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The energy momentum tensor in the presence of body forces and the Peach–Koehler force on a dislocation

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Abstract

A modified energy momentum tensor, in the presence of body forces, is introduced and used to construct the nonconserved J, M, and L integrals, and to derive the energetic forces associated with a defect motion within the material. The J integral is then applied to evaluate the Peach–Koehler force on an inclined edge dislocation within a large block due to its own weight. The equilibrium position of the dislocation is determined for different boundary conditions of interest in geomechanics.

Keywords: Body force; Dislocation; Energy momentum tensor; Geomechanics; Gravity load; J integral; Peach–Koehler force; Potential energy

1. Introduction

The study of conservation integrals in linear and nonlinear elasticity, with its application to elastic and inelastic fracture mechanics and materials science, has been a topic of active research during the past 50 years, following the landmark papers by Eshelby (1951, 1956), Rice (1968), Knowles and Sternberg (1972), and Budiansky and Rice (1973). The reviews by Rice (1985), Maugin (1995), and Herrmann and Kienzler (2001) offer an extensive list of pertinent references, mostly for problems without body forces. In the presence of body forces, the stress tensor and the energy momentum tensor are not divergence-free tensors, which precludes the existence of the J, M, and L conservation laws (Kishimoto et al., 1980; Atluri, 1982; Kirchner, 1999). In most of the previous work with body forces, the energy momentum tensor was defined by the same expression as in the absence of body forces, which leads to less appealing relationships to the energy release rates and the configurational forces on moving defects. In the present paper, a modified energy momentum tensor is introduced, which includes a work term due to the body forces, and which yields simple expressions

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for the configurational forces on defects, in terms of the integrals evaluated over the unloaded surface of the defect. The derived J integral is then applied to evaluate the Peach–Koehler force on an edge dislocation residing on an inclined slip plane within a large block of the material due to its own weight. The equilibrium position of the dislocation is determined for different boundary conditions of interest in the study of the underground stress systems in geomechanics.

2. The energy momentum tensor in the presence of body forces

The infinitesimal strain components are expressed in terms of the displacement components u_i as $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$. The equilibrium equations are $\sigma_{ji,j} + b_i = 0$, where b_i are the components of body forces per unit volume. In terms of the elastic strain energy $W = W(\epsilon_{ij})$, the constitutive equations are $\sigma_{ij} = \partial W/\partial\epsilon_{ij}$. A spatial gradient of the strain energy is then

$$W_{,k} = \frac{\partial W}{\partial \epsilon_{ij}}\epsilon_{ij,k} = \sigma_{ji}u_{i,jk}. \tag{1}$$

This can be recast by using the equilibrium equations as

$$(W\delta_{jk} - \sigma_{ji}u_{i,k})_{,j} = b_i u_{i,k}. \tag{2}$$

Since

$$(b_i u_i)_{,k} = b_{i,k}u_i + b_i u_{i,k}, \tag{3}$$

the subtraction of (3) from (2) yields

$$[(W - b_i u_i)\delta_{jk} - \sigma_{ji}u_{i,k}]_{,j} = -b_{i,k}u_i. \tag{4}$$

This defines the energy momentum tensor in the presence of body forces, P_{jk}, as

$$P_{jk} = (W - b_i u_i)\delta_{jk} - \sigma_{ji}u_{i,k}, \quad P_{jk,j} = -b_{i,k}u_i. \tag{5}$$

In Section 4 it is shown that this definition of the energy momentum tensor is most simply related to the release rates of the potential energy associated with a defect motion within the material in the presence of body forces. It should be noted that the body force term is also included in the structure of the J_P integral used in the study of the progressive failure of over-consolidated clay (Palmer and Rice, 1973), and in the J-integral applications to free-boundary flows in fluid mechanics (Ben Amar and Rice, 2002). The inclusion of the body force term in the structure of P_{jk} is also reminiscent of the energy momentum tensor in dynamic fracture mechanics (Freund, 1990).

3. Nonconserved *J*, *M*, and *L* integrals in the presence of body forces

The J integrals are defined in terms of the energy momentum tensor (5) by

$$J_k = \int_S P_{jk}n_j\,\mathrm{d}S = -\int_V b_{i,k}u_i\,\mathrm{d}V, \tag{6}$$

where S is the bounding surface of the volume V which does not include any singularity of defect. For the two-dimensional elasticity, the surface S is replaced by the contour C, and the volume V by the area A, and the indices j, $k = 1, 2$. In general, the right-hand side of (6) is not equal to zero, and thus $J_k \neq 0$.

If the energy momentum tensor was defined without the $b_i u_i \delta_{jk}$ term, *i.e.*, $\overline{P}_{jk} = W\delta_{jk} - \sigma_{ji}u_{i,k}$, we would have another nonconserved integral, given by $\overline{J}_k = \int_S \overline{P}_{jk}n_j\,\mathrm{d}S = \int_V b_i u_{i,k}\,\mathrm{d}V$, which is the form more frequently used in the literature (Huang et al., 2002; Liang et al., 2003).[1]

When the body forces are spatially uniform ($b_{i,k} = 0$), there is a conservation law

[1] In the two-dimensional context, the latter authors consider a path-independent integral $J_B = \int_C \overline{P}_{1\beta}n_\beta\,\mathrm{d}C - \int_A b_\alpha u_{\alpha,1}\,\mathrm{d}A$.

$$J_k = \int_S P_{jk} n_j \, \mathrm{d}S = \int_S [(W - b_i u_i) n_k - T_i u_{i,k}] \, \mathrm{d}S = 0, \tag{7}$$

for any surface S that does not enclose a singularity or defect. In the absence of body forces, (7) reduces to the well-known result by Rice (1968).[2]

If the strain energy is a homogeneous function of degree $1 < p \leqslant 2$ in the strain components, then

$$W = \frac{1}{p} \sigma_{jk} \epsilon_{jk}. \tag{8}$$

The energy momentum tensor then satisfies the equation

$$(P_{jk} x_k)_{,j} - P_{kk} = -u_i b_{i,k} x_k, \tag{9}$$

where

$$P_{kk} = m(W - b_k u_k) - \sigma_{jk} u_{k,j}. \tag{10}$$

The parameter $m = 3$ for the three-dimensional elasticity, and $m = 2$ for the two-dimensional elasticity (plane strain or plane stress). In view of (10) and (8), we have

$$P_{kk} = \frac{m - p}{p} (\sigma_{jk} u_k)_{,j} + \frac{m(1 - p) - p}{p} b_k u_k, \tag{11}$$

and the substitution into (9) gives

$$\left(P_{jk} x_k - \frac{m - p}{r} \sigma_{jk} u_k \right)_{,j} = u_i \left[\frac{m(1 - p) - p}{p} b_i - b_{i,k} x_k \right]. \tag{12}$$

Upon the application of the Gauss divergence theorem, this specifies the M integral as

$$M = \int_S \left(P_{jk} x_k - \frac{3 - p}{p} \sigma_{jk} u_k \right) n_j \, \mathrm{d}S = \int_V u_i \left(\frac{3 - 4p}{p} b_i - b_{i,k} x_k \right) \mathrm{d}V, \tag{13}$$

for the three-dimensional case, and

$$M = \int_C \left(P_{jk} x_k - \frac{2 - p}{p} \sigma_{jk} u_k \right) n_j \, \mathrm{d}C = \int_A u_i \left(\frac{2 - 3p}{p} b_i - b_{i,k} x_k \right) \mathrm{d}A, \tag{14}$$

for the two-dimensional case. In the absence of body forces, the M integral vanishes for any closed surface that does not embrace a singularity or defect (Günther, 1962; Knowles and Sternberg, 1972).

Finally, the L_k integrals of linear isotropic elasticity, in the presence of body forces, are

$$L_k = e_{kij} \int_S (P_{li} x_j + \sigma_{li} u_j) n_l \, \mathrm{d}S = -e_{kij} \int_V u_l (\delta_{lj} b_i + b_{l,i} x_j) \, \mathrm{d}V. \tag{15}$$

In the case of two-dimensional elasticity, parallel to (x_1, x_2) plane, the only nontrivial integral in (15) is the L_3 integral.

4. Relationships to the energy release rates

In this section we show that the J, M, and L integrals, evaluated over the unloaded surface of a defect, are related to the potential energy release rates and the corresponding configurational forces on a moving defect. By extending the analysis of Budiansky and Rice (1973), let the body of volume V be loaded by the surface tractions $T_i = \overline{T}_i$ over the portion S_T of its external surface S. The displacements $u_i = \bar{u}_i$ are prescribed over the remaining part S_u. Suppose that within a body there is a cavity with the traction-free bounding surface S_0. The potential energy of such body is

[2] For the conservation integrals in couple stress and micropolar elasticity, see Lubarda and Markenscoff (2000, 2003), where the reference to related work can also be found.

$$\Pi = \int_V W \,\mathrm{d}V - \int_{S_T} \overline{T}_i u_i \,\mathrm{d}S - \int_V b_i u_i \,\mathrm{d}V. \tag{16}$$

The rate of change of the potential energy associated with the spatial variation of the surface of the cavity, described by its velocity field $\dot{u}_i^0$, is

$$\dot{\Pi} = \int_V \dot{W} \,\mathrm{d}V - \int_{S_0} W \dot{u}_i^0 n_i \,\mathrm{d}S - \int_{S_T} \overline{T}_i \dot{u}_i \,\mathrm{d}S - \int_V b_i \dot{u}_i \,\mathrm{d}V + \int_{S_0} b_j u_j \dot{u}_i^0 n_i \,\mathrm{d}S, \tag{17}$$

where $\dot{u}_i$ is the velocity field within $V(t)$ due to imposed velocity $\dot{u}_i^0$. Body forces are assumed to be unaffected by the cavity motion (dead body forces). The surface integrals over S_0 on the right-hand side follow from the Reynolds transport theorem, where n_i is the unit normal to S_0 directed into the material. Assuming that $\dot{u}_i$ is a kinematically admissible field within $V(t)$, we have

$$\dot{W} = \sigma_{ij}\dot{\epsilon}_{ij} = (\sigma_{ij}\dot{u}_j)_{,i} + b_j \dot{u}_j. \tag{18}$$

Since S_0 is unloaded and $\dot{u}_j = 0$ on S_u, the application of the Gauss divergence theorem gives

$$\int_V \dot{W} \,\mathrm{d}V = \int_{S_T} \overline{T}_j \dot{u}_j \,\mathrm{d}S + \int_V b_j \dot{u}_j \,\mathrm{d}V, \tag{19}$$

and the substitution into (17) yields

$$\dot{\Pi} = -\int_{S_0} (W - b_j u_j) \dot{u}_i^0 n_i \,\mathrm{d}S. \tag{20}$$

The rate of the energy release due to spatial variation of S_0, specified by a prescribed velocity field $\dot{u}_i^0$, is $f = -\dot{\Pi}$. This represents an energetic or configurational force on the cavity (defect). Since $(W - b_j u_j) n_i = P_{ji} n_j$ over the unloaded S_0, we obtain

$$f = -\dot{\Pi} = \int_{S_0} P_{ji} \dot{u}_i^0 n_j \,\mathrm{d}S. \tag{21}$$

If the cavity translates with a unit velocity in the k-direction, then $\dot{u}_i^0$ can be replaced by δ_{ik}, and (21) gives the rate of energy release per unit cavity translation in the k-direction,

$$f_k = \int_{S_0} P_{jk} n_j \,\mathrm{d}S = J_k(S_0). \tag{22}$$

Since the cavity is unloaded, this is equal to J_k evaluated over S_0.[3]

By applying the Gauss divergence theorem to the surface $S_0 + S$ bounding a region between S_0 and any closed surface S around the cavity, and by using (5), the configurational force f_k is also equal to

$$f_k = J_k(S) + \int_V b_{j,k} u_j \,\mathrm{d}V. \tag{23}$$

If the body forces are spatially uniform, there is a conservation law $J_k = 0$ over the closed surface that does not enclose a cavity, so that $f_k = J_k(S_0) = J_k(S)$.

If the cavity transforms such that $\dot{u}_i^0 = x_i$,

$$f = \int_{S_0} P_{ji} x_i n_j \,\mathrm{d}S = M(S_0). \tag{24}$$

Alternatively, by using any other closed surface S around the cavity,

$$f = M(S) + \int_V u_i \left(\frac{4p - 3}{p} b_i + b_{i,k} x_k \right) \mathrm{d}V. \tag{25}$$

[3] If the energy momentum tensor in Eq. (5) did not include the body force term, so that $\overline{P}_{jk} = P_{jk} + b_i u_i \delta_{jk}$, the configurational force would be given by a less appealing expression, $f_k = \overline{J}_k(S_0) - \int_{S_0} b_i u_i n_k \,\mathrm{d}S$, where $\overline{J}_k = \int_{S_0} \overline{P}_{jk} n_j \,\mathrm{d}S$.

If the absence of body forces, there is a conservation law $M = 0$ over the closed surface that does not enclose a cavity, so that $f = M(S_0) = M(S)$. The conservation law $M = 0$ also holds if the body forces are homogeneous functions of degree $-(4 - 3/p)$ in spatial coordinates x_k.

If b_i are homogeneous functions of degree one in the coordinates x_k, so that $b_{i,k}x_k = b_i$, then

$$f = M(S_0) = M(S) + \left(5 - \frac{3}{p}\right)\int_V b_i u_i \, dV. \tag{26}$$

The two-dimensional counterparts of (25) and (26) are

$$f = M(C_0) = M(C) + \int_A u_i \left(\frac{3p-2}{p} b_i + b_{i,k}x_k\right) dA, \tag{27}$$

and

$$f = M(C_0) = M(C) + \left(4 - \frac{2}{p}\right)\int_A b_i u_i \, dA. \tag{28}$$

If the cavity is given a unit angular velocity around the k-axis, then $\dot{u}_i^0$ in (21) can be replaced by $-e_{kil}x_l$, and

$$f_k = -e_{kil}\int_{S_0} P_{ji} x_l n_j \, dS = -L_k(S_0), \tag{29}$$

When expressed in terms of the surface integral over S, this is

$$f_k = -L_k(S) - e_{kij}\int_V u_l(\delta_{lj} b_i + b_{l,i} x_j) \, dV. \tag{30}$$

If the absence of body forces, there is a conservation law $L_k = 0$ over the closed surface that does not enclose a cavity, so that $f_k = L_k(S_0) = L_k(S)$; Budiansky and Rice (1973).

5. Peach–Koehler force on a dislocation due to gravity load

We show in this section that the J_x integral, evaluated along a closed contour around the dislocation in a large heavy block of the material, whose specific weight is γ, is equal to the Peach–Koehler force on the dislocation. For simplicity, we first assume that the traction-free boundaries of the block are far away from the dislocation, so that near the core of the dislocation, the stress and displacement fields are as if the dislocation was in an infinite medium. The Burgers vector of the edge dislocation is b_x, and its center is at distance $h \gg b$ from the upper unloaded surface of the block (Fig. 1). The displacement is prescribed to be zero at the center

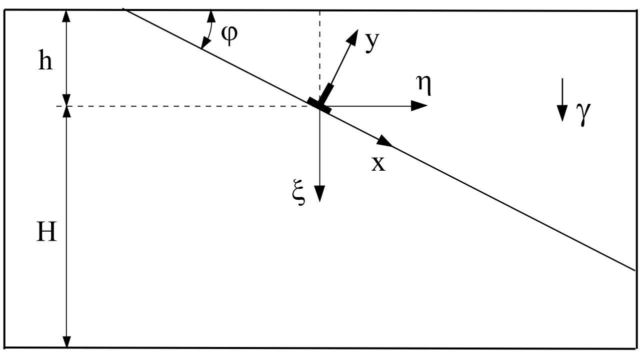

Fig. 1. An edge dislocation of Burgers vector b along the glide plane at an angle φ within a large block of the material whose specific weight is γ. The dislocation is at the depth $h \gg b$ from the unloaded upper side of the block. The lateral sides of the block are unloaded and sufficiently far from the dislocation. The weight of the block is supported by the uniform traction along the lower side of the block, at distance $H \gg h$ from the dislocation. The zero displacement is imposed at the center of the lower edge of the block, directly under the dislocation.

of the remote supporting surface at lower end ($H \gg h$). The total stress and displacement fields in the block are then the sum of the two fields, due to dislocation alone and due to gravity alone, *i.e.*,

$$u_\alpha = u_\alpha^{\rm d} + u_\alpha^{\rm g}, \quad \sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{\rm d} + \sigma_{\alpha\beta}^{\rm g}. \tag{31}$$

The stress field due to gravity is

$$\sigma_{\xi\xi}^{\rm g} = -\gamma(\xi + h), \quad \sigma_{\eta\eta}^{\rm g} = \sigma_{\xi\eta}^{\rm g} = 0, \tag{32}$$

or, with respect to the coordinate axes (x, y),

$$\sigma_{xx}^{\rm g} = \sigma_{\xi\xi}^{\rm g} \sin^2\varphi, \quad \sigma_{yy}^{\rm g} = \sigma_{\xi\xi}^{\rm g} \cos^2\varphi, \quad \sigma_{xy}^{\rm g} = -\frac{1}{2}\sigma_{\xi\xi}^{\rm g} \sin 2\varphi. \tag{33}$$

The corresponding displacement field is

$$\begin{aligned} u_\xi^{\rm g} &= \frac{(1-\nu)\gamma}{4\mu}[H^2 - \xi^2 + 2h(H-\xi)] - \frac{\nu\gamma}{4\mu}\eta^2, \\ u_\eta^{\rm g} &= \frac{\nu\gamma}{2\mu}(\xi + h)\eta, \end{aligned} \tag{34}$$

with the imposed condition $u_\xi^{\rm g}(H, 0) = 0$. The displacement components in the x, y directions are

$$u_x^{\rm g} = u_\xi^{\rm g} \sin\varphi + u_\eta^{\rm g} \cos\varphi, \quad u_y^{\rm g} = -u_\xi^{\rm g} \cos\varphi + u_\eta^{\rm g} \sin\varphi, \tag{35}$$

with

$$\xi = x\sin\varphi - y\cos\varphi, \quad \eta = x\cos\varphi + y\sin\varphi. \tag{36}$$

Finally, the displacement gradients needed for the evaluation of the J_x integral, are

$$\begin{aligned} u_{x,x}^{\rm g} &= \frac{\gamma}{2\mu}(\nu - \sin^2\varphi)(\xi + h), \\ u_{y,x}^{\rm g} &= \frac{\gamma}{2\mu}\left[\frac{1}{2}(\xi + h)\sin 2\varphi + \nu\eta\right]. \end{aligned} \tag{37}$$

The stress components, sufficiently close to the dislocation center (which is far away from the free boundary $\xi = -h$, $h \gg b$), can be approximated by the infinite medium dislocation field, which is (Hirth and Lothe, 1982)

$$\begin{aligned} \sigma_{xx}^{\rm d} &= -\frac{\mu b}{2\pi(1-\nu)} \frac{y(3x^2 + y^2)}{(x^2 + y^2)^2}, \\ \sigma_{yy}^{\rm d} &= \frac{\mu b}{2\pi(1-\nu)} \frac{y(x^2 - y^2)}{(x^2 + y^2)^2}, \\ \sigma_{xy}^{\rm d} &= \frac{\mu b}{2\pi(1-\nu)} \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}. \end{aligned} \tag{38}$$

The displacement components (with the displacement discontinuity along the $+x$ axis), are

$$\begin{aligned} u_x^{\rm d} &= \frac{b}{2\pi}\left[\tan^{-1}\frac{y}{x} + \frac{1}{2(1-\nu)}\frac{xy}{x^2 + y^2}\right], \\ u_y^{\rm d} &= -\frac{b}{2\pi}\frac{1}{4(1-\nu)}\left[(1-2\nu)\ln\frac{x^2 + y^2}{b^2} + \frac{x^2 - y^2}{x^2 + y^2}\right]. \end{aligned} \tag{39}$$

The displacement gradients, needed for the evaluation of the J_x integral, are

$$u_{x,x}^{\mathrm{d}} = \frac{b}{2\pi}\left[-\frac{y}{x^2+y^2} + \frac{1}{2(1-\nu)}\frac{y(y^2-x^2)}{(x^2+y^2)^2}\right],$$
$$u_{y,x}^{\mathrm{d}} = -\frac{b}{2\pi}\frac{1}{2(1-\nu)}\left[(1-2\nu)\frac{x}{x^2+y^2} + \frac{2xy^2}{(x^2+y^2)^2}\right]. \tag{40}$$

The J_x integral is defined by

$$J_x = \int_C (W - b_x u_x - b_y u_y) n_x \, \mathrm{d}C - \int_C (T_x u_{x,x} + T_y u_{y,x}) \, \mathrm{d}C. \tag{41}$$

The strain energy density is

$$W = \frac{1}{2\mu}\left[\frac{1-\nu}{2}(\sigma_{xx}+\sigma_{yy})^2 - \sigma_{xx}\sigma_{yy} + \sigma_{xy}^2\right]. \tag{42}$$

By using the superposition of the dislocation and gravity fields, this can be expressed as

$$W = W^{\mathrm{d}} + W^{\mathrm{g}} + W^{\mathrm{d,g}}, \tag{43}$$

where

$$W^{\mathrm{d}} = \frac{1}{2\mu}\left[\frac{1-\nu}{2}(\sigma_{xx}^{\mathrm{d}}+\sigma_{yy}^{\mathrm{d}})^2 - \sigma_{xx}^{\mathrm{d}}\sigma_{yy}^{\mathrm{d}} + \sigma_{xy}^{\mathrm{d}2}\right],$$
$$W^{\mathrm{g}} = \frac{1}{2\mu}\left[\frac{1-\nu}{2}(\sigma_{xx}^{\mathrm{g}}+\sigma_{yy}^{\mathrm{g}})^2 - \sigma_{xx}^{\mathrm{g}}\sigma_{yy}^{\mathrm{g}} + \sigma_{xy}^{\mathrm{g}2}\right], \tag{44}$$

and

$$W^{\mathrm{d,g}} = \frac{1}{2\mu}\left[(1-\nu)(\sigma_{xx}^{\mathrm{d}}+\sigma_{yy}^{\mathrm{d}})(\sigma_{xx}^{\mathrm{g}}+\sigma_{yy}^{\mathrm{g}}) - (\sigma_{xx}^{\mathrm{d}}\sigma_{yy}^{\mathrm{g}} + \sigma_{xx}^{\mathrm{g}}\sigma_{yy}^{\mathrm{d}}) + 2\sigma_{xy}^{\mathrm{d}}\sigma_{xy}^{\mathrm{g}}\right]. \tag{45}$$

Similarly, we can write

$$T_x u_{x,x} + T_y u_{y,x} = (T_x^{\mathrm{d}} u_{x,x}^{\mathrm{d}} + T_y^{\mathrm{d}} u_{y,x}^{\mathrm{d}}) + (T_x^{\mathrm{g}} u_{x,x}^{\mathrm{g}} + T_y^{\mathrm{g}} u_{y,x}^{\mathrm{g}}) + (T_x^{\mathrm{d}} u_{x,x}^{\mathrm{g}} + T_y^{\mathrm{d}} u_{y,x}^{\mathrm{g}} + T_x^{\mathrm{g}} u_{x,x}^{\mathrm{d}} + T_y^{\mathrm{g}} u_{y,x}^{\mathrm{d}}), \tag{46}$$

and

$$b_x u_x + b_y u_y = (b_x u_x^{\mathrm{d}} + b_y u_y^{\mathrm{d}}) + (b_x u_x^{\mathrm{g}} + b_y u_y^{\mathrm{g}}). \tag{47}$$

Consequently, the J_x integral can be expressed as the sum of three contributions,

$$J_x = J_x^{\mathrm{d}} + J_x^{\mathrm{g}} + J_x^{\mathrm{d,g}}, \tag{48}$$

which are evaluated as follows. First, along a closed contour near the dislocation center, we have

$$J_x^{\mathrm{d}} = \int_C W^{\mathrm{d}} n_x \, \mathrm{d}C - \int_C (T_x^{\mathrm{d}} u_{x,x}^{\mathrm{d}} + T_y^{\mathrm{d}} u_{y,x}^{\mathrm{d}}) \, \mathrm{d}C = 0, \tag{49}$$

which is the well-known result for the dislocation in an infinite medium, where the dislocation does not exert a force on itself. Secondly,

$$J_x^{\mathrm{g}} = \int_C (W^{\mathrm{g}} - b_x u_x^{\mathrm{g}} - b_y u_y^{\mathrm{g}}) n_x \, \mathrm{d}C - \int_C (T_x^{\mathrm{g}} u_{x,x}^{\mathrm{g}} + T_y^{\mathrm{g}} u_{y,x}^{\mathrm{g}}) \, \mathrm{d}C = 0, \tag{50}$$

because the gravity field is uniform, with the corresponding continuous stress and displacement fields, so that $J_x^{\mathrm{g}} = 0$ around any closed contour. The remaining contribution to the J_x integral is

$$J_x^{\mathrm{d,g}} = \int_C (W^{\mathrm{d,g}} - b_x u_x^{\mathrm{d}} - b_y u_y^{\mathrm{d}}) n_x \, \mathrm{d}C - \int_C (T_x^{\mathrm{d}} u_{x,x}^{\mathrm{g}} + T_y^{\mathrm{d}} u_{y,x}^{\mathrm{g}} + T_x^{\mathrm{g}} u_{x,x}^{\mathrm{d}} + T_y^{\mathrm{g}} u_{y,x}^{\mathrm{d}}) \, \mathrm{d}C. \tag{51}$$

We calculate the $J_x^{\mathrm{d,g}}$ integral around the rectangular contour shown in Fig. 2, whose dimensions are $2a \times 2a$, where $a \ll h$. This contour is selected to facilitate the integration; any other contour close to the dislocation

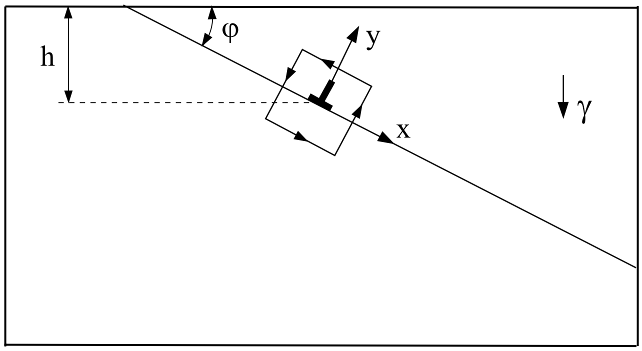

Fig. 2. A rectangular contour of dimensions $2a \times 2a$ ($a \ll h$), symmetrically positioned around the dislocation, used to evaluate the J_x integral, or the gravity-induced force on the dislocation.

can be used, since body force γ is constant, and thus there is a path-independence in the evaluation of the J_x integral. By taking the advantage in the integration process of the even or odd nature of the involved stress and displacement fields, it readily follows that

$$\begin{aligned}
&\int_C W^{\mathrm{d,g}} n_x \, \mathrm{d}C = \frac{\gamma b h}{2\pi(1-\nu)} \sin 2\varphi, \\
&\int_C (b_x u_x^{\mathrm{d}} + b_y u_y^{\mathrm{d}}) n_x \, \mathrm{d}C = 0, \\
&\int_C T_x^{\mathrm{d}} u_{x,x}^{\mathrm{g}} \, \mathrm{d}C = \int_C T_y^{\mathrm{d}} u_{y,x}^{\mathrm{g}} \, \mathrm{d}C = 0, \\
&\int_C T_x^{\mathrm{g}} u_{x,x}^{\mathrm{d}} \, \mathrm{d}C = \int_C T_y^{\mathrm{g}} u_{y,x}^{\mathrm{d}} \, \mathrm{d}C = -\frac{\gamma b h}{4\pi} \left(\pi - \frac{1}{1-\nu} \right) \sin 2\varphi.
\end{aligned} \tag{52}$$

Note that the values of the above nonvanishing integrals do not depend on the actual size of a, provided that $a \ll h$. By adding all contributions, the J_x integral is found to be

$$J_x = \frac{1}{2} \gamma b h \sin 2\varphi = \sigma_{xy}^{\mathrm{g}}(0,0) b, \quad \sigma_{xy}^{\mathrm{g}}(0,0) = \frac{1}{2} \gamma h \sin 2\varphi, \tag{53}$$

which is indeed the Peach–Koehler force on the dislocation due to the gravity field.

If the dislocation is close to the boundary, the Head's solution for the dislocation near a free boundary can be used, instead of that for the dislocation in an infinite medium (Lubarda and Kouris, 1996; Lubarda, 1997; Asaro and Lubarda, 2006). The resulting J_x integral in this case is

$$J_x = \sigma_{xy}^{\mathrm{d+g}}(0,0) b, \quad \sigma_{xy}^{\mathrm{d+g}}(0,0) = -\frac{\mu}{4\pi(1-\nu)} \frac{b \sin \varphi}{h} + \frac{1}{2} \gamma h \sin 2\varphi. \tag{54}$$

The nonsingular dislocation field contribution to the glide force on the dislocation is due to the image field associated with the free boundary at distance h from the dislocation. This is evidently equal to zero if the dislocation Burgers vector is parallel to the free surface. For a nonzero value of φ, the dislocation is an equilibrium position when $J_x = 0$, which gives

$$h = \left[\frac{\mu b}{4\pi(1-\nu)\gamma \cos \varphi} \right]^{1/2}, \quad 0 < \varphi < \pi/2. \tag{55}$$

If the lateral sides of the large block are constrained,[4] with so that $u_\eta = 0$ throughout the block, the shear stress at the center of the dislocation, due to the gravity load, is

[4] Various assumptions about the state of stress underground (geological stress systems) are discussed by Jaeger and Cook (1976).

$$\sigma^{g}_{xy}(0,0) = \frac{1-2\nu}{2(1-\nu)}\gamma h \sin 2\varphi. \tag{56}$$

In this case the dislocation is in an equilibrium position at the distance

$$h = \left[\frac{\mu b}{4\pi(1-2\nu)\gamma\cos\varphi}\right]^{1/2}, \quad 0 < \varphi < \pi/2. \tag{57}$$

6. Conclusion

A modified energy momentum tensor in the presence of body forces is introduced, which includes a work term due to body forces, and which yields simple expressions for the energetic forces associated with a defect motion within the material. The nonconserved J, M, and L integrals are related to the volume integrals whose integrands depend on the body forces and their gradients. The configurational forces, associated with particular defect motions, are expressed in terms of the integrals evaluated over the unloaded surface of the defect. When the body forces are spatially uniform, the conservation law $J_k = 0$ holds for both two-dimensional and three-dimensional problems. The conservation law $M = 0$ holds if the body forces are absent, or if they are homogeneous functions of particular degree in spatial coordinates. The derived J integral is applied to evaluate the Peach–Koehler force on an edge dislocation residing on an inclined slip plane within a large block of the material due to its own weight. It is shown that this is the shear stress along the slip plane at the center of the dislocation, due to the gravity load only, multiplied by the magnitude of the dislocation Burgers vector. The equilibrium position of the dislocation is determined for different boundary conditions that are of interest in the analysis of geological stress systems. Other applications of the J integral in the presence of body forces are also possible, some of which have already been reported in the literature, *e.g.*, in the study of the propagation of a concentrated shear band in heavily over-consolidated clays (Palmer and Rice, 1973), and for the computation of the energy release rate and the direction of crack growth in a thin film bonded to an elastic or viscous layer, where an effective body force in the film is due to deformation of the underlying substrate (Huang et al., 2002; Liang et al., 2003). It is also noted that, while in the problems without body forces the configurational force associated with a self similar defect expansion can be calculated by evaluating the M integral around any closed surface surrounding the defect, in the problems with body forces there is an extra term given by the volume integral on the right-hand side of (25), which depends on the displacement field in the region between the bounding surface of the defect and the selected surface surrounding the defect. Consequently, the short-cut calculations of the stress intensity factors in crack problems based on the evaluation of the M integral around a conveniently selected contour, without solving the corresponding boundary value problem (Eshelby, 1975; Freund, 1978; Rice, 1985), is in general not possible for problems with body forces. The presented analysis of the nonconserved integrals in the presence of body forces can also be extended to dual or complementary integrals, which are related to release rates of the complementary potential energy (Lubarda and Markenscoff, 2007a,b, 2008).

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