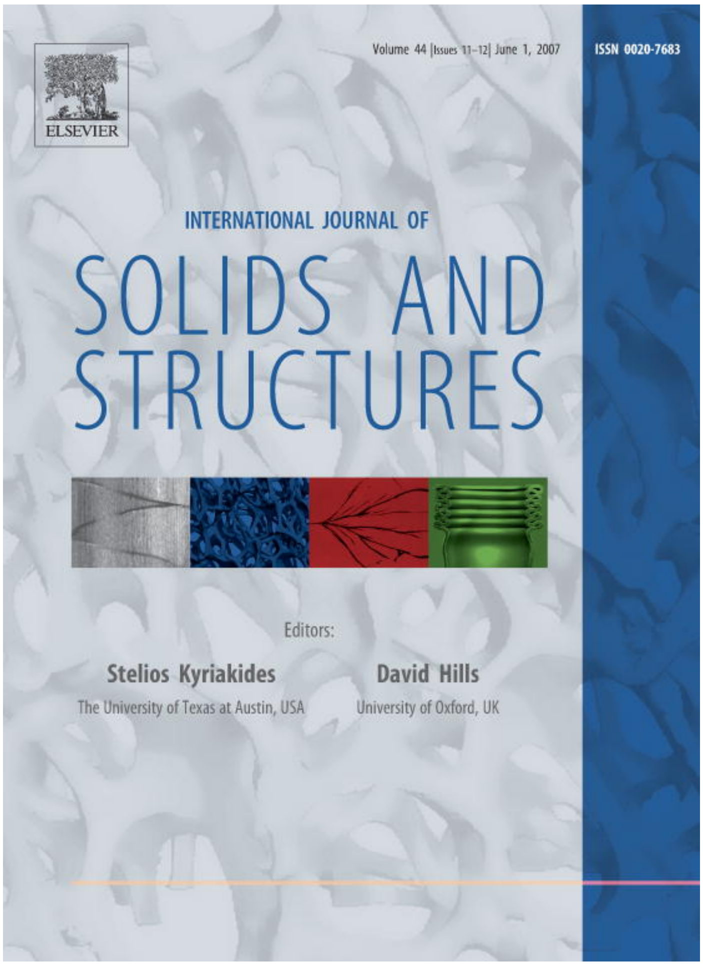
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Dual conservation integrals and energy release rates

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Abstract

Simple derivation of the dual conservation integrals in small strain elasticity is presented, without the aid of Noether's theorem on invariant variational principles. The derived integrals are related to the release rates of the potential and complementary potential energy associated with the defect motion. The analysis corrects the errors in earlier derivation of the relationship between the dual integrals and the release rates of the complementary potential energy. Selected examples in plane and anti-plane strain illustrate the calculation of dual integrals and their application. It is shown that the evaluation of dual integrals is of similar complexity to that of classical integrals, so that either can be used to determine the stress intensity factors or the forces between defects, without solving the corresponding boundary value problems. An advantage of combining the two calculations is discussed.

Keywords: Complementary energy; Conservation integrals; Dual integrals; Energy momentum tensor; Potential energy

1. Introduction

In the wake of Eshelby's (1951, 1956) work on the energy momentum tensor and configurational forces on moving defects, a significant amount of research has been devoted to the analysis of conservation integrals in infinitesimal and finite strain elasticity. The derived conservation integrals are expressed in terms of spatial gradients of displacements, and are related to the release rates of the potential energy associated with defect motion (Knowles and Sternberg, 1972; Budiansky and Rice, 1973). The dual or complementary conservation integrals are related to the release rates of the complementary potential energy, and are expressed in terms of spatial gradients of stresses. Bui (1973, 1974) introduced a dual $\hat{J}$ integral, and compared it with the original Rice's (1968) J integral of fracture mechanics. Shortly after Bui, and in the context of elastodynamics, Carlsson (1974) also studied the structure and physical significance of dual conservation integrals. Sun (1985) and Li (1988) derived the expressions for the dual $\hat{J}_k, \hat{L}_k$ and $\hat{M}$ integrals, but they related them to the release rates of the complementary potential energy in an incorrect way. The reason for this was that the original analysis

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of Budiansky and Rice (1973), based on the potential energy, required a more careful extension in the case of the complementary energy considerations, with the appropriate incorporation of the rates of stress and the change of the surface orientation. This is discussed in Section 5 of this paper. In Sections 2–4 we construct the dual conservation integrals (in the absence of body forces) by exploring the divergence free property of the stress and energy momentum tensors, without the aid of Noether's theorem on invariant variational principles. The plain strain and the anti-plane strain versions of the dual conservation integrals are given in Appendix A. Selected examples illustrate the calculation of the dual integrals and their applications. In many cases the evaluation of dual integrals is of similar complexity to that of classical integrals, and thus either can be used in specific problems to, for example, determine the stress intensity factors or the forces between defects, without solving the corresponding boundary value problems. An advantage of combining the two calculations is discussed.

A brief outline of the basic concepts of infinitesimal elasticity used in the body of this paper is a follows. Small deformations of elastic material are geometrically described by the displacement vector whose rectangular components are u_i. The surface forces are in equilibrium with the symmetric Cauchy stress σ_{ij}, such that $T_i = n_j\sigma_{ji}$, where n_j are the components of the unit vector orthogonal to the surface element under consideration. In the absence of body forces, the integral conditions of equilibrium are

$$\int_S T_i \, dS = 0, \quad \int_S e_{ijk} x_j T_k \, dS = 0, \tag{1}$$

where e_{ijk} are the components of the permutation tensor. The corresponding differential equations of equilibrium are

$$\sigma_{ji,j} = 0, \quad \sigma_{ij} = \sigma_{ji}. \tag{2}$$

The elastic strain energy is $W = W(\epsilon_{ij})$, with the complementary strain energy

$$\Phi(\sigma_{ij}) = \sigma_{ij}\epsilon_{ij} - W(\epsilon_{ij}), \tag{3}$$

where the small strain tensor ϵ_{ij} is the symmetric part of the displacement gradient

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{4}$$

The stress–strain relations are

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}, \quad \epsilon_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}}. \tag{5}$$

2. Dual *J* integrals

A simple derivation of the dual J integrals for small strain nonlinear elasticity is as follows. A spatial gradient of the strain energy $W = W(\epsilon_{ij})$ is

$$W_{,k} = \frac{\partial W}{\partial \epsilon_{ij}} \epsilon_{ij,k} = \sigma_{ij}\epsilon_{ij,k}. \tag{6}$$

By using (4), this can be rewritten as

$$W_{,j}\delta_{jk} - \sigma_{ji}u_{i,jk} = 0, \tag{7}$$

which, in view of equilibrium conditions (2), reduces to

$$(W\delta_{jk} - \sigma_{ji}u_{i,k})_{,j} = 0. \tag{8}$$

This defines a divergence-free energy momentum tensor, or Eshelby stress,

$$P_{jk} = W\delta_{jk} - \sigma_{ji}u_{i,k}, \quad P_{jk,j} = 0. \tag{9}$$

Thus the conservation law

$$J_k = \int_S P_{jk} n_j \, dS = \int_S (W n_k - T_j u_{j,k}) \, dS = 0 \tag{10}$$

for any closed surface S which does not enclose a singularity or a defect. This result is originally due to Eshelby (1951, 1956).

In a dual analysis, we consider a spatial gradient of the complementary strain energy $\Phi = \Phi(\sigma_{ij})$, which is

$$\Phi_{,k} = \frac{\partial \Phi}{\partial \sigma_{ij}} \sigma_{ij,k} = \epsilon_{ij} \sigma_{ij,k}. \tag{11}$$

In view of the symmetry of ϵ_{ij}, this becomes

$$\Phi_{,j} \delta_{jk} - u_{i,j} \sigma_{ji,k} = 0. \tag{12}$$

Having in mind the equilibrium conditions, (11) can be recast as

$$(\Phi \delta_{jk} - u_i \sigma_{ji,k})_{,j} = 0, \tag{13}$$

which defines a divergence-free dual energy momentum tensor

$$\hat{P}_{jk} = \Phi \delta_{jk} - u_i \sigma_{ji,k}, \quad \hat{P}_{jk,j} = 0, \tag{14}$$

and a dual conservation law

$$\hat{J}_k = \int_S \hat{P}_{jk} n_j \, dS = \int_S (\Phi n_k - u_j \sigma_{ij,k} n_i) \, dS = 0 \tag{15}$$

for any closed surface S that does not embrace a singularity or a defect.

While J_k in (10) is expressed in terms of spatial gradients of displacement, $\hat{J}_k$ in (15) is expressed in terms of the stress gradients. A dual $\hat{J}_k$ conservation integral is originally due to Bui (1973, 1974) (for planar elasticity) and Carlsson (1974) (for three-dimensional elasticity). It is noted that

$$\begin{aligned} P_{jk} + \hat{P}_{jk} &= (W + \Phi)\delta_{jk} - (\sigma_{ji} u_i)_{,k}, \\ P_{kk} &= W, \quad \hat{P}_{kk} = 3\Phi, \end{aligned} \tag{16}$$

the first of these being also noted by Li and Gupta (2006).

If the strain energy W is a homogeneous function of degree r in strain components ($1 < r \leqslant 2$), the complementary strain energy Φ is a homogeneous function of degree $s = r/(r-1)$ in stress components ($s \geqslant 2$), and $\Phi = rW/s$. In this case it readily follows that

$$rJ_k - s\hat{J}_k = \int_S (s u_j \sigma_{ij,k} - r \sigma_{ij} u_{j,k}) n_i \, dS. \tag{17}$$

As shown in Section 5, if S encloses a defect then $\hat{J}_k = -J_k \neq 0$ and

$$J_k = \int_S \left(\frac{1}{r} u_j \sigma_{ij,k} - \frac{1}{s} \sigma_{ij} u_{j,k} \right) n_i \, dS. \tag{18}$$

For linear elasticity $r = s = 2$, this reduces to Bui's (1994) reciprocal relation. The application of this representation for J_k is discussed in Appendix A.

3. Dual M integrals

Let the strain energy $W = W(\epsilon_{ij})$ be a homogeneous function of degree r in strain components, so that

$$W = \frac{1}{r} \sigma_{jk} \epsilon_{jk}. \tag{19}$$

The energy momentum tensor (9), being a divergence-free tensor, satisfies the equation

$$(P_{jk} x_k)_{,j} - P_{kk} = 0. \tag{20}$$

In view of (19), we have

$$P_{kk} = \frac{3-r}{r}\sigma_{jk}u_{k,j}, \tag{21}$$

and the substitution into (20) gives

$$\left(P_{jk}x_k - \frac{3-r}{r}\sigma_{jk}u_k\right)_{,j} = 0. \tag{22}$$

Upon the application of the Gauss divergence theorem, this yields the M conservation law

$$M = \int_S \left(P_{jk}x_k - \frac{3-r}{r}\sigma_{jk}u_k\right)n_j\,\mathrm{d}S = 0 \tag{23}$$

for any closed surface that does not embrace a singularity or a defect.

A dual energy momentum tensor (14) is also divergence-free tensor and thus it satisfies the equation

$$(\hat{P}_{jk}x_k)_{,j} - \hat{P}_{kk} = 0. \tag{24}$$

The complementary strain energy, corresponding to (19), is

$$\Phi = \frac{1}{s}\sigma_{jk}\epsilon_{jk}, \quad s = \frac{r}{r-1}, \tag{25}$$

so that

$$\hat{P}_{kk} = \frac{3}{s}(u_k\sigma_{jk})_{,j}. \tag{26}$$

The substitution into (24) gives

$$\left(\hat{P}_{jk}x_k - \frac{3}{s}u_k\sigma_{jk}\right)_{,j} = 0. \tag{27}$$

Consequently, there is a dual $\hat{M}$ conservation law

$$\hat{M} = \int_S \left(\hat{P}_{jk}x_k - \frac{3}{s}u_k\sigma_{jk}\right)n_j\,\mathrm{d}S = 0 \tag{28}$$

for any closed surface that does not embrace a singularity or a defect. The duality is such that M is expressed in terms of spatial gradients of displacements, while $\hat{M}$ is in terms of the stress gradients. By using a different approach, $\hat{M}$ integral was derived in this form by Sun (1985), and for linear elasticity ($s = 2$) by Carlsson (1974).

4. Dual L integrals

For isotropic elasticity the principal directions of stress and strain coincide, so that $\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}$ is a commutative product and thus a symmetric tensor. Consequently,

$$c_k = 2e_{kij}\sigma_{il}\epsilon_{lj} = e_{kij}(\sigma_{il}u_{l,j} + \sigma_{li}u_{j,l}) = 0. \tag{29}$$

In view of (9), c_k can be rewritten as

$$c_k = e_{kij}(P_{ji} + \sigma_{li}u_{j,l}). \tag{30}$$

In the absence of body forces, the energy momentum and stress tensors are both divergence-free ($P_{li,l} = 0$, $\sigma_{li,l} = 0$), and thus

$$c_k = d_{kl,l}, \quad d_{kl} = e_{kij}(P_{li}x_j + \sigma_{li}u_j). \tag{31}$$

This establishes a conservation law

$$L_k = e_{kij}\int_S (P_{li}x_j + \sigma_{li}u_j)n_l\,\mathrm{d}S = 0 \tag{32}$$

for any closed surface S that does not embrace a singularity or a defect. A derivation of (32), as well as (10) and (23), based on Noether's theorem on invariant variational principles was first given by Günther (1962), and Knowles and Sternberg (1972). To derive a dual $\hat{L}$ integral, introduce the components of a dual vector $\hat{c}_k$, defined by $\hat{c}_k + c_k = 0$, so that, from (29),

$$\hat{c}_k = e_{kij}(u_{l,i}\sigma_{jl} + u_{i,l}\sigma_{lj}). \tag{33}$$

Incorporating a dual energy momentum tensor (14), $\hat{c}_k$ can be rewritten as

$$\hat{c}_k = e_{kij}(\hat{P}_{ji} + u_{i,l}\sigma_{lj} + u_{l,i}\sigma_{jl} + u_l\sigma_{jl,i}). \tag{34}$$

Since a dual energy momentum tensor and the stress tensor are divergence-free ($\hat{P}_{li,l} = 0$, $\sigma_{lj,l} = 0$), (34) becomes

$$\hat{c}_k = \hat{d}_{kl,l}, \quad \hat{d}_{kl} = e_{kij}(\hat{P}_{li}x_j + u_i\sigma_{lj} + \delta_{il}u_r\sigma_{jr}). \tag{35}$$

Thus, there is a dual conservation law

$$\hat{L}_k = e_{kij}\int_S (\hat{P}_{li}x_j + u_i\sigma_{lj} + \delta_{il}u_r\sigma_{jr})n_l\,\mathrm{d}S = 0 \tag{36}$$

for any closed surface that does not embrace a singularity or a defect. An equivalent form of this integral was first obtained by Sun (1985).

5. Dual integrals and energy release rates

The physical interpretation of the dual conservation integrals follows from the consideration of the release rates of the complementary potential energy, by extending the analysis of Budiansky and Rice (1973) on the relationship between the conservation integrals and the release rates of the potential energy. The Budiansky and Rice analysis is summarized is Section 5.1, while the release rates of the complementary potential energy are considered in Section 5.2.

5.1. Release rates of potential energy

Consider a body of volume V loaded by surface tractions $T_i = \bar{T}_i$ over the portion S_T of its external surface S. The displacements $u_i = \bar{u}_i$ are prescribed over the remaining part S_u. Suppose that within a body there is an unloaded cavity (or crack) of the bounding surface S_0. The potential energy of such body is

$$\Pi = \int_V W\,\mathrm{d}V - \int_{S_T} \bar{T}_i u_i\,\mathrm{d}S. \tag{37}$$

Without changing the boundary conditions on S, the rate of change of the potential energy associated with the spatial variation of the surface S_0, described by its velocity field $\dot{u}_i^0$, is

$$\dot{\Pi} = \int_V \dot{W}\,\mathrm{d}V - \int_{S_0} W\dot{u}_i^0 n_i\,\mathrm{d}S - \int_{S_T} \bar{T}_i\dot{u}_i\,\mathrm{d}S, \tag{38}$$

where $\dot{u}_i$ is the associated velocity field within $V(t)$ due to imposed velocity $\dot{u}_i^0$. The second integral on the right-hand side follows from the Reynolds transport theorem, where n_i is the unit normal to S_0 directed out of the cavity (*i.e.*, into the material). This choice for the positive direction of the unit normal to S_0 is selected for the later convenience; if the opposite choice is made only the sign in front of the second integral on the right-hand side of (38) is changed. The positive direction of the unit normal to the external surface S is as usual, in the direction of the outward normal to S. Assuming that $\dot{u}_i$ is a kinematically admissible field within $V(t)$, we have

$$\dot{W} = \sigma_{ij}\dot{\epsilon}_{ij} = (\sigma_{ij}\dot{u}_j)_{,i}. \tag{39}$$

Since S_0 is unloaded and $\dot{u}_j = 0$ on S_u, the application of the Gauss divergence theorem gives

$$\int_V \dot{W}\,\mathrm{d}V = \int_{S_T} \bar{T}_j \dot{u}_j\,\mathrm{d}S, \tag{40}$$

and the substitution into (38) yields

$$\dot{\Pi} = -\int_{S_0} W \dot{u}_i^0 n_i\,\mathrm{d}S. \tag{41}$$

The rate of energy release due to spatial variation of S_0, specified by a prescribed velocity field $\dot{u}_i^0$, is $f = -\dot{\Pi}$. This represents the energetic or configurational force on the cavity (defect). Since $Wn_i = P_{ji}n_j$ over the unloaded S_0, the rate of the strain energy is

$$f = -\dot{\Pi} = \int_{S_0} P_{ji} \dot{u}_i^0 n_j\,\mathrm{d}S. \tag{42}$$

If the cavity merely translates with a unit velocity in the k-direction, then $\dot{u}_i^0$ can be replaced by δ_{ik}, and (42) gives the rate of energy release per unit cavity translation in the k-direction,

$$f_k = \int_{S_0} P_{jk} n_j\,\mathrm{d}S = J_k(S_0). \tag{43}$$

Since the cavity is unloaded, this is equal to J_k evaluated over S_0. By the conservation law $J_k = 0$ applied to the surface $S_0 + S$ bounding a region between S_0 and any closed surface S surrounding the cavity, the configurational force is also equal to J_k evaluated over S, *i.e.*, $f_k = J_k(S_0) = J_k(S)$.

If the cavity expands uniformly such that $\dot{u}_i^0 = x_i$, then[1]

$$f = \int_{S_0} P_{ji} x_i n_j\,\mathrm{d}S = M(S_0). \tag{44}$$

By the conservation law $M = 0$ applied over $S_0 + S$, the configurational force is also equal to M evaluated over S, *i.e.*, $f = M(S_0) = M(S)$.

If the cavity is given a unit angular velocity around the k-axis, then $\dot{u}_i^0$ in (42) can be replaced by $-e_{kil}x_l$, and

$$f_k = -e_{kil} \int_{S_0} P_{ji} x_l n_j\,\mathrm{d}S = -L_k(S_0). \tag{45}$$

By the conservation law $L_k = 0$ applied over $S_0 + S$, the configurational force is also equal to $-L_k$ evaluated over S, *i.e.*, $f_k = -L_k(S_0) = -L_k(S)$.

5.2. Release rates of complementary potential energy

The complementary potential energy is defined by

$$\Omega = \int_V \Phi\,\mathrm{d}V - \int_{S_u} \bar{u}_i T_i\,\mathrm{d}S \tag{46}$$

such that $\Pi + \Omega = 0$. Indeed, since S_0 is unloaded,

$$\Pi + \Omega = \int_V (W + \Phi)\,\mathrm{d}V - \int_S T_i u_i\,\mathrm{d}S = 0, \tag{47}$$

[1] A transformation $\dot{u}_i^0 = x_i$ in general includes a self-similar expansion around the centroid of the cavity and a translation, unless the coordinate origin is placed at the centroid of the cavity's interior, in which case it gives rise to a uniform expansion around the centroid only. The dependence of the M integral on the coordinate origin has been effectively used in fracture mechanics to determine various stress intensity factors, *e.g.*, Eshelby (1975), Freund (1978), Rice (1985).

which follows from $W+\Phi=\sigma_{ij}\epsilon_{ij}$ with the help of equilibrium conditions (2), geometric relationship (4), and the Gauss divergence theorem. The rate of the complementary potential energy associated with spatial variation of the cavity surface due to its velocity field $\dot{u}_i^0$ is

$$\dot{\Omega}=\int_V \dot{\Phi}\,\mathrm{d}V-\int_{S_0}\Phi\dot{u}_i^0 n_i\,\mathrm{d}S-\int_{S_u}\bar{u}_i\dot{T}_i\,\mathrm{d}S, \tag{48}$$

where $\dot{T}_i$ is the induced loading rate on S_u due to infinitesimal motion of S_0. In a geometrically linear theory, the change of S due to $\dot{u}_i^0$ is ignored. Assuming the stress rate field within $V(t)$ is statically admissible ($\dot{\sigma}_{ji,j}=0$), the rate of the complementary strain energy is

$$\dot{\Phi}=\epsilon_{ij}\dot{\sigma}_{ij}=(u_j\dot{\sigma}_{ij})_{,i}. \tag{49}$$

The stress rate $\dot{\sigma}_{ij}$ is the stress rate at fixed points in space, *i.e.*, a local (nonconvected) stress rate. Thus,

$$\int_V \dot{\Phi}\,\mathrm{d}V=\int_S u_j\dot{\sigma}_{ij}n_i\,\mathrm{d}S-\int_{S_0}u_j\dot{\sigma}_{ij}n_i\,\mathrm{d}S. \tag{50}$$

Since for a geometrically linear theory, $\dot{\sigma}_{ij}n_i=\dot{T}_j$ on S ($\dot{T}_j$ being equal to zero on S_T), (50) is rewritten as

$$\int_V \dot{\Phi}\,\mathrm{d}V=\int_{S_u}\bar{u}_j\dot{T}_j\,\mathrm{d}S-\int_{S_0}u_j\dot{\sigma}_{ij}n_i\,\mathrm{d}S, \tag{51}$$

and the substitution into (48) gives

$$\dot{\Omega}=-\int_{S_0}(\Phi\dot{u}_i^0+u_j\dot{\sigma}_{ij})n_i\,\mathrm{d}S. \tag{52}$$

The surface of the cavity is unloaded, so that its traction $T_j=n_i\sigma_{ij}$ remains zero throughout the motion. Thus,

$$\frac{\mathrm{d}T_j}{\mathrm{d}t}=\frac{\mathrm{d}n_i}{\mathrm{d}t}\sigma_{ij}+n_i\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}t}=0, \tag{53}$$

where $\mathrm{d}/\mathrm{d}t$ designates the material time derivative, following the material particle. Expressing the material derivative of stress as the sum of its local ($\dot{\sigma}_{ij}$) and convected ($\sigma_{ij,l}\dot{u}_l^0$) part, (53) gives

$$n_i\dot{\sigma}_{ij}=-\frac{\mathrm{d}n_i}{\mathrm{d}t}\sigma_{ij}-n_i\sigma_{ij,l}\dot{u}_l^0. \tag{54}$$

If the cavity translates or expands in a self-similar manner, then $\mathrm{d}n_i/\mathrm{d}t=0$ and

$$n_i\dot{\sigma}_{ij}=-n_i\sigma_{ij,l}\dot{u}_l^0. \tag{55}$$

This type of expression is well-known from fracture mechanics and the analysis of crack advance (Moran and Shih, 1987). When (55) is introduced in (52), there follows

$$\dot{\Omega}=\int_{S_0}(-\Phi\delta_{il}+u_j\sigma_{ij,l})n_i\dot{u}_l^0\,\mathrm{d}S=-\int_{S_0}\hat{P}_{il}n_i\dot{u}_l^0\,\mathrm{d}S, \tag{56}$$

which is the correct expression for the rate of the complementary potential energy. The analysis based on the rate of the complementary potential energy presented in Section 4 of Sun (1985) is incorrect, because $\sigma_{ij,l}n_i$ is not equal to zero over S_0, as there tacitly assumed; see his equation (4.3). A similar remark applies to the analysis in Section 6 of Li (1988).

Recalling from (47) that $\Pi+\Omega=0$, the release rate of the complementary potential energy due to spatial variation of S_0 is

$$f=-\dot{\Pi}=\dot{\Omega}=-\int_{S_0}\hat{P}_{il}n_i\dot{u}_l^0\,\mathrm{d}S. \tag{57}$$

This is a dual expression to (42), in the case of translation or self-similar expansion of the cavity.

If the cavity translates with a unit velocity in the k-direction, then $\dot{u}_l^0$ is replaced by δ_{kl}, and (57) gives the release rate of the complementary potential energy per unit cavity translation in the k-direction,

$$f_k = -\int_{S_0} \hat{P}_{ik} n_i \, \mathrm{d}S = -\hat{J}_k(S_0). \tag{58}$$

By the conservation law $\hat{J}_k = 0$ applied over $S_0 + S$, the configurational force is also equal to $-\hat{J}_k$ evaluated over S, *i.e.*, $f_k = -\hat{J}_k(S_0) = \hat{J}_k(S)$. Furthermore, by comparing with (43), we conclude that $\hat{J}_k = -J_k$ over S_0, or any other closed surface surrounding the cavity.

If the cavity expands uniformly such that $\dot{u}_l^0 = x_l$, the energy release rate is

$$f = -\int_{S_0} \hat{P}_{il} n_i x_l \, \mathrm{d}S = -\hat{M}(S_0). \tag{59}$$

By the conservation law $\hat{M} = 0$ applied over $S_0 + S$, the configurational force is also equal to $-\hat{M}$ evaluated over S, *i.e.*, $f = -\hat{M}(S_0) = \hat{M}(S)$. Furthermore, by comparing with (44), we conclude that $\hat{M} = -M$ over any closed surface surrounding the cavity.

If the cavity rotates within the material, then

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -n_j Q_{ji}, \tag{60}$$

where Q_{ji} are the components of antisymmetric spin matrix, and $\dot{u}_i^0 = Q_{ij} x_j$. When this is introduced into (54), there follows[2]

$$n_i \dot{\sigma}_{ij} = (\delta_{ik}\sigma_{lj} - \sigma_{ij,k} x_l) n_i Q_{kl}, \tag{61}$$

and (52) gives

$$f = \dot{\Omega} = -\int_{S_0} (\hat{P}_{ik} x_l + \delta_{ik} u_j \sigma_{lj}) n_i Q_{kl} \, \mathrm{d}S. \tag{62}$$

If the spin is of unit magnitude and about the k-axis, then $Q_{ij} = -e_{ijk}$ and from (62) the corresponding configurational force is

$$f_k = e_{ijk} \int_{S_0} (\hat{P}_{li} x_j + \delta_{li} u_r \sigma_{jr}) n_l \, \mathrm{d}S. \tag{63}$$

When this is compared with $\hat{L}_k(S_0)$ from (36), it follows that

$$f_k = \hat{L}_k, \tag{64}$$

where $\hat{L}_k$ is evaluated over S_0 or any other closed surface surrounding the cavity. By comparing with (45), we identify the relationship $\hat{L}_k = -L_k$, for any closed surface surrounding the traction-free cavity or crack.

6. Conclusion

We have presented in this paper a simple derivation of dual conservation integrals in small strain elasticity, without using Noether's theorem on invariant variational integrals. The dual integrals $\hat{J}_k$, $\hat{L}_k$, and $\hat{M}$ are related to the release rates of complementary potential energy by extending the Budiansky and Rice derivation of the relationship between the integrals J_k, L_k, and M and the release rates of potential energy. This required an adequate incorporation of the rates of stress and the change of the surface orientation, which was not properly done in earlier published work. Plane strain and anti-plane strain versions of dual integrals are also derived and used to solve selected examples. They illustrate that the evaluation of dual integrals is of similar complex-

[2] The expression for the rate of stress used to derive the $\hat{L}_k$ integral by Carlsson (1974) does not include the first term between the brackets of (61). Consequently, his expression (8) differs from our expressions (36) or (62). In fact, Carlsson's integral (8) is not a conservation integral.

ity to that of classical integrals, and thus either can be used to determine the stress intensity factors or the forces between defects, without solving the corresponding boundary value problems. An advantage of combining the two calculations is also discussed.

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Appendix A. Dual conservation integrals in 2D elasticity

A.1. Dual conservation integrals for plane strain

In the case of plane strain, the energy momentum tensor and its dual are

$$P_{\alpha\beta} = W\delta_{\alpha\beta} - \sigma_{\alpha\gamma}u_{\gamma,\beta}, \quad P_{33} = W,$$
$$\hat{P}_{\alpha\beta} = \Phi\delta_{\alpha\beta} - u_\gamma\sigma_{\alpha\gamma,\beta}, \quad \hat{P}_{33} = \Phi,$$

where the Greek subscripts range from 1 to 2. The dual J integrals are

$$J_\beta = \int_C P_{\alpha\beta}n_\alpha\,\mathrm{d}C, \quad \hat{J}_\beta = \int_C \hat{P}_{\alpha\beta}n_\alpha\,\mathrm{d}C.$$

If a closed contour C does not surround a defect, the above integrals vanish. The dual M integrals are

$$M = \int_C \left(P_{\alpha\beta}x_\beta - \frac{2-r}{r}\sigma_{\alpha\beta}u_\beta\right)n_\alpha\,\mathrm{d}C,$$
$$\hat{M} = \int_C \left(\hat{P}_{\alpha\beta}x_\beta - \frac{2}{s}u_\beta\sigma_{\alpha\beta}\right)n_\alpha\,\mathrm{d}C,$$

while the dual L integrals take the form

$$L_3 = e_{\alpha\beta3}\int_C (P_{\gamma\alpha}x_\beta + \sigma_{\gamma\alpha}u_\beta)n_\gamma\,\mathrm{d}C,$$
$$\hat{L}_3 = e_{\alpha\beta3}\int_C (\hat{P}_{\gamma\alpha}x_\beta + u_\alpha\sigma_{\gamma\beta} + \delta_{\alpha\gamma}u_\delta\sigma_{\beta\delta})n_\gamma\,\mathrm{d}C.$$

Example 1. Dual M integrals around an edge dislocation

Consider an edge dislocation with the Burgers vector b_x in an infinite linearly elastic and isotropic material. For a circular contour surrounding the dislocation, it readily follows that

$$M = r^2\int_0^{2\pi}\left(W - \sigma_{rr}\frac{\partial u_r}{\partial r} - \sigma_{r\theta}\frac{\partial u_\theta}{\partial r}\right)\mathrm{d}\theta = r^2\int_0^{2\pi} W\,\mathrm{d}\theta,$$
$$\hat{M} = r^2\int_0^{2\pi}\left[\Phi - u_r\left(\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\sigma_{rr}\right) - u_\theta\left(\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\sigma_{r\theta}\right)\right]\mathrm{d}\theta = r^2\int_0^{2\pi}\Phi\,\mathrm{d}\theta.$$

The simple expressions on the right-hand sides follow from the structure of the dislocation stress and displacement fields (*e.g.*, Hirth and Lothe, 1982), which gives[3]

[3] Thus, in this case the calculation of $\hat{M}$ involves less integration than the calculation of M integral.

$$\int_0^{2\pi}\left(\sigma_{rr}\frac{\partial u_r}{\partial r}+\sigma_{r\theta}\frac{\partial u_\theta}{\partial r}\right)\mathrm{d}\theta=0,$$

and

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\sigma_{rr}=0,\quad \frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\sigma_{r\theta}=0.$$

Since for an edge dislocation in a linearly elastic isotropic material,

$$W=\Phi=\frac{1}{2\mu}\left[(1-2\nu)\sigma_{rr}^2+\sigma_{r\theta}^2\right]=\frac{\mu b_x^2}{8\pi^2(1-\nu)^2}\frac{1}{r^2}(1-2\nu\sin^2\theta),$$

it readily follows that

$$M=\hat{M}=\frac{\mu b_x^2}{4\pi(1-\nu)}.$$

The so-calculated value of M or $\hat{M}$ can, for example, be conveniently used to determine the force on the dislocation exerted by the nearby crack faces (*e.g.*, Asaro and Lubarda, 2006). The dual J and L integrals around the dislocation all vanish.

Example 2. Dual M integrals around the crack tip under remote loading

Along a circular path around the crack tip, we have

$$\frac{\partial u_\beta}{\partial x_\alpha}x_\alpha=\frac{1}{2}u_\beta,$$

because from the asymptotic crack fields it is known that u_β are homogeneous functions of degree 1/2 in x_α. Since $x_\alpha n_\alpha=r$, we obtain

$$M=\int_C\left(Wn_\alpha x_\alpha-T_\beta\frac{\partial u_\beta}{\partial x_\alpha}x_\alpha\right)\mathrm{d}C=\int_0^{2\pi}\left(Wr-\frac{1}{2}T_\beta u_\beta\right)r\,\mathrm{d}\theta.$$

Recalling that $T_\beta\sim r^{-1/2}$, $u_\beta\sim r^{1/2}$, $W\sim r^{-1}$, and by taking the limit as $r\to 0$, we obtain $M=0$. Because along the traction free crack faces $n_\alpha x_\alpha=0$ and $T_\beta=0$, it follows that $M=0$ around the crack tip regardless of the value of r. Similarly, when evaluating a dual $\hat{M}$ integral, we have

$$\frac{\partial\sigma_{\alpha\gamma}}{\partial x_\beta}x_\beta=-\frac{1}{2}\sigma_{\alpha\gamma},$$

because the stress components are homogeneous functions of degree $-1/2$ in x_β. Thus,

$$\hat{M}=\int_C(\Phi x_\alpha-u_\gamma\sigma_{\alpha\gamma,\beta}x_\beta-u_\beta\sigma_{\alpha\beta})n_\alpha\,\mathrm{d}C=\int_0^{2\pi}\left(\Phi r-\frac{1}{2}u_\beta T_\beta\right)r\,\mathrm{d}\theta=0.$$

Evaluation of dual M integrals for other geometries and loading configurations, such as those considered by Freund (1978), and their implications for the calculation of the stress intensity factors or the forces between defects can be done similarly. In this analysis, it may be useful to note that under the translation of the coordinate system, the dual M integrals transform according to the following rule:

If the values of M and $\hat{M}$ integrals, defined with the coordinate origin placed at the point O, are M_0 and $\hat{M}_0$, then the values of the M and $\hat{M}$ integrals, defined with the coordinate origin at the point A, with coordinates x_i^A with respect to O, are

$$M_A=M_0-x_i^AJ_i,\quad \hat{M}_A=\hat{M}_0-x_i^A\hat{J}_i.$$

Regarding the calculation of the dual J_x integrals around the crack tip, it readily follows that

$$J_x=-\hat{J}_x=\frac{1-\nu^2}{E}(K_\mathrm{I}^2+K_\mathrm{II}^2),$$

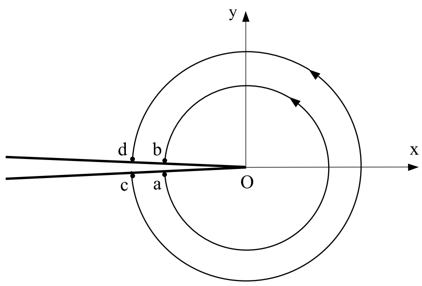

Fig. 1. The dual J_x and $\hat{J}_x$ integrals are both path-independent around the crack tip, *i.e*, $J_x^{ab} = J_x^{cd}$ and $\hat{J}_x^{ab} = \hat{J}_x^{cd}$.

where K_I and K_{II} are the mode I and II stress intensity factors. Since J_x and $\hat{J}_x$ vanish along the horizontal segments *bd* and *ac* of the unloaded crack faces (Fig. 1), J_x and $\hat{J}_x$ are both path-independent around the crack tip, *i.e.*,

$$J_x^{ab} = J_x^{cd}, \quad \hat{J}_x^{ab} = \hat{J}_x^{cd}.$$

In evaluating these integrals, the unit normal **n** is directed relative to the integration contours as shown in the figure. The details of the calculation of the dual J_x integrals around the crack tip are given for the mode III loading below.

A.2. Dual conservation integrals for anti-plane strain

In the case of anti-plane strain, the dual energy momentum tensors are

$$P_{\alpha\beta} = W\delta_{\alpha\beta} - \sigma_{\alpha3}u_{3,\beta}, \quad \hat{P}_{\alpha\beta} = \Phi\delta_{\alpha\beta} - u_3\sigma_{\alpha3,\beta}.$$

The corresponding dual integrals are given by

$$J_\beta = \int_C P_{\alpha\beta}n_\alpha \, dC, \quad \hat{J}_\beta = \int_C \hat{P}_{\alpha\beta}n_\alpha \, dC,$$

$$M = \int_C \left(P_{\alpha\beta}x_\beta - \frac{2-r}{r}\sigma_{\alpha3}u_3\right)n_\alpha \, dC,$$

$$\hat{M} = \int_C \left(\hat{P}_{\alpha\beta}x_\beta - \frac{2}{s}u_3\sigma_{\alpha3}\right)n_\alpha \, dC,$$

$$L_3 = e_{\alpha\beta3}\int_C P_{\gamma\alpha}x_\beta n_\gamma \, dC,$$

$$\hat{L}_3 = e_{\alpha\beta3}\int_C (\hat{P}_{\gamma\alpha}x_\beta + \delta_{\alpha\gamma}u_3\sigma_{\beta3})n_\gamma \, dC.$$

Example 3. Dual J integrals around the crack tip under mode III loading

The J_x integral around the circle with the center at the coordinate origin is

$$J_x = \int_{-\pi}^{\pi} \left[Wn_x - (\sigma_{zx}n_x + \sigma_{zy}n_y)u_{z,x}\right] r \, d\theta.$$

By using the well-known asymptotic stress and displacement fields near the crack tip of a semi-infinite crack under remote mode III loading,

$$\sigma_{zx} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}, \quad \sigma_{zy} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}, \quad u_z = \frac{2K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2},$$

we obtain

$$W = \frac{1}{2\mu}\left(\sigma_{zx}^2 + \sigma_{zy}^2\right) = \frac{K_{III}^2}{4\pi\mu r},$$

so that

$$J_x = \frac{K_{III}^2}{2\mu}.$$

A dual $\hat{J}_x$ integral is

$$\hat{J}_x = \int_{-\pi}^{\pi} \left[\Phi n_x - (\sigma_{zx,x} n_x + \sigma_{zy,x} n_y) u_z\right] r \, \mathrm{d}\theta.$$

For linear elasticity $\Phi = W$, and

$$\hat{J}_x = -\frac{K_{III}^2}{2\mu}.$$

Since the remote traction is assumed to be applied, the energies are $\Pi = -\Omega = -$total strain energy. Thus, $J_x = -\partial\Pi/\partial l > 0$, and $\hat{J}_x = -\partial\Omega/\partial l < 0$ (l being the crack length), because the total strain energy increases with the crack extension at constant load.

Example 4. *Dual J integrals around the crack tip of a rectangular strip*

The upper side of the rectangular strip in Fig. 2 is given a uniform out-of-plane displacement w, while the lower side is fixed. The J_x integral for the closed path *abc...ha* vanishes. Since both J_x and $\hat{J}_x$ vanish along the unloaded horizontal crack faces *ha* and *cb*, we have the path-independent properties

$$J_x^{ab} = J_x^{hgfedc}, \quad \hat{J}_x^{ab} = \hat{J}_x^{hgfedc}.$$

The stresses along *hg* and *dc* and (infinitely remote from the crack tip) are zero, whereas $\sigma_{yz} = \mu w/H$ along infinitely remote *fe*. Since $u_{z,x} = 0$ along the horizontal parts of the integration contours, we obtain

$$J_x^{ab} = J_x^{fe} = \int_f^e W \, \mathrm{d}y, \tag{A.1}$$

$$\hat{J}_x^{ab} = \hat{J}_x^{fe} + \hat{J}_x^{ed} = \int_f^e \Phi \, \mathrm{d}y - \int_e^d w\sigma_{yz,x}(-\mathrm{d}x). \tag{A.2}$$

Since $W = \Phi = \sigma_{yz}^2/2\mu = \mu w^2/2H^2$ along the segment *fe*, while the line integral over *ed* is equal to $w\sigma_{yz}(e) = \mu w^2/H$, the above gives

$$J_x^{ab} = -\hat{J}_x^{ab} = \frac{\mu w^2}{2H}.$$

The non-vanishing displacements are prescribed in this problem, so that the energies are $\Pi = -\Omega =$ Total Strain Energy. Thus, $J_x^{ab} = -\partial\Pi/\partial l > 0$, and $\hat{J}_x^{ab} = -\partial\Omega/\partial l < 0$, because the total strain energy decreases as the crack extends at constant prescribed displacement.

The evaluation of the integral of the strain energy along the segment *ef* can be circumvented in the calculation of dual J integrals by using the following combination of two calculations. When (A.2) is subtracted from (A.1), there follows

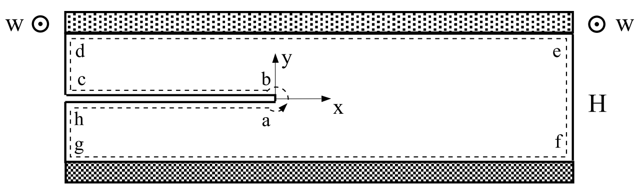

Fig. 2. An infinitely long strip of thickness H with a semi-infinite crack. The lower side of the strip is fixed and the upper is given a uniform out-of-plane displacement w.

$$J_x^{ab} - \hat{J}_x^{ab} = -w \int_e^d \sigma_{yz,x}\,\mathrm{d}x = w\sigma_{yz}(e) = \frac{\mu w^2}{H}.$$

Since $\hat{J}_x = -J_x$, the above gives $J_x^{ab} = \mu w^2/2H$. This result can also be obtained directly from the reciprocal relation (18), with $r = s = 2$. Indeed,

$$J_x = \frac{1}{2}\int_C [(u_z\sigma_{xz,x} - \sigma_{xz}u_{z,x})n_x + (u_z\sigma_{yz,x} - \sigma_{yz}u_{z,x})n_y]\,\mathrm{d}C.$$

The only nonvanishing contribution to this integral along the path *hgfedcb* is from the segment *ed*, so that

$$J_x = \frac{1}{2}\int_e^d w\sigma_{yz,x}(-\mathrm{d}x) = \frac{1}{2}w\sigma_{yz}(e) = \frac{\mu w^2}{2H}.$$

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