does not appear in the structure of the higher-order stresses q_{ij} and τ_{ijk}, or $\overline{q}$ and $\overline{\tau}_k$, and the corresponding variational principles. This is similar to classical plasticity, without strain gradient effects, in which η does not affect isothermal stress and strain fields either. The use of $\eta=\eta(e_{\mathrm{p}})$ in (35) and (39) may, however, be of importance for non-isothermal elastoplastic analysis at large strains. The development of non-isothermal strain gradient plasticity theory, in which temperature dependence of free energy must be added to the theory, is a worthwhile goal of future investigation. Towards that goal, the thermomechanical analysis of non-isothermal elastic–plastic deformation from classical plasticity (e.g., Prager, 1958; Lee, 1969; Lubarda, 1982; Rosakis et al., 2000), appropriately modified to incorporate the plastic strain-gradient effects, such as that developed by Anand et al. (2015) within strain gradient crystal plasticity, can be of help.

The total free energy consists of the elastic strain energy associated with elastic strains $\varepsilon_{ij}^{\mathrm{e}}$, and the locked-in strain energy around statistically stored and geometrically necessary dislocations, so that

$$\psi=\psi_{\mathrm{e}}\left(\varepsilon_{ij}^{\mathrm{e}}\right)+\psi_{\mathrm{p}}(e_{\mathrm{p}},E_{\mathrm{p}}),\quad \psi_{\mathrm{e}}\left(\varepsilon_{ij}^{\mathrm{e}}\right)=\mu\varepsilon_{ij}^{\mathrm{e}\prime}\varepsilon_{ij}^{\mathrm{e}\prime}+\frac{1}{2}\kappa\varepsilon_{kk}^{\mathrm{e}\;2}.\tag{41}$$

5.1. *Free energy in the case of alternative measures of effective plastic strain*

The dissipative part of the rate of plastic work is again assumed to be given by (35), with the corresponding affinities as in (36). In view of this and the expression for the rate of plastic work (20), the plastic contribution to the rate of free energy is

$$\dot{\psi}_{\mathrm{p}}=\dot{w}_{\mathrm{p}}^{\mathrm{rec}}=\dot{w}_{\mathrm{p}}-\dot{w}_{\mathrm{p}}^{\mathrm{dis}}=[1-\eta(e_{\mathrm{p}})]\sigma_0(e_{\mathrm{p}})\dot{e}_{\mathrm{p}}+\sigma_0(\mathbb{E}_{\mathrm{p}})\dot{\mathbb{E}}_{\mathrm{p}}-\sigma_0(\varepsilon_{\mathrm{p}})\dot{\varepsilon}_{\mathrm{p}}.\tag{42}$$

The overall plastic contribution to the free energy is

$$\psi_{\mathrm{p}}(e_{\mathrm{p}},\varepsilon_{\mathrm{p}},\mathbb{E}_{\mathrm{p}})=\int_0^{e_{\mathrm{p}}}[1-\eta(\epsilon_{\mathrm{p}})]\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}}+\int_0^{\mathbb{E}_{\mathrm{p}}}\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}}-\int_0^{\varepsilon_{\mathrm{p}}}\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}}.\tag{43}$$

The first term on the right-hand side accounts for the locked-in strain energy around statistically stored dislocations, while the last two terms together represent the free energy contribution due to plastic strain gradient effects from the current network of geometrically necessary dislocations. If all plastic work in the model of classical plasticity is assumed to be dissipated ($\eta=1$), (43) reduces to the expression originally proposed by Fleck et al. (2014).

For the sake of comparison with (40), the expression for the plastic contribution to the free energy (43) is rewritten as

$$\psi_{\mathrm{p}}(e_{\mathrm{p}},\varepsilon_{\mathrm{p}},\mathbb{E}_{\mathrm{p}})=w_{\mathrm{p}}^{\mathrm{rec}}(e_{\mathrm{p}},\varepsilon_{\mathrm{p}},\mathbb{E}_{\mathrm{p}})=\int_0^{e_{\mathrm{p}}}[1-\eta(\epsilon_{\mathrm{p}})]\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}}+\int_{\varepsilon_{\mathrm{p}}}^{\mathbb{E}_{\mathrm{p}}}\sigma_0(\epsilon_{\mathrm{p}})\mathrm{d}\epsilon_{\mathrm{p}}.\tag{44}$$

Thus, the two expressions differ by the different integration limits used in the second integral on the right-hand side of (40) and (44), which accounts for the strain energy around geometrically necessary dislocations.

Other, more involved representations of the free energy have been considered. Gurtin and Anand (2005a,2005b) represented the free energy as the sum of the classical elastic strain energy and the defect energy, the latter assumed to depend on the Burgers tensor defined as a measure of the incompatibility of the plastic part of the deformation gradient field. See also Gurtin (2004), and Gurtin and Reddy (2009).

6. Yield condition and plastic loading conditions

In the considered framework of strain gradient plasticity, the yield condition is of the von Mises (J_2) type in the deviatoric Cauchy stress space, $\sigma_{eq} = \sigma_Y$, where σ_Y specifies the radius of the current yield surface (Hutchinson, 2012). The loading/unloading conditions from the state on the yield surface are

$$\sigma'_{ij}\dot{\varepsilon}_{ij}\begin{cases} \leq 0, & \text{elastic unloading } (\dot{e}_p = 0)\ , \\ > 0, & \text{plastic loading } (\dot{e}_p > 0)\ . \end{cases} \tag{45}$$

The consistency condition during plastic loading is $(3/2)\sigma'_{ij}\dot{\sigma}'_{ij} = \sigma_Y\dot{\sigma}_Y$, in which $\dot{\sigma}_Y$ can be positive or negative, i.e., the yield surface in the stress space can expand or shrink, depending on the plastic strain gradients. It is also noted from (33) that $\sigma_{eq} = \bar{q} - \bar{\tau}_{k,k}$, so that the right-hand side could be used to specify the size of the yield surface. The formulation of the entire incremental boundary value problem, based on an appropriate functional with a specified rate potential function and the prescribed boundary conditions on $\dot{u}_i$ and $\dot{e}_p$, is discussed by Hutchinson (2012) and Fleck et al. (2014, 2015).

An alternative approach to phenomenological strain gradient plasticity (the so-called "lower-order" strain gradient plasticity), in which additional boundary conditions on plastic strain are not required, has been proposed by Acharya and Bassani (1996), Bassani (2001) and Kuroda and Tvergaard (2010).

7. Recoverable and dissipative parts of the work-conjugates

Hutchinson (2012) and Fleck et al. (2014) partitioned the work conjugates $(\bar{q}, \bar{\tau}_k)$ and (q_{ij}, τ_{ijk}) to their recoverable and unrecoverable (dissipative) parts. This partition was motivated by the observation by Gudmundson (2004) and Gurtin and Anand (2009) that the incremental theory of strain gradient plasticity proposed by Fleck and Hutchinson (2001) can violate thermodynamic restriction of non-negative plastic dissipation for certain nonproportional strain histories.

The partition proceeds by first expressing the rate of plastic work as the sum of a dissipative and recoverable parts,

$$\dot{w}_p = \bar{q}\dot{e}_p + \bar{\tau}_k\dot{e}_{p,k} \equiv \dot{w}_p^{dis} + \dot{w}_p^{rec}, \quad \dot{w}_p^{rec} = \dot{\psi}_p\ . \tag{46}$$

The dissipative part (35) can be expressed as

$$\dot{w}_p^{dis} = \eta(e_p)\sigma_0(e_p)\dot{e}_p \equiv \bar{q}^{dis}\dot{e}_p + \bar{\tau}_k^{dis}\dot{e}_{p,k}, \tag{47}$$

from which

$$\bar{q}^{dis} = \eta(e_p)\sigma_0(e_p), \quad \bar{\tau}_k^{dis} = 0. \tag{48}$$

The rate of the recoverable part of plastic work is a non-dissipative portion of the rate of plastic work ($\dot{w}_p^{rec} = \dot{\psi}_p$). If this is written as

$$\dot{w}_p^{rec} = \bar{q}^{rec}\dot{e}_p + \bar{\tau}_k^{rec}\dot{e}_{p,k}, \tag{49}$$

the recoverable potions of $\bar{q}$ and $\bar{\tau}_k$ are found from (15) and (48) to be

$$\bar{q}^{rec} = \bar{q} - \bar{q}^{dis} = \sigma_0(E_p)\frac{e_p}{E_p} - \eta(e_p)\sigma_0(e_p), \quad \bar{\tau}_k^{rec} = \bar{\tau}_k - \bar{\tau}_k^{dis} = l^2\sigma_0(E_p)\frac{e_{p,k}}{E_p}\ . \tag{50}$$

If it is assumed that η=1 throughout plastic deformation, the expressions (48) and (50) reduce to those of Hutchinson (2012).

The dissipative (unrecoverable) parts of q_{ij} and τ_{ijk} can be recognized from (35). Since $\dot{e}_p = (2/3)m_{ij}\dot{\varepsilon}_{ij}^p$, one has

$$\dot{w}_p^{dis} = \frac{2}{3}\eta(e_p)\sigma_0(e_p)m_{ij}\dot{\varepsilon}_{ij}^p \equiv q_{ij}^{dis}\dot{\varepsilon}_{ij}^p + \tau_{ijk}^{dis}\dot{\varepsilon}_{ij,k}^p. \tag{51}$$

Thus, up to their immaterial workless terms,

$$q_{ij}^{dis} = \frac{2}{3}\eta(e_p)\sigma_0(e_p)m_{ij}, \quad \tau_{ijk}^{dis} = 0. \tag{52}$$

The recoverable parts $q_{ij}^{rec} = q_{ij} - q_{ij}^{dis}$ and $\tau_{ijk}^{rec} = \tau_{ijk} - \tau_{ijk}^{dis}$ appear in the representation

$$\dot{w}_p^{rec} = q_{ij}^{rec}\dot{\varepsilon}_{ij}^p + \tau_{ijk}^{rec}\dot{\varepsilon}_{ij,k}^p. \tag{53}$$

In view of (18) and (52), they are

$$q_{ij}^{\rm rec}=\frac{2}{3}\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}e_{\rm p}m_{ij}-\frac{2}{3}\eta(e_{\rm p})\sigma_0(e_{\rm p})m_{ij},\quad \tau_{ijk}^{\rm rec}=\frac{2}{3}l^2\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}e_{{\rm p},k}m_{ij}. \tag{54}$$

In particular, by comparing (50) and (54), it follows that

$$\bar{q}^{\rm rec}=q_{ij}^{\rm rec}m_{ij},\quad \bar{q}^{\rm dis}=q_{ij}^{\rm dis}m_{ij},\quad \bar{\tau}_k^{\rm rec}=\tau_{ijk}^{\rm rec}m_{ij}, \tag{55}$$

as anticipated from (19). In the case of classical plasticity, $q_{ij}=\sigma'_{ij}$, $q_{ij}^{\rm dis}=\eta\sigma'_{ij}$, and $q_{ij}^{\rm rec}=(1-\eta)\sigma'_{ij}$, while $\bar{q}^{\rm dis}=\eta\sigma_{\rm eq}$ and $\bar{q}^{\rm rec}=(1-\eta)\sigma_{\rm eq}$.

The evaluation of dissipative and recoverable parts of the higher order stresses q_{ij} and τ_{ijk} within considered simple models of strain gradient plasticity may be a helpful guide in the construction of more involved models of strain gradient plasticity, such as those which account for anisotropic strain hardening during plastic deformation under non-proportional loading. They may also be used to study the conditions under which the incremental and nonincremental theories of strain gradient plasticity give rise to elastic loading gaps at initial yield and under nonproportional loading histories (Fleck et al., 2015).

7.1. Partition in the case of alternative measures of effective plastic strain

The recoverable potions of $\bar{q}$ and $\bar{\tau}_k$ in this case are

$$\bar{q}^{\rm rec}=[1-\eta(e_{\rm p})]\sigma_0(e_{\rm p})+\frac{2}{3}\left[\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}-\frac{\sigma_0(\varepsilon_{\rm p})}{\varepsilon_{\rm p}}\right]m_{ij}\varepsilon_{ij}^{\rm p}+\frac{2}{3}l^2\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}m_{ij,k}\varepsilon_{ij,k}^{\rm p}, \tag{56}$$

$$\bar{\tau}_k^{\rm rec}=\frac{2}{3}l^2\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}m_{ij}\varepsilon_{ij,k}^{\rm p}, \tag{57}$$

while the dissipative parts are still given by (48). The recoverable parts of q_{ij} and τ_{ijk} are

$$q_{ij}^{\rm rec}=\frac{2}{3}[1-\eta(e_{\rm p})]\sigma_0(e_{\rm p})m_{ij}+\frac{2}{3}\left[\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}-\frac{\sigma_0(\varepsilon_{\rm p})}{\varepsilon_{\rm p}}\right]\varepsilon_{ij}^{\rm p},\quad \tau_{ijk}^{\rm rec}=\frac{2}{3}l^2\frac{\sigma_0(\mathbb{E}_{\rm p})}{\mathbb{E}_{\rm p}}\varepsilon_{ij,k}^{\rm p}, \tag{58}$$

the dissipative parts $q_{ij}^{\rm dis}$ and $\tau_{ijk}^{\rm dis}$ being given by (52)

By comparing (56) and (57) with (58), the relationships hold

$$\bar{q}^{\rm rec}=q_{ij}^{\rm rec}m_{ij}+\tau_{ijk}^{\rm rec}m_{ij,k}\ ,\quad \bar{q}^{\rm dis}=q_{ij}^{\rm dis}m_{ij}\ ,\quad \bar{\tau}_k^{\rm rec}=\tau_{ijk}^{\rm rec}m_{ij}\ , \tag{59}$$

in accord with (28).

There are formulations of strain gradient plasticity in which $\tau_{ijk}^{\rm dis}\neq 0$, e.g., Nielsen and Niordson (2014), and Nielsen et al. (2014). In the theory by Fleck and Willis (2009b), both recoverable and dissipative stresses are considered in order to develop a kinematic hardening theory. Their analysis also includes interfacial terms, which account for elastic energy stored and plastic work dissipated at internal interfaces. The theory of Gurtin and Anand (2005a,2005b) involves a recoverable higher order stress, that is the defect stress work-conjugate to the Burgers tensor, under the assumption of irrotational plastic flow. See also the analyses by Gurtin (2004) and Gudmundson (2004).

7.2. Other cases

The recoverable and dissipative parts of higher order stresses can be similarly identified in the case of other representations of the dissipative and recoverable parts of plastic work. For example, if the plastic gradient contribution to the free energy is assumed to be quadratic in plastic gradients $(1/2)\beta l^2 e_{{\rm p},k}e_{{\rm p},k}$, as in Mühlhaus and Aifantis (1991), one can define $w_{\rm p}(e_{\rm p},e_{{\rm p},k})=w_{\rm p}^{\rm dis}(e_{\rm p})+\psi_{\rm p}(e_{\rm p},e_{{\rm p},k})$, where

$$w_{\rm p}^{\rm dis}(e_{\rm p})=\int_0^{e_{\rm p}}\eta(\epsilon_{\rm p})\sigma_0(\epsilon_{\rm p}){\rm d}\epsilon_{\rm p},\quad \psi_{\rm p}\left(e_{\rm p},e_{{\rm p},k}\right)=\int_0^{e_{\rm p}}[1-\eta(\epsilon_{\rm p})]\sigma_0(\epsilon_{\rm p}){\rm d}\epsilon_{\rm p}+\frac{1}{2}\beta l^2 e_{{\rm p},k}e_{{\rm p},k}. \tag{60}$$

The material parameter β has the dimension of stress. As in the two cases from Sections 2 and 5, while $\dot{w}_{\rm p}^{\rm dis}>0$ during plastic deformation, the rates $\dot{\psi}_{\rm p}$ and $\dot{w}_{\rm p}$ can, in general, be either positive or negative (for proportional straining, $\dot{\psi}_{\rm p}>0$ and $\dot{w}_{\rm p}>0$). It readily follows that

$$q_{ij}^{\rm dis}=\frac{2}{3}\eta(e_{\rm p})\sigma_0(e_{\rm p})m_{ij},\quad \tau_{ijk}^{\rm dis}=0\,;\quad q_{ij}^{\rm rec}=\frac{2}{3}[1-\eta(e_{\rm p})]\sigma_0(e_{\rm p})m_{ij},\quad \tau_{ijk}^{\rm rec}=\frac{2}{3}\beta l^2 e_{{\rm p},k}m_{ij}, \tag{61}$$

while $\overline{q}=q_{ij}m_{ij}$ and $\overline{\tau}_k=\tau_{ijk}m_{ij}$ for both dissipative and recoverable parts. If $\beta=\beta(e_{\rm p})$, as considered by Gurtin and Anand (2009) in a simple generalization of Mühlhaus and Aifantis (1991) theory, then $q_{ij}^{\rm rec}$ in (61) is replaced with

$$q_{ij}^{\rm rec}=\frac{2}{3}\overline{q}^{\rm rec}m_{ij}\,,\quad \overline{q}^{\rm rec}=[1-\eta(e_{\rm p})]\sigma_0(e_{\rm p})+\frac{1}{2}l^2\frac{{\rm d}\beta}{{\rm d}e_{\rm p}}e_{{\rm p},k}e_{{\rm p},k}. \tag{62}$$

7.3. Yield surface in $q_{ij}^{\rm dis}$ space

For the considered types of effective plastic strain measures, the yield surface can be defined in the $q_{ij}^{\rm dis}$ space. In view of (52), this is

$$\Phi=\left(\frac{3}{2}q_{ij}^{\rm dis}q_{ij}^{\rm dis}\right)^{1/2}-f=0\,,\quad f=\eta(e_{\rm p})\sigma_0(e_{\rm p}), \tag{63}$$

such that the rate of plastic strain obeys the normality rule

$$\dot{\varepsilon}_{ij}^{\rm p}=\dot{e}_{\rm p}\frac{\partial\Phi}{\partial q_{ij}^{\rm dis}}. \tag{64}$$

Since for the hardening material, $\sigma_0=\sigma_0(e_{\rm p})$ is a monotonically non-decreasing function, the yield surface (63) expands isotropically in the $q_{ij}^{\rm dis}$ space during plastic deformation. The radius of this surface is specified by the affinity f conjugate to the flux $\dot{e}_{\rm p}$, such that the internal energy dissipation is $\mathscr{D}=f\dot{e}_{\rm p}$. If $l=0$, then $q_{ij}^{\rm dis}=\eta(e_{\rm p})\sigma'_{ij}$ and (63) reduces to the von Mises yield surface of classical plasticity $[(3/2)\sigma'_{ij}\sigma'_{ij}]^{1/2}=\sigma_0(e_{\rm p})$ (Fleck et al., 2014). The corresponding recoverable part of q_{ij} in this case is $q_{ij}^{\rm rec}=[1-\eta(e_{\rm p})]\sigma'_{ij}$, so that $q_{ij}=\sigma'_{ij}$ and $\overline{q}=q_{ij}m_{ij}=\sigma_{\rm eq}$.

8. Proportional plastic straining

In the case of proportional plastic straining it is assumed that the plastic strain components and their gradients monotonically increase in proportion (Fleck et al., 2014). The effective plastic strain and the gradient-enhanced effective plastic strain are then

$$e_{\rm p}=\left(\frac{2}{3}\varepsilon_{ij}^{\rm p}\varepsilon_{ij}^{\rm p}\right)^{1/2},\quad E_{\rm p}=\left(\frac{2}{3}\varepsilon_{ij}^{\rm p}\varepsilon_{ij}^{\rm p}+\frac{2}{3}l^2\varepsilon_{ij,k}^{\rm p}\varepsilon_{ij,k}^{\rm p}\right)^{1/2}. \tag{65}$$

The plastic work and the plastic contribution to the free energy are

$$w_{\rm p}=\int_0^{E_{\rm p}}\sigma_0(\epsilon_{\rm p}){\rm d}\epsilon_{\rm p},\quad \psi_{\rm p}=\int_0^{E_{\rm p}}\sigma_0(\epsilon_{\rm p}){\rm d}\epsilon_{\rm p}-\int_0^{e_{\rm p}}\eta(\epsilon_{\rm p})\sigma_0(\epsilon_{\rm p})\,{\rm d}\epsilon_{\rm p}, \tag{66}$$

with $\dot{w}_{\rm p}=\sigma_0(E_{\rm p})\dot{E}_{\rm p}>0$. It readily follows that

$$q_{ij}=\frac{2}{3}\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}\varepsilon_{ij}^{\rm p},\quad \tau_{ijk}=\frac{2}{3}l^2\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}\varepsilon_{ij,k}^{\rm p}. \tag{67}$$

Since the plastic strain components are $\varepsilon_{ij}^{\rm p}=e_{\rm p}m_{ij}$, from (33) and (67) the work conjugates to $e_{\rm p}$ and $e_{{\rm p},k}$ are found to be

$$\overline{q}=\sigma_0(E_{\rm p})\frac{e_{\rm p}}{E_{\rm p}}+l^2\frac{\sigma_0(E_{\rm p})}{E_{\rm p}e_{\rm p}}\left(e_{\rm p}^{*\,2}-e_{{\rm p},k}e_{{\rm p},k}\right),\quad \overline{\tau}_k=l^2\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}e_{{\rm p},k}. \tag{68}$$

For brevity, in the expression for $\overline{q}$ the notation is used $e_{\rm p}^{*\,2}=(2/3)\varepsilon_{ij,k}^{\rm p}\varepsilon_{ij,k}^{\rm p}$.

The dissipative and recoverable parts are

$$q_{ij}^{\rm dis} = \frac{2}{3}\bar{q}^{\rm dis} m_{ij}\,, \quad q_{ij}^{\rm rec} = q_{ij} - q_{ij}^{\rm dis}\,; \quad \tau_{ijk}^{\rm dis} = 0\,, \quad \tau_{ijk}^{\rm rec} = \tau_{ijk}\,, \tag{69}$$

with $\bar{q}^{\rm dis} = \eta(e_{\rm p})\sigma_0(e_{\rm p})$ and $\bar{q}^{\rm rec} = \bar{q} - \bar{q}^{\rm dis}$.

9. Torsion of a hollow circular tube

The effects of strain gradients on plastic response have been studied for various problems at the micron-scale, notably in the bending and torsion analyses of thin beams and wires (e.g., Fleck et al., 1994; Gao et al., 1999; Huang et al., 2000; Gudmundson, 2004). An additional insight can be gained by considering torsion of a hollow circular tube made of a rigid-plastic material. This is appealing because the problem allows a closed-form analytical solution, from which the solutions for a solid circular rod and a thin-walled circular tube follow as spacial cases. The mid-radius of the tube is denoted by a and its thickness by δ, so that the outer and inner radii of the tube are $a \pm \delta/2$. If the tube is subjected to an applied torque $T > T_{\rm Y}$, the non-vanishing strain is $\varepsilon_{z\varphi} = r\theta/2$, where θ is the angle of twist (per unit length of the tube), and (r,φ,z) are the cylindrical coordinates with z along the central axis of the tube. Since elastic component of strain is absent in the case of rigid-plastic model, there is no deformation for $T \le T_{\rm Y}$, while for $T > T_{\rm Y}$ the entire deformation is plastic. The threshold torque $\rm T_Y$ will be determined in the sequel. Adopting the small strain framework and proportional straining, (65) give

$$\begin{aligned} e_{\rm p} &= \frac{2\varepsilon_{z\varphi}}{\sqrt{3}} = \frac{r\theta}{\sqrt{3}}\,, \quad e_{{\rm p},r} = \frac{\theta}{\sqrt{3}}\,, \quad e_{\rm p}^* = \sqrt{\frac{2}{3}}\,\theta, \\ E_{\rm p} &= \frac{\theta}{\sqrt{3}}\left(r^2 + 2l^2\right)^{1/2}\,, \quad E_{{\rm p},r} = \frac{\theta}{\sqrt{3}}\,\frac{r}{\left(r^2+2l^2\right)^{1/2}}. \end{aligned} \tag{70}$$

In deriving (70) from (65), it is noted that

$$\varepsilon_{z\varphi,r} = \frac{\partial\varepsilon_{z\varphi}}{\partial r} = \frac{\theta}{2}\,, \quad \varepsilon_{zr,\varphi} = -\frac{1}{r}\varepsilon_{z\varphi} = -\frac{\theta}{2}. \tag{71}$$

The substitution of (70) into (68) yields

$$\bar{q} = \sigma_0(E_{\rm p})\,\frac{r^2+l^2}{r\left(r^2+2l^2\right)^{1/2}}\,, \quad \bar{\tau}_r = \sigma_0(E_{\rm p})\,\frac{l^2}{\left(r^2+2l^2\right)^{1/2}}\,. \tag{72}$$

The divergence of the vector $\bar{\tau}_k$ is

$$\bar{\tau}_{k,k} = \bar{\tau}_{r,r} + \frac{1}{r}\bar{\tau}_r = \sigma_0(E_{\rm p})\,\frac{2l^4}{r\left(r^2+2l^2\right)^{3/2}} + \frac{r\theta}{\sqrt{3}}\,\frac{{\rm d}\sigma_0}{{\rm d}E_{\rm p}}\,\frac{l^2}{r^2+2l^2}. \tag{73}$$

The non-vanishing component of the shear stress is obtained by substituting (73) into (33), which gives $\sigma_{z\varphi} = (\bar{q} - \bar{\tau}_{k,k})/\sqrt{3}$, i.e.,

$$\sigma_{z\varphi} = \frac{\sigma_0(E_{\rm p})}{\sqrt{3}}\,\frac{r\left(r^2+3l^2\right)}{\left(r^2+2l^2\right)^{3/2}} - \frac{r\theta}{3}\,\frac{{\rm d}\sigma_0}{{\rm d}E_{\rm p}}\,\frac{l^2}{r^2+2l^2}. \tag{74}$$

Alternatively, by (30), this expression can be obtained from $\sigma_{z\varphi} = q_{z\varphi} - \tau_{z\varphi r,r}$, where $q_{z\varphi}$ is the work-conjugate to $\varepsilon_{z\varphi}$ and $\tau_{z\varphi r}$ to $\varepsilon_{z\varphi,r}$. From the first of (67),

$$q_{z\varphi} = \frac{r\theta}{3}\,\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}. \tag{75}$$

For a rigid-plastic incompressible material, there is a line force along the outer and inner radius of the cross-section of the tube ($r = a \pm \delta/2$), where the lateral cylindrical surface intersects the cross-section of the tube. This can be determined from

$$p_k = \left[n_i^{(1)}k_j^{(1)}\tau_{kij} - k_k^{(1)}n_i^{(1)}n_j^{(1)}n_l^{(1)}\tau_{lij}\right] + \left[n_i^{(2)}k_j^{(2)}\tau_{kij} - k_k^{(2)}n_i^{(2)}n_j^{(2)}n_l^{(2)}\tau_{lij}\right], \tag{76}$$

where $\mathbf{n}^{(i)}$ is the unit outward normal to surface $S^{(i)}$ $(i = 1,2)$, and $\mathbf{k}^{(i)} = \mathbf{c}^{(i)} \times \mathbf{n}^{(i)}$. The vector $\mathbf{c}^{(i)}$ is the unit tangent vector along the intersecting edge of the two surfaces, with $S^{(i)}$ to the left. The first subscript in τ_{ijk} specifies the normal to the surface over which the τ_{ijk} component acts, the second index specifies the orientation of the forces, and the third index specifies the orientation of the lever arm between the two forces of the doublet. The expression (76) can be derived by an analogous analysis to one used by Fleck and Hutchinson (1997) in their earlier version of strain gradient plasticity; see also Mindlin (1964) and the expression (13) of Huang et al. (2000). It readily follows that $p_\varphi = \tau_{\varphi zr}$ along the outer radius $r = a + \delta/2$, and $p_\varphi = -\tau_{\varphi zr}$ along the inner radius $r = a - \delta/2$.

The non-vanishing components of the third-order moment stress tensor in cylindrical coordinates are, from (67) and (71),

$$\begin{aligned}\tau_{z\varphi r} = \tau_{\varphi zr} &= \frac{2}{3} l^2 \frac{\sigma_0(E_\mathrm{p})}{E_\mathrm{p}} \varepsilon^\mathrm{p}_{z\varphi,r} = \frac{\theta}{3} l^2 \frac{\sigma_0(E_\mathrm{p})}{E_\mathrm{p}} = \frac{l^2}{r} q_{z\varphi},\\ \tau_{zr\varphi} = \tau_{rz\varphi} &= \frac{2}{3} l^2 \frac{\sigma_0(E_\mathrm{p})}{E_\mathrm{p}} \varepsilon^\mathrm{p}_{zr,\varphi} = -\frac{\theta}{3} l^2 \frac{\sigma_0(E_\mathrm{p})}{E_\mathrm{p}} = -\frac{l^2}{r} q_{z\varphi}.\end{aligned} \tag{77}$$

Therefore, the circumferential line forces are

$$p_\varphi(a \pm \delta/2) = \pm \frac{\theta}{3} l^2 \left[\frac{\sigma_0(E_\mathrm{p})}{E_\mathrm{p}}\right]_{a \pm \delta/2} = \pm \frac{l^2}{a \pm \delta/2} q_{z\varphi}(a \pm \delta/2). \tag{78}$$

On the lateral surfaces of the tube $r = a \pm \delta/2$, there is a nonvanishing but uniform moment stress component $\tau_{rz\varphi}$, so that the adjacent force components of the doublets constituting $\tau_{rz\varphi}$ cancel each other and the traction-free boundary condition on the lateral surfaces of the twisted tube is satisfied in that sense. For the general formulation of the independent higher-order boundary conditions in rigid-hardening plasticity, see Fleck and Willis (2009b).

9.1. Torque expressions

The torque required to produce a given twist θ is obtained from the moment equilibrium equation. This can be written as either

$$T(\theta) = 2\pi \int_{a-\delta/2}^{a+\delta/2} r^2 \sigma_{z\varphi}\, \mathrm{d}r + 2\pi\left[(a+\delta/2)^2 p_\varphi(a+\delta/2) + (a-\delta/2)^2 p_\varphi(a-\delta/2)\right], \tag{79}$$

or

$$T(\theta) = 2\pi \int_{a-\delta/2}^{a+\delta/2} r^2 q_{z\varphi}\, \mathrm{d}r + 2\pi \int_{a-\delta/2}^{a+\delta/2} r\left(\tau_{z\varphi r} - \tau_{zr\varphi}\right)\mathrm{d}r, \tag{80}$$

with $\sigma_{z\varphi}$ given by (74), $q_{z\varphi}$ given by (75), and $p_\varphi(a \pm \delta/2)$ by (78). This can be evaluated for an assumed functional form $\sigma_0 = \sigma_0(E_\mathrm{p})$ representing the stress–strain curve in uniaxial tension test. For example,

$$\sigma_0(E_\mathrm{p}) = \sigma^0_\mathrm{Y}\left(1 + \frac{m}{n} E_\mathrm{p}\right)^n, \quad 0 < n \le 1, \tag{81}$$

where the initial yield stress is σ^0_Y and the parameter $m = h_0/\sigma^0_\mathrm{Y}$, with h_0 being the initial hardening rate. Other types of the stress–strain curve can be used, such as $\sigma_0(E_\mathrm{p}) = \sigma^0_\mathrm{Y}(1 + c\, E^n_\mathrm{p})$, ($c = \mathrm{const.}$), which is characterized by an infinite initial hardening rate (for $n < 1$).

The threshold value of the torque for the onset of deformation is obtained from (79) by using the corresponding expressions for the shear stress and the line forces at the onset of plastic deformation. From (74), (78) and (81), these are

$$\sigma^0_{z\varphi} = \frac{\sigma^0_\mathrm{Y}}{\sqrt{3}} \frac{r(r^2 + 3l^2)}{(r^2 + 2l^2)^{3/2}}, \quad p^0_\varphi(a \pm \delta/2) = \pm \frac{\sigma^0_\mathrm{Y}}{\sqrt{3}} \frac{l^2}{\left[(a \pm \delta/2)^2 + 2l^2\right]^{1/2}}. \tag{82}$$

Fig. 1 shows the variation of $\sigma^0_{z\varphi}$ with r for three selected values of the material length l and two selected values of the ratio δ/a. As $l \to 0$, the shear stress approaches a constant value $\sigma^0_\mathrm{Y}/\sqrt{3}$. The departure from this uniform stress distribution increases with the increase of l. There is a much smaller effect of l on $\sigma^0_{z\varphi}(a + \delta/2)$ (outer radius of the tube) than on $\sigma^0_{z\varphi}(a - \delta/2)$ (inner radius of the tube), as seen from Fig. 1a. Fig. 1b corresponds to a solid circular rod of radius $2a$.

The substitution of (82) into (79) and integration gives $T_Y = T(0) = T_\sigma(0) + T_p(0)$, where the torque contributions from the shear stress and the line forces are

$$T_\sigma(0)=\frac{2\pi}{\sqrt{3}}\sigma_Y^0\left[\frac{1}{3}\left(r^2+2l^2\right)^{3/2}-\frac{l^2r^2}{\left(r^2+2l^2\right)^{1/2}}\right]_{a-\delta/2}^{a+\delta/2},$$
$$T_p(0)=\frac{2\pi}{\sqrt{3}}l^2\sigma_Y^0\left\{\frac{(a+\delta/2)^2}{\left[(a+\delta/2)^2+2l^2\right]^{1/2}}-\frac{(a-\delta/2)^2}{\left[(a-\delta/2)^2+2l^2\right]^{1/2}}\right\}. \quad (83)$$

The threshold value of the torque for the onset of deformation in the absence of strain gradient effects ($l=0$) is

$$T_Y^0=\frac{2\pi}{\sqrt{3}}\sigma_Y^0\delta\left(a^2+\frac{1}{12}\delta^2\right). \quad (84)$$

Fig. 2a shows the variation of the normalized torque $T(0)/T_Y^0$ with the length parameter l/a for three selected values of the ratio δ/a. The ratio $\delta/a=2$ corresponds to a solid rod of radius $2a$. Fig. 2b shows the variations of the torque contributions from the shear stress and the line forces alone. While the contribution $T_\sigma(0)$ decreases with the increase of l, the total torque $T(0)$, with the included contribution from the line forces, increases with the increase of l. The plots are for a hollow tube of thickness $\delta=a/4$.

9.2. Torque–twist relationship

In view of (77), the integrand of the second integral in (80) is equal to $2l^2q_{z\varphi}$, so that the expression for $T(\theta)$ in (80) simplifies to

$$T(\theta)=2\pi\int_{a-\delta/2}^{a+\delta/2}\left(r^2+2l^2\right)q_{z\varphi}\,\mathrm{d}r, \quad (85)$$

where

$$q_{z\varphi}=\frac{\sigma_Y^0}{\sqrt{3}}\frac{r\left[1+\alpha\left(r^2+2l^2\right)^{1/2}\right]^n}{\left(r^2+2l^2\right)^{1/2}}, \quad \alpha=\frac{m\theta}{n\sqrt{3}}. \quad (86)$$

The variation of $q_{z\varphi}$ (lighter curves) with r for a circular tube of thickness $\delta=a$ and the three indicated values of $m\gamma$ is shown in Fig. 3a. The length parameter was taken to be $l=0.25a$, and the hardening parameter $n=0.05$. For the comparison, the darker curves represent the corresponding variation of $\sigma_{z\varphi}=q_{z\varphi}-\tau_{z\varphi r,r}$. The corresponding variation of the moment stress $\tau_{z\varphi r}$ is shown in Fig. 3b. Since $\tau_{z\varphi r,r}<0$, it follows that $\sigma_{z\varphi}>q_{z\varphi}$ at any given r and $m\gamma$. For a solid rod with nonvanishing l, the gradient $\tau_{z\varphi r,r}=0$ at the center of the rod.

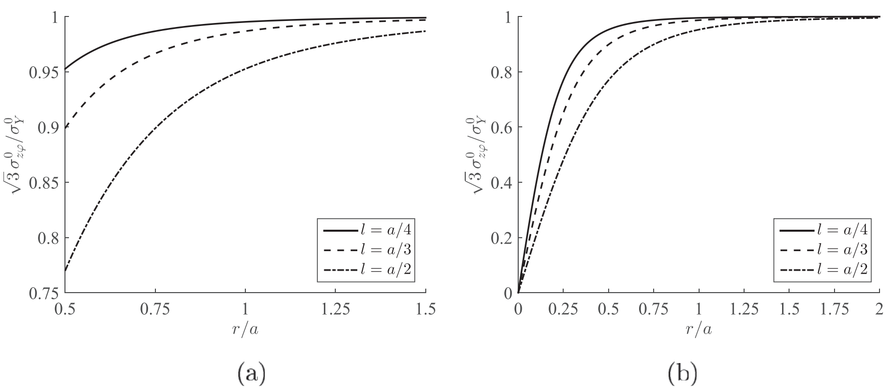

Fig. 1. The variation of the shear stress $\sigma^0_{z\varphi}$ (normalized by $\sigma^0_Y/\sqrt{3}$) with r/a, for the three selected values of the material length l. Part (a) is for the thickness of the tube $\delta=a$. Part (b) corresponds to solid (non-hollow) rod of radius $2a$.

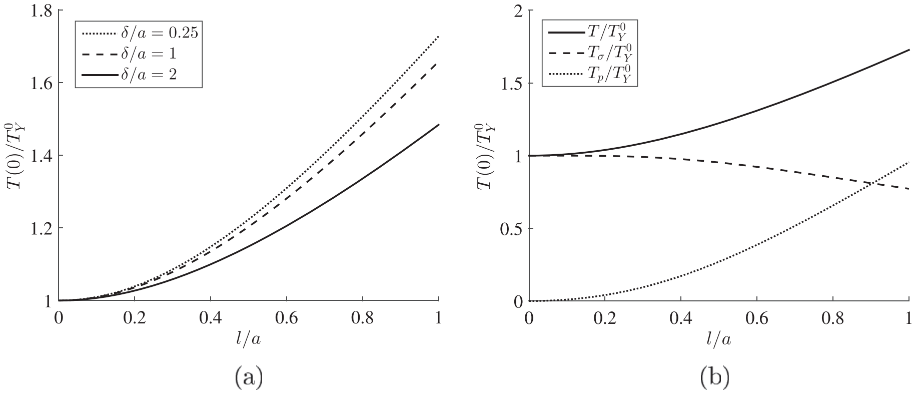

Fig. 2. The variation of the normalized yield threshold value of the torque $T(0)/T_Y^0$ with the length parameter l/a in the case of three different thickness ratios δ/a. (b) The variation of $T(0)/T_Y^0$ (solid curve) with the length parameter l/a in the case $\delta = 0.25a$, so that $l/\delta = 4(l/a)$. The dashed and dotted curves specify the variations of the torque contributions from the shear stress and the line forces separately.

The integral in (85) can be evaluated in closed form. For $\theta \neq 0$, the result is

$$T(\theta) = \frac{2\pi}{\sqrt{3}}\frac{\sigma_Y^0}{\alpha^3}\left(\frac{\rho^{n+3}}{n+3} - 2\frac{\rho^{n+2}}{n+2} + \frac{\rho^{n+1}}{n+1}\right)\Bigg|_{a-\delta/2}^{a+\delta/2}, \quad \rho = 1 + \alpha\left(r^2 + 2l^2\right)^{1/2}. \tag{87}$$

For $\theta = 0$, the threshold torque at the onset of deformation is

$$T(0) = \frac{2\pi}{\sqrt{3}}\frac{\sigma_Y^0}{3}\left\{\left[(a+\delta/2)^2 + 2l^2\right]^{3/2} - \left[(a-\delta/2)^2 + 2l^2\right]^{3/2}\right\}. \tag{88}$$

Note that $T(\theta)$ is not a homogeneous function of some degree in θ. The torque-twist expressions (87) and (88) can be compared with the related expressions reported in the literature. The torque-twist expression in the deformation theory of plasticity model of Fleck et al. (1994), based on the assumed power–law relationship between the introduced effective stress measure and its conjugate gradient-enhanced effective strain measure, is given by their expression (6.9). See also the expressions (2.27)–(2.29) of Fleck and Hutchinson (1997). The torque-twist expression in the constitutive model adopted by Huang et al. (2000) is given by their eq. (35). In the elastic-plastic constitutive model used by Gudmundson (2004), the torque–twist relationship was determined numerically from his expressions (56) and (57).

Fig. 4 shows the variation of the normalized torque $T(\theta)/T_Y^0$ with the strain measure $\gamma=(a+\delta/2)\theta$ at the outer radius of the tube, calculated from (74), (79) and (81), as the twist θ monotonically increases. The tube thickness is taken to be $\delta = a$, so that the inner radius of the tube is $a/2$ and the outer radius $3a/2$. The plots are made with respect to $m\gamma$ as an independent variable to avoid any prior numerical specification of the parameter $m > 0$. The range of $m\gamma$ is extended up to 1/2; for smaller values of m (say $m < 10$), only portion of this range corresponds to the small strain regime (say $\gamma < 5\%$). The three curves correspond to

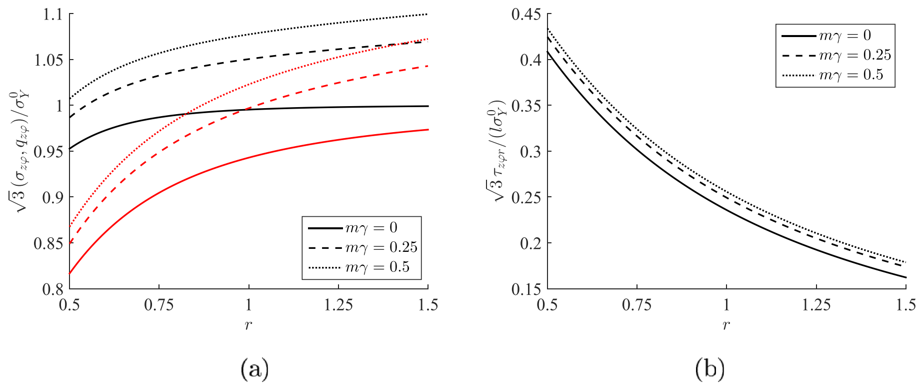

Fig. 3. (a) The variation of $q_{z\varphi}$ (lighter curves) with r for a circular tube of thickness $\delta = a$ and the three indicated values of $m\gamma$. The length parameter is $l = 0.25a$, and the hardening parameter $n = 0.05$. The darker curves represent the corresponding variation of $\sigma_{z\varphi}$. (b) The corresponding variation of the moment stress $\tau_{z\varphi r}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

three selected values of the material length l. The normalizing factor is T_Y^0. Fig. 3a is for the hardening parameter $n = 0.05$, while Fig. 4b corresponds to linear hardening ($n = 1$).

Fig. 5 shows the variation of $T(\theta)/T_Y^0$ with $m\gamma$ for a solid circular rod of radius $2a$. Fig. 5a is for the hardening parameter $n = 0.05$ and Fig. 5b for $n = 1$. Fig. 6 shows the results for a thin-walled tube of thickness $\delta = a/5$. There are no micro-scale experimental results on the torsion of thin-walled circular tubes in strain gradient plasticity reported as of yet; such experiments are needed to verify the analytical predictions presented in this paper. The comparison with pseudo-experimental results from discrete dislocation dynamics simulations (e.g., Bittencourt et al., 2003; Bardella et al., 2013) may also be of help. Fleck and Hutchinson (1997) reported that for each solid rod (wire) tested, a power–law relationship between $T(\theta)$ and the surface strain γ gave a good fit to the experimental data. The power exponent in this relationship was related to the power dependence of the strain energy on the gradient-enhanced effective strain, and the coefficient was dependent on the ratio of the material length scale and the radius of the wire.

9.3. Recoverable and dissipative parts

From (48) and (81), the dissipative part of the work conjugate to $e_{\rm p}$, at an arbitrary radius r and a given amount of twist θ, is

$$\overline{q}^{\rm dis} = \overline{q}_Y^{\rm dis}\left(1 + \frac{mr\theta}{n\sqrt{3}}\right)^n, \quad \overline{q}_Y^{\rm dis} = \eta(r\theta)\sigma_Y^0, \tag{89}$$

while $q_{z\varphi}^{\rm dis} = \overline{q}^{\rm dis}/\sqrt{3}$, $\overline{\tau}_r^{\rm dis} = 0$, and $\tau_{z\varphi r}^{\rm dis} = \tau_{zr\varphi}^{\rm dis} = 0$. The variation of $\overline{q}^{\rm dis}/\sigma_Y^0$ at the inner, outer, and mid-radius of a hollow tube with the shear parameter γ is shown in Fig. 7a. The thickness of the tube $\delta = a$, the length parameter $l = a/4$, and the hardening parameter $n = 0.05$. The factor η is assumed to be constant during the increase of θ and equal to 0.9. The recoverable part of $\overline{q}$ is $\overline{q}^{\rm rec} = \overline{q} - \overline{q}^{\rm dis}$, where

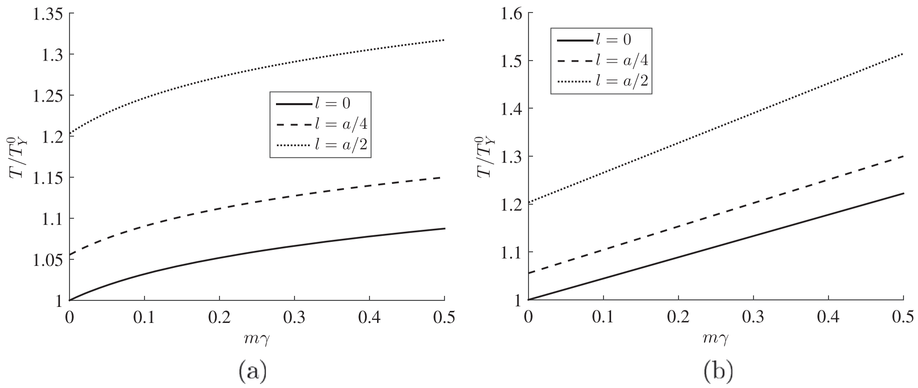

Fig. 4. The variation of the normalized torque T/T_Y^0 with $m\gamma$ for a hollow tube of thickness $\delta = a$ and the three indicated values of l. The hardening parameter is: (a) $n = 0.05$ and (b) $n = 1$. The strain parameter $\gamma = (a + \delta/2)\theta$, where θ is the imposed twist.

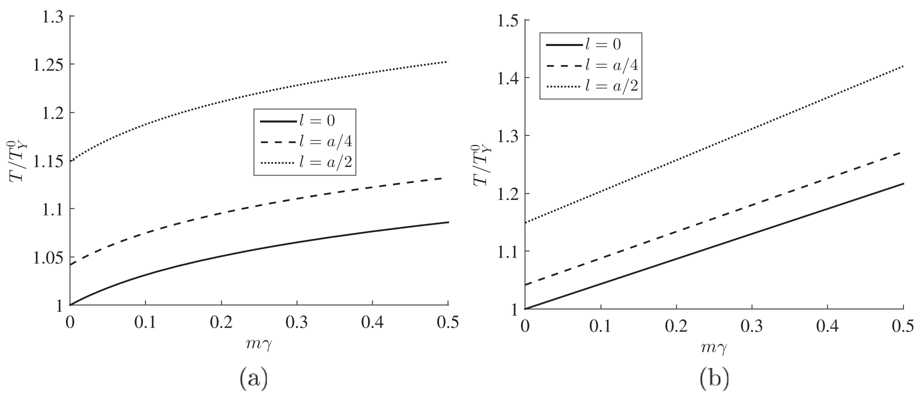

Fig. 5. The variation of the normalized torque T/T_Y^0 with $m\gamma$ for a solid circular rod of radius $2a$ and the three indicated values of l. The hardening parameter is: (a) $n = 0.05$ and (b) $n = 1$.

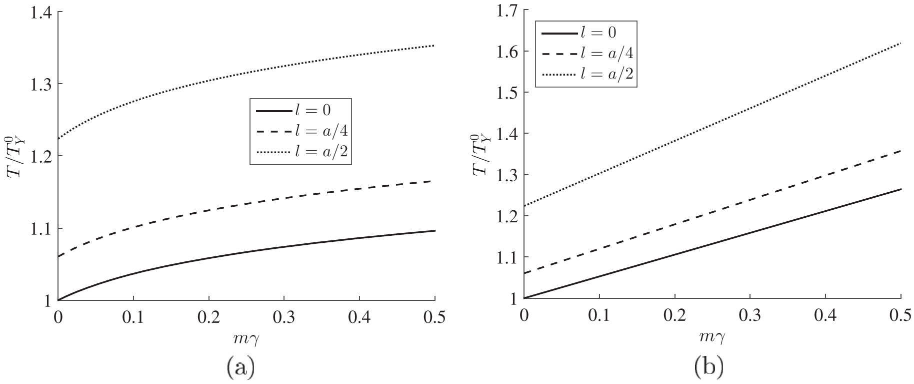

Fig. 6. The variation of the normalized torque T/T_Y^0 with $m\gamma$ for a thin-walled circular tube of thickness $\delta = a/5$ and the three indicated values of l. The hardening parameter is: (a) $n = 0.05$ and (b) $n = 1$.

$$\overline{q} = \overline{q}_Y(r)\left[1 + \frac{m\theta}{n\sqrt{3}}\left(r^2 + 2l^2\right)^{1/2}\right]^n, \quad \overline{q}_Y(r) = \sigma_Y^0 \frac{r^2 + l^2}{r\left(r^2 + 2l^2\right)^{1/2}}. \tag{90}$$

The yield threshold value of $\overline{q}$ is denoted by $\overline{q}_Y(r)$. Fig. 7b and c show the variations of $\overline{q}$ and $\overline{q}^{\rm rec}$ for the same data as used in the plots for $\overline{q}^{\rm dis}$ in Fig. 7a. The large values of $\overline{q}$ and $\overline{q}^{\rm rec}$ at the inner radius of the tube at the onset of deformation are due to the large value of $\overline{q}_Y$ at the inner radius. The plot of $\overline{q}_Y(r)$, obtained from (90), is shown in Fig. 7d.

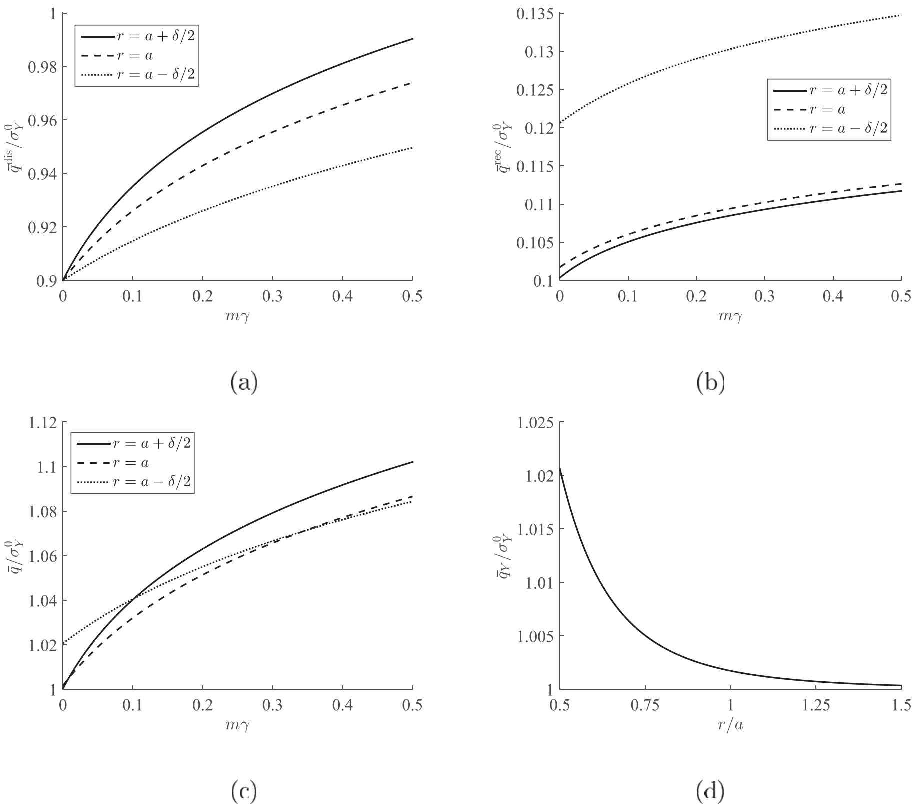

Fig. 7. (a) The variation of $\overline{q}^{\rm dis}$ at the inner, outer, and mid-radius of a hollow tube with γ in the case η=0.9. The thickness of the tube δ=a, the length parameter $l = a/4$, and the hardening parameter $n = 0.05$. The corresponding variation of (b) $\overline{q}^{\rm rec}$, and (c) $\overline{q} = \overline{q}^{\rm rec} + \overline{q}^{\rm dis}$. (d) The plot of $\overline{q}_{\mathbf{Y}}$ with the radius r.

10. Conclusions

There are two main objectives of the present paper. The first objective is to include the locked-in strain energy around statistically stored dislocations in the expression for the free energy of the phenomenological strain gradient plasticity by incorporating in the analysis the strain dependent factor $\eta = \eta(e_p)$. This factor specifies the fraction of the rate of plastic work converted into heat, in accordance with the measurements of latent heat from classical metal plasticity. The expressions for the recoverable and dissipative parts of the higher order stresses, defined as the work-conjugates to plastic strain and its gradient, are then derived. Two formulations of the strain gradient plasticity are considered, one based on the definition of the gradient-enhanced effective plastic strain and the corresponding plastic work expression proposed by Hutchinson (2012), and the other by Fleck et al. (2014). In the first case, the plastic work is given by (5) and the plastic contribution to the free energy by (39). The corresponding recoverable parts of the work-conjugates to plastic strain and its gradient are given by (50) for $\overline{q}^{\mathrm{rec}}$ and $\overline{\tau}_k^{\mathrm{rec}}$, and (54) for q_{ij}^{rec} and $\tau_{ijk}^{\mathrm{rec}}$. In the second case, the plastic work is given by (10), the plastic contribution to the free energy by (43), and the recoverable and dissipative parts of the work-conjugates by (56) and (57) for $\overline{q}^{\mathrm{rec}}$ and $\overline{\tau}_k^{\mathrm{rec}}$, and (58) for q_{ij}^{rec} and $\tau_{ijk}^{\mathrm{rec}}$. The dissipative portion of the plastic work $w_{\mathrm{p}}^{\mathrm{dis}}$ is assumed to be the same in both formulations and given by (37), so that $\overline{q}^{\mathrm{dis}}$ is given in both cases by (48) and q_{ij}^{dis} by (52). In the case of proportional straining, the recoverable and dissipative parts are given by (67). For the early stage of plastic deformation, the factor η can be approximately taken to be constant and equal to 0.9. If it is assumed that $\eta = 1$, so that the recoverable energy due to plastic deformation is entirely associated with the plastic strain gradient effects from the network of geometrically necessary dislocations, the obtained results reduce to those of Hutchinson (2012), and Fleck et al. (2014). The stress and strain fields of isothermal boundary-value problems are not affected by η, since this factor does not appear in the representation of the higher-order stresses q_{ij} and τ_{ijk}, or the structure of the corresponding variational principle. The factor η may, however, be of importance for the non-isothermal analysis in which $\dot{w}_{\mathrm{p}}^{\mathrm{dis}} = \eta(e_{\mathrm{p}})\sigma_0(e_{\mathrm{p}})\dot{e}_{\mathrm{p}}$ acts as an internal heat source. The development of such non-isothermal strain gradient plasticity theory, in which temperature dependence of free energy must be added to the theory, is a worthwhile goal of future investigation.

The second objective of the paper is to apply the presented analysis to evaluate the effects of the strain gradient on the plastic response of twisted hollow circular tubes made of a rigid-plastic material characterized by different types of strain-hardening. The shear stress, the edge line forces, and the applied torque are determined for various values of the material length parameter. These are given by (74), (78), and (87). The results for solid rods, hollow tubes and thin-walled tubes are presented at the onset of plastic yield and at an arbitrary stage of proportional plastic straining. Linear and nonlinear hardening are both considered. Recoverable and dissipative parts of the work conjugates to plastic strain and its gradient are evaluated and discussed for different hardening and length parameters.

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References

Acharya, A., Bassani, J.L., 1996. On non-local flow theories that preserve the classical structure of incremental boundary value problems. In: Pineau, A., Zaoui, A. (Eds.), IUTAM Symposium on Micromechanics of Plasticity and Damage of Multiphase Materials. Kluwer Academic Publishers, Dordrecht, pp. 3–9.
Aifantis, E.C., 1984. On the microstructural origin of certain inelastic modes. Trans. ASME J. Eng. Mat. Technol. 106, 326–130.
Anand, L., Gurtin, M.E., Lele, S.P., Gething, M., 2005. A one-dimensional theory of strain-gradient plasticity: formulation, analysis, numerical results. J. Mech. Phys. Solids 53, 1789–1826.
Anand, L., Gurtin, M.E., Reddy, B.D., 2015. The stored energy of cold work, thermal annealing, and other thermodynamic issues in single crystal plasticity at small length scales. Int. J. Plast. 64, 1–25.
Ashby, M.F., 1970. The deformation of plastically non-homogeneous materials. Philos. Mag. 21, 399–424.
Bardella, L., 2006. A deformation theory of strain gradient crystal plasticity that accounts for geometrically necessary dislocations. J. Mech. Phys. Solids 54, 128–160.
Bardella, L., 2007. Some remarks on the strain gradient crystal plasticity modelling, with particular reference to the material length scales involved. Int. J. Plast. 23, 296–322.
Bardella, L., Panteghini, A., 2015. Modelling the torsion of thin metal wires by distortion gradient plasticity. J. Mech. Phys. Solids 78, 467–492.
Bardella, L., Segurado, J., Panteghini, A., Llorca, J., 2013. Latent hardening size effect in small-scale plasticity. Model. Simul. Mat. Sci. Eng. 21, 055009.
Bassani, J.L., 2001. Incompatibility and a simple gradient theory of plasticity. J. Mech. Phys. Solids 49, 1983–1996.
Bever, M.B., Holt, D.L., Tichener, A.L., 1973. The stored energy of cold work. Prog. Mat. Sci. 17, 1–190.
Bittencourt, E., Needleman, A., Gurtin, M.E., van der Giessen, E., 2003. A comparison of nonlocal continuum and discrete dislocation plasticity predictions. J. Mech. Phys. Solids 51, 281–310.
Clayton, J., 2011. Nonlinear Mechanics of Crystals. Springer, Dordrecht.
Dahlberg, C.F.O., Faleskog, J., Niordson, C.F., Legarth, B.N., 2013. A deformation mechanism map for polycrystals modeled using strain gradient plasticity and interfaces that slide and separate. Int. J. Plast. 43, 177–195.
Danas, K., Deshpande, V.S., Fleck, N.A., 2012. Size effects in the conical indentation of an elasto-plastic solid. J. Mech. Phys. Solids 60, 1605–1625.
Devincre, B., Kubin, L.P., 1997. Mesoscopic simulations of dislocations and plasticity. Mat. Sci. Eng. A 234, 8–14.
Evans, A.G., Hutchinson, J.W., 2009. A critical assessment of theories of strain gradient plasticity. Acta Mater 57, 1675–1688.
Farren, W.S., Taylor, G.I., 1925. The heat developed during plastic extension of metals. Philos. Trans. R. Soc. A 107, 422–451.

Fleck, N.A., Hutchinson, J.W., 1993. A phenomenological theory for strain gradient effects in plasticity. J. Mech. Phys. Solids 41, 1825–1857.
Fleck, N.A., Hutchinson, J.W., 1997. Strain gradient plasticity. Adv. Appl. Mech. 33, 295–361.
Fleck, N.A., Hutchinson, J.W., 2001. A reformulation of strain gradient plasticity. J. Mech. Phys. Solids 49, 2245–2271.
Fleck, N.A., Hutchinson, J.W., Willis, J.R., 2014. Strain gradient plasticity under non-proportional loading. Proc. Roy. Soc. A 470, 20140267.
Fleck, N.A., Hutchinson, J.W., Willis, J.R., 2015. Guidelines for constructing strain gradient plasticity theories. J. Appl. Mech. 82, 071002.
Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W., 1994. Strain gradient plasticity: theory and experiment. Acta Metall. Mater 42, 475–487.
Fleck, N.A., Willis, J.R., 2009a. A mathematical basis for strain gradient plasticity theory. Part I: scalar plastic multiplier. J. Mech. Phys. Solids 57, 161–177.
Fleck, N.A., Willis, J.R., 2009b. A mathematical basis for strain gradient plasticity theory. Part II: tensorial plastic multiplier. J. Mech. Phys. Solids 57, 1045–1057.
Gao, H., Huang, Y., Nix, W.D., Hutchinson, J.W., 1999. Mechanism-based strain gradient plasticity – I. Theory. J. Mech. Phys. Solids 47, 1239–1263.
Gudmundson, P.A., 2004. Unified treatment of strain gradient plasticity. J. Mech. Phys. Solids 52, 1379–1406.
Gurtin, M.E., 2002. A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations. J. Mech. Phys. Solids 50, 5–32.
Gurtin, M.E., 2003. On a framework for small-deformation viscoplasticity: free energy, microforces, strain gradients. Int. J. Plast. 19, 47–90.
Gurtin, M.E., 2004. A gradient theory of small-deformation isotropic plasticity that accounts for the Burgers vector and for dissipation due to plastic spin. J. Mech. Phys. Solids 52, 2545–2568.
Gurtin, M.E., Anand, L., 2005a. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: small deformations. J. Mech. Phys. Solids 53, 1624–1649.
Gurtin, M.E., Anand, L., 2005b. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: finite deformations. J. Mech. Phys. Solids 53, 2297–2318.
Gurtin, M.E., Anand, L., 2009. Thermodynamics applied to gradient theories involving accumulated plastic strain. The theories of Aifantis and Fleck and Hutchinson and their generalizations. J. Mech. Phys. Solids 57, 405–421.
Gurtin, M.E., Fried, E., Anand, L., 2010. The Mechanics and Thermodynamics of Continua. Cambridge University Press, Cambridge.
Gurtin, M.E., Reddy, B.D., 2009. Alternative formulations of isotropic hardening for Mises materials, and associated variational inequalities. Cont. Mech. Thermodyn. 21, 237–250.
Huang, Y., Gao, H., Nix, W.D., Hutchinson, J.W., 2000. Mechanism-based strain gradient plasticity - II. Analysis. J. Mech. Phys. Solids 48, 99–128.
Huang, Y., Qu, S., Hwang, K.C., Li, M., Gao, H., 2004. A conventional theory of mechanism-based strain gradient plasticity. Int. J. Plast. 20, 753–782.
Hutchinson, J.W., 2000. Plasticity at the micron scale. Int. J. Solids Struct. 37, 225–238.
Hutchinson, J.W., 2012. Generalizing J_2 flow theory: fundamental issues in strain gradient plasticity. Acta Mech. Sin. 28, 1078–1086.
Keller, C., Hug, E., Feaugas, X., 2011. Microstructural size effects on mechanical properties of high purity nickel. Int. J. Plast. 27, 635–654.
Kuroda, M., Tvergaard, V., 2010. An alternative treatment of phenomenological higher-order strain-gradient plasticity theory. Int. J. Plast. 26, 507–515.
Kysar, J., Saito, Y., Öztop, M., Lee, D., Huh, W., 2010. Experimental lower bounds on geometrically necessary dislocation density. Int. J. Plast. 26, 1097–1123.
Lee, E.H., 1969. Elastic-plastic deformation at finite strain. J. Appl. Mech. 36, 1–6.
Lele, S.P., Anand, L., 2009. A large-deformation strain-gradient theory for isotropic viscoplastic materials. Int. J. Plast. 25, 420–453.
Liu, D., He, Y., Dunstan, D.J., Zhang, B., Gan, Z., Hu, P., 2013. Toward a further understanding of size effects in the torsion of thin metal wires: an experimental and theoretical assessment. Int. J. Plast. 41, 30–52.
Lubarda, V.A., 1982. Non-isothermal elastic-plastic deformation. Theor. Appl. Mech. 8, 87–93.
Ma, Z.S., Zhou, Y.C., Long, S.G., Lu, C., 2012. On the intrinsic hardness of a metallic film/substrate system: Indentation size and substrate effects. Int. J. Plast. 34, 1–11.
Mayeur, J.R., McDowell, D.L., 2014. A comparison of Gurtin type and micropolar theories of generalized single crystal plasticity. Int. J. Plast. 57, 29–51.
Mindlin, R.D., 1964. Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–78.
Mühlhaus, H.B., Aifantis, E.C., 1991. A variational principle for gradient plasticity. Int. J. Solids Struct. 28, 845–857.
Needleman, A., 2000. Computational mechanics at the mesoscale. Acta Mater 48, 105–124.
Nielsen, K.L., Niordson, C.N., 2014. A numerical basis for strain gradient plasticity theory: rate-independent and rate-dependent formulations. J. Mech. Phys. Solids 63, 113–117.
Nielsen, K.L., Niordson, C.F., Hutchinson, J.W., 2014. Strain gradient effects in periodic flat punch indenting at small scales. Int. J. Solids Struct. 51, 3549–3556.
Niordson, C.F., Hutchinson, J.W., 2011. Basic strain gradient plasticity theories with application to constrained film deformation. J. Mech. Mat. Struct. 6, 395–416.
Nix, W.D., Gao, H., 1998. Indentation size effects in crystalline materials: a law for strain gradient plasticity. J. Mech. Phys. Solids 46, 411–425.
Öztop, M.S., Niordson, C.F., Kysar, J.W., 2013. Length-scale effect due to periodic variation of geometrically necessary dislocation densities. Int. J. Plast. 41, 189–201.
Polizzotto, C., 2009. A nonlocal strain gradient plasticity theory for finite deformations. Int. J. Plast. 25, 1280–1300.
Prager, W., 1958. Non-isothermal plastic deformation. Proc. Koninkl. Nederl. Akad. Wetensch. B 61, 176–182.
Qiu, X., Huang, Y., Wei, Y., Gao, H., Hwang, K.C., 2003. The flow theory of mechanism-based strain gradient plasticity. Mech. Mater. 35, 245–258.
Rosakis, P., Rosakis, A.J., Ravichandran, G., Hodowany, J., 2000. A thermodynamic internal variable model for the partition of plastic work into heat and stored energy in metals. J. Mech. Phys. Solids 48, 581–607.
Senger, J., Weygand, D., Kraft, O., Gumbsch, P., 2011. Dislocation microstructure evolution in cyclically twisted microsamples: a discrete dislocation dynamics simulation. Model. Simul. Mat. Sci. Eng. 19, 074004.
Steinmann, P., 2015. Geometrical Foundations of Continuum Mechanics: an Application to First- and Second-Order Elasticity and Elasto-Plasticity. Springer, Berlin.
Stölken, J.S., Evans, A.G., 1998. A microbend test method for measuring the plasticity length scale. Acta Mater 46, 5109–5115.
Tadmor, E.B., Miller, R., Phillips, R., Ortiz, M., 1999. Nanoindentation and incipient plasticity. J. Mat. Res. 134, 2233–2250.
Taheri-Nassaj, N., Zbib, H.M., 2015. On dislocation-pileups and stress-gradient dependent plastic flow. Int. J. Plast. 74, 1–16.
Taylor, G.I., Quinney, H., 1934. The latent energy remaining in a metal after cold working. Philos. Trans. R. Soc. A 143, 307–326.
Voyiadjis, G.Z., Faghihi, D., 2012. Thermo-mechanical strain gradient plasticity with energetic and dissipative length scales. Int. J. Plast. 30–31, 218–247.
Voyiadjis, G.Z., Pekmezi, G., Deliktas, B., 2010. Nonlocal gradient-dependent modeling of plasticity with anisotropic hardening. Int. J. Plast. 26, 1335–1356.
Wei, Y., Hutchinson, J.W., 2003. Hardness trends in micron scale indentation. J. Mech. Phys. Solids 51, 2037–2056.
Wulfinghoff, S., Böhlke, T., 2015. Gradient crystal plasticity including dislocation-based work-hardening and dislocation transport. Int. J. Plast. 69, 152–169.
Zbib, H.M., Diaz de la Rubia, T., Bulatov, V., 2002. A multiscale model of plasticity based on discrete dislocation dynamics. Trans. ASME J. Eng. Mat. Tech. 124, 78–87.