Tensorial Representation of the Effective Elastic Properties of the Damaged Material

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ABSTRACT: New representations of the effective elastic stiffness and compliance tensors are derived using an appropriate fourth order tensor basis. This paper considers some of the typical crack distributions associated with an initially isotropic matrix and proportional loading. Resulting isotropic and transversely isotropic responses of the damaged material are described, neglecting direct interaction between adjacent cracks. Derived tensor representations enable easy manipulations and are particularly convenient in obtaining explicit expressions for the inverse operators.

KEY WORDS: averaging schemes, compliance tensor, cracks, damage, dilute crack distribution, effective elastic properties, fourth-order tensor basis, stiffness tensor, transverse isotropy.

1. INTRODUCTION

ANALYTICAL DETERMINATION OF the effective elastic properties of cracked solids has been a topic of intensive research in the last several decades. Various phenomenological and micromechanically based models are developed. Within a phenomenological approach, a macroscopic damage variable is introduced as an internal variable which in some average sense represents existing crack distribution. Elastic stress (or strain) potential is then taken to be an isotropic function of both the strain (or stress) and damage tensors (Vakulenko and Kachanov, 1971; Kachanov, 1980). To capture the damage induced anisotropy, vectorial, second-order and fourth-order damage tensors were introduced by various authors (see recent reviews by Krajcinovic, 1989; Kachanov, 1992; Lubarda and Krajcinovic, 1993). Within a micromechanical approach, the effective elastic properties are derived by using the pertinent results of the microconstituent analysis, such as that of a planar crack embedded in an infinite

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medium. Various averaging schemes are utilized in the transition from micro- to macro-response, depending on the density of cracks and their interaction. In this paper, a noninteracting, dilute crack distribution is assumed. An approximate crack interaction is taken into account according to the self-consistent scheme (Budiansky and O'Connell, 1976; Horii and Nemat-Nasser, 1983), originally developed for estimates of the elastic properties of composite materials.

The objective of this paper is to reexamine the mathematical structure of the effective compliances and stiffnesses of the damaged material in the case of the dilute distribution of penny-shaped cracks. The presented tensor analysis simultaneously gives expressions for all components of the elastic compliance or stiffness tensors, and it has an advantage over previous derivations published in the literature. Furthermore, obtained formulae enable easy manipulations and are particularly convenient in derivation of the explicit expressions for needed inverse operators.

2. ELASTIC PROPERTIES OF DAMAGED MATERIAL

Analytical micromechanical estimates of the effective (overall) elastic properties of a solid containing a large number of microcracks are usually derived by considering an isolated crack in an elastic body, and determining the change in the body's compliance due to the presence of the crack. An adequate averaging scheme is then utilized to estimate the overall properties in the presence of many (interacting or noninteracting) cracks. In this section, one of the several available procedures is applied to derive the elastic compliance due to a single crack. The selected procedure utilizes expressions for the crack release energies (Budiansky and O'Connell, 1976), which leads directly to a compliance tensor having a convenient structure. Other procedures, such as one based on the expressions for the crack opening or jump displacements (see, for example, Horii and Nemat-Nasser, 1983 or Ju and Tseng, 1992) can be used as well. Therefore, let a single penny-shaped crack be embedded in an infinite isotropic elastic solid, uniformly loaded at infinity. Decompose this problem into two problems, that of the body without a crack loaded at infinity (^{0}), and that of the body with a crack appropriately loaded over the crack faces (*). Correspondingly, the local strain $\mathbf{e}$ can be written as the sum of the strains belonging to two problems, i.e., $\mathbf{e} = \mathbf{e}^0 + \mathbf{e}^*$. The volume-average strains are decomposed in the same manner, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}^*$. Let $\boldsymbol{\sigma}$ be the remote loading and $\mathcal{M}^0$ the elastic compliance of the virgin material without the crack. Introducing $\mathcal{M}$ as the average elastic compliance of the body with a crack, and denoting by $\mathcal{M}^*$ the compliance mapping $\boldsymbol{\sigma}$ to $\boldsymbol{\epsilon}^*$ ($\boldsymbol{\epsilon}^* = \mathcal{M}^*:\boldsymbol{\sigma}$), it follows that

$$\mathcal{M} = \mathcal{M}^0 + \mathcal{M}^* \tag{1}$$

The compliance $\mathcal{M}^*$, related to the complementary strain energy $\Psi^* = (1/2)\mathcal{M}^*:(\boldsymbol{\sigma} \otimes \boldsymbol{\sigma})$ by

$$\mathcal{M}^* = \frac{\partial^2 \Psi^*}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} \tag{2}$$

can be conveniently determined by observing that Ψ^* is equal to the energy release associated with the self-similar crack growth from zero to the current size a. As shown by Budiansky and Rice (1973), this energy can be expressed as

$$\Psi^* = \int_0^a \frac{M}{a} da \tag{3}$$

where M is the M-conservation integral of fracture mechanics. The M integral can be written in terms of the J integral by means of a line integral along the crack perimeter l

$$M = \oint_l aJdl \tag{4}$$

Substituting Equation (4) into Equation (3), it follows that (Budiansky and O'Connell, 1976)

$$\Psi^* = \int_0^a \left(\oint_l Jdl \right) da \tag{5}$$

In the close neighborhood of the crack edge, the stress and strain states are a combination of plane strain and antiplane shear. Thus, the energy release rate or the J integral can be expressed in terms of the corresponding stress-intensity factors $K_M (M = I, II, III)$ as

$$J = \frac{1 - \nu_0}{2\mu_0}(K_I^2 + K_{II}^2) + \frac{1}{2\mu_0} K_{III}^2 \tag{6}$$

Expression (6) can be rewritten in a compact form as

$$J = C_{MN} K_M K_N \tag{7}$$

where

$$C_{MN} = \frac{1}{2\mu_0}[(1 - \nu_0)\delta_{MN} + \nu_0\delta_{III,M}\delta_{III,N}] \quad (8)$$

Consequently, substituting Equation (7) into Equation (5) gives

$$\Psi^* = \int_0^a \left(\oint_l C_{MN}K_MK_N dl \right) da \quad (9)$$

For a penny-shaped crack, the stress-intensity factors, written in the symmetrized form, are (Tada, 1973):

$$K_I = \frac{2}{\pi}(\pi a)^{1/2}\sigma'_{11}$$

$$K_{II} = \frac{2}{\pi}(\pi a)^{1/2}\frac{1}{2 - \nu_0}[(\sigma'_{12} + \sigma'_{21}) \cos \alpha + (\sigma'_{13} + \sigma'_{31}) \sin \alpha] \quad (10)$$

$$K_{III} = \frac{2}{\pi}(\pi a)^{1/2}\frac{1 - \nu_0}{2 - \nu_0}[(\sigma'_{12} + \sigma'_{21}) \sin \alpha - (\sigma'_{13} + \sigma'_{31}) \cos \alpha]$$

where σ'_{ij} are the stress components in the crack coordinate system, and α is an angle defined in Figure 1. The stress component σ'_{11} is assumed to be tensile. The same procedure was used by Sumarac (1987), except that Expressions (10) were

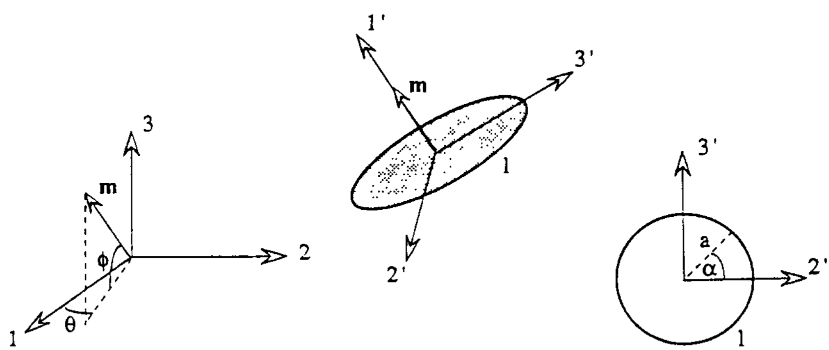

Figure 1. *The penny-shaped crack of radius* a *and circumference* l*. (1′,2′,3′) denotes the local crack coordinate system, direction 1′ being coincident with the normal to the crack plane* **m**. *The angles θ and ϕ define orientation of the vector* **m** *relative to the global coordinate system (1,2,3).*

not written in a symmetrized form, which led to nonsymmetric structure of the compliance tensor. Differentiating (10), it follows that:

$$\frac{\partial K_I}{\partial \sigma'_{ij}} = \frac{2}{\pi}(\pi a)^{1/2}\delta_{i1}\delta_{j1}$$

$$\frac{\partial K_{II}}{\partial \sigma'_{ij}} = \frac{2}{\pi}(\pi a)^{1/2}\frac{1}{2 - \nu_0}[(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})\cos\alpha + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})\sin\alpha]$$

$$\frac{\partial K_{III}}{\partial \sigma'_{ij}} = \frac{2}{\pi}(\pi a)^{1/2}\frac{1 - \nu_0}{2 - \nu_0}[(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})\sin\alpha - (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})\cos\alpha] \tag{11}$$

If σ'_{11} is compressive stress, $K_I = 0$ along with the right-hand side of the first expression in Equation (11).

The components of the compliance tensor $\mathscr{M}^*$ in the local crack coordinate system are

$$\mathscr{M}^{*\prime}_{ijkl} = \frac{\partial^2\Psi^*}{\partial\sigma'_{ij}\partial\sigma'_{kl}} = 2\int_0^a\left(\oint_l C_{MN}\frac{\partial K_M}{\partial\sigma'_{ij}}\frac{\partial K_N}{\partial\sigma'_{kl}}dl\right)da \tag{12}$$

Since

$$C_{MN}\frac{\partial K_M}{\partial\sigma'_{ij}}\frac{\partial K_N}{\partial\sigma'_{kl}} = \frac{1-\nu_0}{2\mu_0}\left(\frac{\partial K_I}{\partial\sigma'_{ij}}\frac{\partial K_I}{\partial\sigma'_{kl}} + \frac{\partial K_{II}}{\partial\sigma'_{ij}}\frac{\partial K_{II}}{\partial\sigma'_{kl}}\right) + \frac{1}{2\mu_0}\frac{\partial K_{III}}{\partial\sigma'_{ij}}\frac{\partial K_{III}}{\partial\sigma'_{kl}} \tag{13}$$

substitution of Equations (11) and (13) into Equation (12) gives

$$\mathscr{M}^{*\prime}_{ijkl} = \frac{16}{3}\frac{1-\nu_0}{2\mu_0}a^3\Big\{(\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1})H(\sigma'_{11}) + \frac{1}{2(2-\nu_0)} \times [(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(\delta_{k1}\delta_{l2} + \delta_{k2}\delta_{l1}) + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})(\delta_{k1}\delta_{l3} + \delta_{k3}\delta_{l1})]\Big\} \tag{14}$$

The Heaviside step function, $H(\sigma'_{11})$, is introduced in Equation (14) to enable simultaneous consideration of tensile and compressive stress component σ'_{11}. In the local (crack) coordinate system, the normal to the crack plane has the components

$$m'_i = \delta_{i1} \tag{15}$$

The expression for the compliance tensor [Equation (14)] thus becomes

$$\mathcal{M}^{*\prime}_{ijkl} = \frac{16}{3}\frac{1-\nu_0}{2\mu_0}a^3\Bigg\{(m_i' m_j' m_k' m_l')H(\sigma_{11}') + \frac{1}{2(2-\nu_0)}[(m_i'\delta_{j2} + \delta_{i2}m_j')(m_k'\delta_{l2} + \delta_{k2}m_l') + (m_i'\delta_{j3} + \delta_{i3}m_j')(m_k'\delta_{l3} + \delta_{k3}m_l')]\Bigg\} \tag{16}$$

Expression (16) can be rearranged as

$$\mathcal{M}^{*\prime}_{ijkl} = \frac{16}{3}\frac{1-\nu_0}{2\mu_0}a^3\Bigg[(m_i' m_j' m_k' m_l')H(\sigma_{11}') + \frac{1}{2(2-\nu_0)}(\delta_{ik}m_j' m_l' + m_i' m_k'\delta_{jl} + \delta_{il}m_j' m_k' + m_i' m_l'\delta_{jk} - 4m_i' m_j' m_k' m_l')\Bigg] \tag{17}$$

The required symmetry properties, $\mathcal{M}^{*\prime}_{ijkl} = \mathcal{M}^{*\prime}_{jikl} = \mathcal{M}^{*\prime}_{ijlk} = \mathcal{M}^{*\prime}_{klij}$, clearly hold. Expression (17) defines the change of elastic compliance due to the presence of a single penny-shaped crack, expressed in a local-crack coordinate system.

3. REPRESENTATION IN TERMS OF FOURTH-ORDER TENSOR BASIS

In the subsequent analysis, it is advantageous to introduce the following set of the fourth order tensors:

$$I^1_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad I^2_{ijkl} = \delta_{ij}\delta_{kl}$$

$$I^3_{ijkl} = \delta_{ij}m_k m_l \qquad I^4_{ijkl} = m_i m_j \delta_{kl}$$

$$I^5_{ijkl} = \frac{1}{4}(\delta_{ik}m_j m_l + m_i m_k \delta_{jl} + \delta_{il}m_j m_k + m_i m_l \delta_{jk}) \qquad I^6_{ijkl} = m_i m_j m_k m_l \tag{18}$$

This set of tensors, combining products of the Kronecker delta tensor and a unit vector **m**, form the basis for the fourth-order tensors that are symmetric with respect to the first and second pairs of indices, but not necessarily symmetric with respect to permutation of the pairs (Kunin, 1981). All tensors of the fourth-order made up of Kronecker delta and the unit vector, possessing described symmetry, can be represented as a linear combination of the basic tensors $\mathbf{I}^i$. The linear tensor space spanned by this basis is closed with respect to the trace product, forming an algebra. For example, it can be shown that $\mathbf{I}^3{:}\mathbf{I}^2 = \mathbf{I}^2$, $\mathbf{I}^4{:}\mathbf{I}^2 = 3\mathbf{I}^4$, $\mathbf{I}^5{:}\mathbf{I}^2 = \mathbf{I}^4$, and $\mathbf{I}^6{:}\mathbf{I}^2 = \mathbf{I}^4$.

Using Equation (18), Expression (17) can be rewritten in a compact form as

$$\mathcal{M}^{*\prime}_{ijkl} = \frac{16}{3}\frac{1-\nu_0}{2-\nu_0}\frac{1}{2\mu_0}a^3\{2I^{5m\prime}_{ijkl} + [(2-\nu_0)H(\sigma'_{11}) - 2]I^{6m\prime}_{ijkl}\} \qquad (19)$$

In Equation (19), $I^{5m\prime}_{ijkl}$ and $I^{6m\prime}_{ijkl}$ are the tensors defined by Equation (18), in which the components of the normal **m** are expressed relative to the local crack coordinate system ($m'_i = \delta_{1i}$).

If $\sigma'_{11} \geq 0$, the expressions for the compliance attributable to cracks [Equation (19)] further reduces to

$$\mathcal{M}^{*\prime}_{ijkl} = \frac{16}{3}\frac{1-\nu_0}{2-\nu_0}\frac{1}{2\mu_0}a^3(2I^{5m\prime}_{ijkl} - \nu_0 I^{6m\prime}_{ijkl}) \qquad (20)$$

The expanded form of this expression was used by Krajcinovic and Fanella (1986) to model the concrete behavior.

The compliance tensor expressed in Equation (19) relative to the local crack coordinate system can be written in the global coordinate system using the coordinate transformation

$$\mathcal{M}^{*}_{ijkl} = Q_{i\alpha}Q_{j\beta}\mathcal{M}^{*\prime}_{\alpha\beta\gamma\delta}Q_{k\gamma}Q_{l\delta} \qquad (21)$$

where **Q** is the orthogonal tensor of the transformation between the two coordinate systems. Using Equation (21), Expression (20) can be written in terms of the global coordinates as

$$\mathcal{M}^{*}_{ijkl} = \frac{16}{3}\frac{1-\nu_0}{2-\nu_0}\frac{1}{2\mu_0}a^3\{2I^{5m}_{ijkl} + [(2-\nu_0)H(\mathbf{m}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{m}) - 2]I^{6m}_{ijkl}\} \qquad (22)$$

where:

$$I^{5m}_{ijkl} = \frac{1}{4}(\delta_{ik}m_jm_l + m_im_k\delta_{jl} + \delta_{il}m_jm_k + m_im_l\delta_{jk})$$
$$I^{6m}_{ijkl} = m_im_jm_km_l \qquad (23)$$

are the fourth order tensors combining the Kronecker delta and the unit vector **m**, expressed in the global coordinate system as

$$\mathbf{m} = \{\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi\} \tag{24}$$

If $\sigma'_{11} = \mathbf{m}^T \cdot \boldsymbol{\sigma} \cdot \mathbf{m} \geq 0$ ($\boldsymbol{\sigma}$ is the stress tensor with the components in global coordinate system), Equation (22) reduces to

$$\mathcal{M}^*_{ijkl} = \frac{16}{3}\frac{1-\nu_0}{2-\nu_0}\frac{1}{2\mu_0}a^3(2I^{5m}_{ijkl} - \nu_0 I^{6m}_{ijkl}) \tag{25}$$

This is the same expression as Equation (20), except that in Equation (20) the components of normal **m** are expressed in the local crack coordinate system, while in Equation (25) they are expressed relative to the global coordinate system. The superscript m indicates the reference to the plane of the crack (Figure 1).

4. TRANSVERSELY ISOTROPIC CRACK DISTRIBUTION

Consider now the case of a solid containing many cracks. If all cracks have the same normal **m**, and if the direct crack interaction is neglected (dilute crack concentration), the average compliance is

$$\bar{\mathcal{M}}^* = \frac{16}{3}\frac{1-\nu_0}{2-\nu_0}\frac{1}{2\mu_0}\omega(2\mathbf{I}^{5m} - \nu_0\mathbf{I}^{6m}) \tag{26}$$

The nondimensional scalar quantity $\omega = N\langle a^3\rangle$ is a micromechanical damage parameter (Budiansky and O'Connell, 1976), which defines the density of the considered crack distribution within a representative unit volume ($\langle\rangle$ denotes the average value). Material response corresponding to Equation (26) is transversely isotropic.

Consider next a crack distribution in which all normals to the crack planes have the same angle ϕ = const. Neglecting direct interaction between adjacent cracks (dilute distribution of cracks), the average compliance attributable to cracks is

$$\bar{\mathcal{M}}^*_{ijkl} = \frac{N}{2\pi}\int_0^{2\pi}\mathcal{M}^*_{ijkl}d\theta$$

$$= \frac{8}{3\pi}\frac{1-\nu_0}{2-\nu_0}\frac{1}{2\mu_0}\omega\left(2\int_0^{2\pi}I^{5m}_{ijkl}d\theta - \nu_0\int_0^{2\pi}I^{6m}_{ijkl}d\theta\right) \tag{27}$$

In the derivation of Equation (27), the stress state is assumed to be such that $\mathbf{m}^T \cdot \boldsymbol{\sigma} \cdot \mathbf{m} \geq 0$ for all $\mathbf{m}$, allowing the representation [Equation (25)] for the components of the compliance tensor $\mathcal{M}^*_{ijkl}$.

In general, for an arbitrary angle ϕ,

$$m_i m_j = \delta_{i1}\delta_{j1} \cos^2\phi \cos^2\theta + \delta_{i2}\delta_{j2} \cos^2\phi \sin^2\theta + \delta_{i3}\delta_{j3} \sin^2\phi$$
$$+ (\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) \cos^2\phi \sin\theta \cos\theta + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1}) \sin\phi \cos\phi \cos\theta$$
$$+ (\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}) \sin\phi \cos\phi \sin\theta \quad (28)$$

Consider the cylindrical crack distribution, $\phi = 0$. The normal $\mathbf{m}$ to a crack plane has components $\{\cos\theta, \sin\theta, 0\}$. In this case, Equation (28) reduces to

$$m_i m_j = \delta_{i1}\delta_{j1} \cos^2\theta + \delta_{i2}\delta_{j2} \sin^2\theta + (\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) \sin\theta \cos\theta \quad (29)$$

Hence,

$$\int_0^{2\pi} m_i m_j d\theta = \pi(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) = \pi(\delta_{ij} - \delta_{i3}\delta_{j3}) \quad (30)$$

Introduce the vector $\mathbf{n}$, normal to $\mathbf{m}$, having the components in the global coordinate system

$$n_i = \delta_{i3} \quad (31)$$

The integral [Equation (30)] can be accordingly written as

$$\int_0^{2\pi} m_i m_j d\theta = \pi(\delta_{ij} - n_i n_j) \quad (32)$$

such that

$$\int_0^{2\pi} I^{5m}_{ijkl} d\theta = \frac{\pi}{4}[2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
$$- (\delta_{ik} n_j n_l + n_i n_k \delta_{jl} + \delta_{il} n_j n_k + n_i n_l \delta_{jk})] \quad (33)$$

Using the fourth order tensor basis $\mathbf{I}^i$ introduced in Equation (18), the integral [Equation (33)] can be written in a compact form as

$$\int_0^{2\pi} \mathbf{I}^{5m} d\theta = \pi(\mathbf{I}^1 - \mathbf{I}^{5n}) \quad (34)$$

To evaluate the integral

$$\int_0^{2\pi} I^{6m}_{ijkl} d\theta = \int_0^{2\pi} m_i m_j m_k m_l d\theta \quad (35)$$

expressions of the type [Equation (29)] are substituted for products, $m_i m_j$ and $m_k m_l$. Performing integration, it follows that

$$\begin{aligned}\int_0^{2\pi} I^{6m}_{ijkl} d\theta = \frac{\pi}{4}[3(\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2}) \\ + (\delta_{i1}\delta_{j2}\delta_{k1}\delta_{l2} + \delta_{i1}\delta_{j2}\delta_{k2}\delta_{l1} + \delta_{i2}\delta_{j1}\delta_{k1}\delta_{l2} \\ + \delta_{i2}\delta_{j1}\delta_{k2}\delta_{l1} + \delta_{i1}\delta_{j1}\delta_{k2}\delta_{l2} + \delta_{i2}\delta_{j2}\delta_{k1}\delta_{l1})]\end{aligned} \quad (36)$$

After a somewhat delicate rearrangement of terms, described in the Appendix, Equation (36) can be cast in a remarkably simple form

$$\begin{aligned}\int_0^{2\pi} I^{6m}_{ijkl} d\theta = \frac{\pi}{4}[(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l) \\ + (\delta_{jk} - n_j n_k)(\delta_{il} - n_i n_l) + (\delta_{ki} - n_k n_i)(\delta_{jl} - n_j n_l)]\end{aligned} \quad (37)$$

or

$$\begin{aligned}\int_0^{2\pi} I^{6m}_{ijkl} d\theta = \frac{\pi}{4}[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ij}\delta_{kl} - (\delta_{ij} n_k n_l + n_i n_j \delta_{kl}) \\ - (\delta_{ik} n_j n_l + n_i n_k \delta_{jl} + \delta_{il} n_j n_k + n_i n_l \delta_{jk}) + 3(n_i n_j n_k n_l)]\end{aligned} \quad (38)$$

Utilizing the fourth order tensors forming the basis of Equation (18), Equation (38) can be written as

$$\int_0^{2\pi} \mathbf{I}^{6m} d\theta = \frac{\pi}{4}(2\mathbf{I}^1 + \mathbf{I}^2 - \mathbf{I}^{3n} - \mathbf{I}^{4n} - 4\mathbf{I}^{5n} + 3\mathbf{I}^{6n}) \tag{39}$$

The average compliance attributable to the considered crack distribution is finally derived substituting Equations (33) and (38) into Equation (27)

$$\bar{\mathcal{M}}^*_{ijkl} = \frac{2}{3}\frac{1 - \nu_0}{2 - \nu_0}\frac{1}{2\mu_0}\omega[(4 - \nu_0)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \nu_0\delta_{ij}\delta_{kl}$$
$$+ \nu_0(\delta_{ij}n_kn_l + n_in_j\delta_{kl}) - (2 - \nu_0)(\delta_{ik}n_jn_l + n_in_k\delta_{jl} + \delta_{il}n_jn_k + n_in_l\delta_{jk})$$
$$- 3\nu_0(n_in_jn_kn_l)] \tag{40}$$

Note that $\bar{\mathcal{M}}^*$ is linearly proportional to the damage parameter ω, i.e., $\bar{\mathcal{M}}^* = \omega\mathbf{M}^*$, where $\mathbf{M}^*$ is the constant tensor given by

$$\mathbf{M}^* = \frac{2}{3}\frac{1 - \nu_0}{2 - \nu_0}\frac{1}{2\mu_0}[2(4 - \nu_0)\mathbf{I}^1 - \nu_0\mathbf{I}^2 + \nu_0(\mathbf{I}^{3n} + \mathbf{I}^{4n})$$
$$- 4(2 - \nu_0)\mathbf{I}^{5n} - 3\nu_0\mathbf{I}^{6n}] \tag{41}$$

5. ISOTROPIC CRACK DISTRIBUTION

If the crack distribution is dilute and isotropic, the average compliance is

$$\bar{\mathcal{M}}^* = \frac{N}{4\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \mathcal{M}^* \cos\phi \, d\phi \, d\theta \tag{42}$$

By using Equations (23) and (28), it can be shown that:

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \mathbf{I}^{5m} \cos\phi \, d\phi \, d\theta = \frac{4\pi}{3}\mathbf{I}^1$$
$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \mathbf{I}^{6m} \cos\phi \, d\phi \, d\theta = \frac{4\pi}{15}(2\mathbf{I}^1 + \mathbf{I}^2) \tag{43}$$

The average compliance is derived substituting Equation (43) into Equations (25) and (42)

$$\bar{\mathcal{M}}^* = \frac{16}{45}\frac{1 - \nu_0}{2 - \nu_0}\frac{1}{2\mu_0}\omega[2(5 - \nu_0)\mathbf{I}^1 - \nu_0\mathbf{I}^2] \tag{44}$$

It is easily shown that Equation (44) produces the compliance components in agreement to those given by Equation (20) of Horii and Nemat-Nasser (1983).

6. EFFECTIVE COMPLIANCE TENSOR

The compliance tensor of the undamaged, isotropic, and homogeneous elastic matrix is

$$\mathcal{M}^0 = \frac{1}{2\mu_0}\left(\mathbf{I}^1 - \frac{\nu_0}{1 + \nu_0}\mathbf{I}^2\right) \tag{45}$$

The effective (overall) compliance is derived by superposing Equation (45) and the expression for the average compliance $\bar{\mathcal{M}}^*$. For an isotropic crack distribution, i.e., for $\bar{\mathcal{M}}^*$ defined by Equation (44), the overall compliance becomes

$$\mathcal{M} = \frac{1}{2\mu}\left(\mathbf{I}^1 - \frac{\nu}{1 + \nu}\mathbf{I}^2\right) \tag{46}$$

where the damage dependent shear modulus and Poisson's ratio are given by:

$$\mu = \frac{45(2 - \nu_0)}{45(2 - \nu_0) + 32(1 - \nu_0)(5 - \nu_0)\omega}\mu_0$$
$$\nu = \frac{45(2 - \nu_0) + 16(- \nu_0^2)\omega}{45(2 - \nu_0) + 16(1 - \nu_0^2)(10 - 3\nu_0)\omega}\nu_0 \tag{47}$$

The above expressions are in agreement with Expressions (2.24) given in Kachanov (1992). In the case of cylindrical crack distribution such that Equation (40) applies, it follows that

$$\mathcal{M} = \frac{1}{2\mu_0}\sum_{i=1}^{6} C_i\mathbf{I}^i \tag{48}$$

where the parameters C_i are given by:

$$C_1 = 1 + \frac{4}{3}\frac{(4 - \nu_0)(1 - \nu_0)}{2 - \nu_0}\omega \qquad C_2 = -\frac{\nu_0}{1 + \nu_0} + \frac{2}{3}\frac{\nu_0(1 - \nu_0)}{2 - \nu_0}\omega$$

$$C_3 = \frac{4}{3}\frac{1 - \nu_0}{2 - \nu_0}\omega \qquad C_4 = \frac{4}{3}\frac{1 - \nu_0}{2 - \nu_0}\omega \tag{49}$$

$$C_5 = -\frac{8}{3}(1 - \nu_0)\omega \qquad C_6 = -2\nu_0\frac{1 - \nu_0}{2 - \nu_0}\omega$$

In the expanded form, Equation (48) is

$$\begin{aligned}\mathcal{M}_{ijkl} = \frac{1}{2\mu_0}\Bigg\{&\left[1 + \frac{4}{3}\frac{(4 - \nu_0)(1 - \nu_0)}{2 - \nu_0}\omega\right]\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &- \left[\frac{\nu_0}{1 + \nu_0} + \frac{2}{3}\frac{\nu_0(1 - \nu_0)}{2 - \nu_0}\omega\right]\delta_{ij}\delta_{kl} + \frac{2}{3}\frac{1 - \nu_0}{2 - \nu_0}\omega[2(\delta_{ij}n_kn_l + n_in_j\delta_{kl}) \\ &- (2 - \nu_0)(\delta_{ik}n_jn_l + n_in_k\delta_{jl} + \delta_{il}n_jn_k + n_in_l\delta_{jk}) - 3\nu_0(n_in_jn_kn_l)]\Bigg\}\end{aligned} \tag{50}$$

recognized as the compliance tensor of a transversely isotropic material with the plane of symmetry normal to the direction **n**. Indeed, written in terms of the elastic moduli and Poisson's ratios in two orthogonal directions, the parameters $\hat{C}_i = C_i/2\mu_0$ appearing in Equation (48) are:

$$\hat{C}_1 = \frac{1 + \nu}{E} = \frac{1}{2\mu} \qquad \hat{C}_2 = -\frac{\nu}{E}$$

$$\hat{C}_3 = \frac{\nu}{E} - \frac{\nu'}{E'} \qquad \hat{C}_4 = \frac{\nu}{E} - \frac{\nu'}{E'} \tag{51}$$

$$\hat{C}_5 = \frac{1}{\mu'} - \frac{1}{\mu} \qquad \hat{C}_6 = \frac{1 + 2\nu'}{E'} - \frac{\nu}{E} + \frac{1}{2\mu} - \frac{1}{\mu'}$$

where E and E' are the Young's moduli in the plane of isotropy and in the direction normal to it. Also, ν is the Poisson's ratio characterizing transverse contraction in the plane of isotropy when tension is applied in the same plane. In contrast, ν' is the Poisson's ratio when tension is applied normal to the plane of isotropy. Finally, μ' is the shear modulus for any plane perpendicular to the plane of isotropy (Lekhnitskii, 1981). These five material parameters can, therefore, be expressed from Equation (51) in terms of the parameters C_i as:

$$E = \frac{2\mu_0}{C_1 + C_2}, \quad \nu = -\frac{C_2}{C_1 + C_2}$$

$$E' = \frac{2\mu_0}{C_1 + C_2 + 2C_3 + C_5 + C_6}, \quad \nu' = -\frac{C_2 + C_3}{C_1 + C_2 + 2C_3 + C_5 + C_6}$$

$$\mu' = \frac{2\mu_0}{2C_1 + C_5} \tag{52}$$

7. EFFECTIVE STIFFNESS TENSOR

To derive the expression for the stiffness tensor, i.e., the inverse of the compliance tensor [Equation (48)], it is convenient to use the fourth-order tensors of the new basis $\mathbf{J}^i$, defined in terms of the old basis $\mathbf{I}^i$ ($i = 1,2,\ldots,6$) through the linear transformation (Kunin, 1981):

$$\mathbf{J}^1 = \frac{1}{2}(\mathbf{I}^2 - \mathbf{I}^3 - \mathbf{I}^4 + 3\mathbf{I}^6) \quad \mathbf{J}^2 = \frac{1}{2}(-\mathbf{I}^2 + \mathbf{I}^3 + \mathbf{I}^4 + \mathbf{I}^6)$$

$$\mathbf{J}^3 = \frac{1}{2}(-2\mathbf{I}^3 - \mathbf{I}^4 + 3\mathbf{I}^6) \quad \mathbf{J}^4 = \frac{1}{2}(-2\mathbf{I}^3 + \mathbf{I}^4 + \mathbf{I}^6) \tag{53}$$

$$\mathbf{J}^5 = 2(\mathbf{I}^5 - \mathbf{I}^6) \quad \mathbf{J}^6 = \frac{1}{2}(2\mathbf{I}^1 - \mathbf{I}^2 + \mathbf{I}^3 + \mathbf{I}^4 - 4\mathbf{I}^5 + \mathbf{I}^6)$$

The tensor [Equation (40)] written in the basis [Equation (53)] takes the form

$$\bar{\mathcal{M}}^* = \frac{4}{3}\frac{1 - \nu_0}{2 - \nu_0}\frac{1}{2\mu_0}\omega[(2 - \nu_0)(\mathbf{J}^{1n} - \mathbf{J}^{2n}) + 2\mathbf{J}^{5n} + (4 - \nu_0)\mathbf{J}^{6n}] \tag{54}$$

The compliance of the undamaged solid $\mathcal{M}^0$, defined by Equation (45), can also be expressed in terms of $\mathbf{J}^i$ basis. The effective overall compliance, $\mathcal{M} = \mathcal{M}^0 + \bar{\mathcal{M}}^*$, consequently becomes

$$\mathcal{M} = \frac{1}{2\mu_0}\sum_{i=1}^{6} c_i \mathbf{J}^i \tag{55}$$

In Equation (55), the parameters c_i are given by:

$$c_1 = \frac{2 - \nu_0}{2(1 + \nu_0)} + \frac{4}{3}(1 - \nu_0)\omega, \qquad c_2 = \frac{\nu_0}{2(1 + \nu_0)} - \frac{4}{3}(1 - \nu_0)\omega$$

$$c_3 = \frac{3\nu_0}{2(1 + \nu_0)} \qquad c_4 = -\frac{\nu_0}{2(1 + \nu_0)} \tag{56}$$

$$c_5 = 1 + \frac{8(1 - \nu_0)}{3(2 - \nu_0)}\omega \qquad c_6 = 1 + \frac{4(1 - \nu_0)(4 - \nu_0)}{3(2 - \nu_0)}\omega$$

The representation [Equation (55)] in terms of the basis $\mathbf{J}^i$ is very useful because it provides the explicit inverse representation

$$\mathscr{L} = \mathscr{M}^{-1} = 2\mu_0 \sum_{i=1}^{6} b_i \mathbf{J}^i \tag{57}$$

As discussed in Kunin (1981), the coefficients b_i are related to coefficients c_i by

$$\{b_1, \ldots, b_6\} = \left\{\frac{c_1}{\Delta}, -\frac{c_2}{\Delta}, -\frac{c_3}{\Delta}, -\frac{c_4}{\Delta}, \frac{1}{c_5}, \frac{1}{c_6}\right\} \tag{58}$$

where $\Delta = c_1^2 - c_2^2 - c_3^2 + c_4^2$.

Returning to the original $\mathbf{I}^i$ basis, Equation (57) becomes

$$\mathscr{L} = 2\mu_0 \sum_{i=1}^{6} a_i \mathbf{I}^i \tag{59}$$

where the coefficients a_i are related to coefficients b_i by:

$$\begin{aligned}
a_1 &= b_6 \\
a_2 &= \frac{1}{2}(b_1 - b_2 - b_6) \\
a_3 &= \frac{1}{2}(-b_1 + b_2 - 2b_3 - 2b_4 + b_6) \\
a_4 &= \frac{1}{2}(-b_1 + b_2 - b_3 + b_4 + b_6) \\
a_5 &= 2(b_5 - b_6) \\
a_6 &= \frac{1}{2}(3b_1 + b_2 + 3b_3 + b_4 - 4b_5 + b_6)
\end{aligned} \tag{60}$$

Note that $c_4 - c_3 = -2(c_3 + c_4)$, hence $b_4 - b_3 = -2(b_3 + b_4)$ and $a_3 = a_4$, which assures the self-adjoint symmetry of the tensor $\mathscr{L}$.

Expression (59) is the exact inverse of Equation (48). If the damage parameter ω is sufficiently small so that the quadratic and higher order terms in ω can be neglected, coefficients [Equation (60)] can be simplified to:

$$a_1 = 1 - \frac{4}{3}\frac{(1 - \nu_0)(4 - \nu_0)}{2 - \nu_0}\omega$$

$$a_2 = \frac{\nu_0}{1 - 2\nu_0} - \frac{2}{3}\frac{\nu_0(1 - \nu_0)}{(2 - \nu_0)(1 - 2\nu_0)^2}(15 - 20\nu_0 + 4\nu_0^2)\omega$$

$$a_3 = a_4 = \frac{2}{3}\frac{\nu_0(1 - \nu_0)}{(2 - \nu_0)(1 - 2\nu_0)^2}(7 - 16\nu_0 + 4\nu_0^2)\omega \tag{61}$$

$$a_5 = \frac{8}{3}(1 - \nu_0)\omega$$

$$a_6 = -\frac{2}{3}\frac{\nu_0(1 - \nu_0)}{(2 - \nu_0)(1 - 2\nu_0)}(13 - 2\nu_0)\omega$$

The components of Equation (59), with the parameters a_i defined by Equation (61), are identical to the components of the stiffness tensor derived by Nemat-Nasser and Hori (1990) [their Equation (5.11b)]. Also, from Equations (59) and (61) it clearly follows that $\mathscr{L} = \mathscr{L}^0 + \omega\mathbf{L}^*$, where

$$\mathbf{L}^* = \frac{2\mu_0}{\omega}\left[(a_1 - 1)\mathbf{I}^1 + \left(a_2 - \frac{\nu_0}{1 - 2\nu_0}\right)\mathbf{I}^2 + a_3(\mathbf{I}^3 + \mathbf{I}^4) + a_5\mathbf{I}^5 + a_6\mathbf{I}^6\right] \tag{62}$$

is the constant fourth-order tensor (independent of ω), and $\mathscr{L}^0 = 2\mu_0[\mathbf{I}^1 + \nu_0/(1 - 2\nu_0)\mathbf{I}^2]$.

8. CONCLUSION

This paper contains derivations of the explicit tensorial representations of the effective stiffness and compliance tensors for some typical crack distributions, corresponding to isotropic and transversely isotropic macroresponse. An adequate averaging scheme is utilized to obtain the effective elastic properties of a cracked solid with a dilute distribution of cracks. Derived expressions allow easy manipulations and are compared with some previously published results.

An alternative phenomenological analysis and elaboration on the structure of the elastic potential as a function of strain and appropriate damage tensors will be presented in a separate paper. Future work will also focus on the description of strongly nonproportional loading and associated damage evolution that changes direction in its progression.

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APPENDIX

To evaluate the integral appearing in Expression (35) of the text, i.e.,

$$\int_0^{2\pi} I_{ijkl}^{6m}\, d\theta = \int_0^{2\pi} m_i m_j m_k m_l\, d\theta \tag{A.1}$$

expressions of the type in Equation (29) are substituted for products $m_i m_j$ and $m_k m_l$. Performing integration, it then follows that

$$\begin{aligned}\int_0^{2\pi} I_{ijkl}^{6m}\, d\theta = \frac{\pi}{4}[3(&\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2}) \\ &+ (\delta_{i1}\delta_{j2}\delta_{k1}\delta_{l2} + \delta_{i1}\delta_{j2}\delta_{k2}\delta_{l1} + \delta_{i2}\delta_{j1}\delta_{k1}\delta_{l2} \\ &+ \delta_{i2}\delta_{j1}\delta_{k2}\delta_{l1} + \delta_{i1}\delta_{j1}\delta_{k2}\delta_{l2} + \delta_{i2}\delta_{j2}\delta_{k1}\delta_{l1})]\end{aligned} \tag{A.2}$$

This can be rearranged into

$$\begin{aligned}\int_0^{2\pi} I_{ijkl}^{6m}\, d\theta = \frac{\pi}{4}[&\delta_{i1}\delta_{k1}(\delta_{j1}\delta_{l1} + \delta_{j2}\delta_{l2}) + \delta_{j1}\delta_{k1}(\delta_{i1}\delta_{l1} + \delta_{i2}\delta_{l2}) \\ &+ \delta_{i1}\delta_{l1}(\delta_{j1}\delta_{k1} + \delta_{j2}\delta_{k2}) + \delta_{j1}\delta_{l1}(\delta_{i1}\delta_{k1} + \delta_{i2}\delta_{k2}) \\ &+ \delta_{i2}\delta_{j2}(\delta_{k1}\delta_{l1} + \delta_{k2}\delta_{l2}) + \delta_{k2}\delta_{l2}(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) \\ &+ \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2} - \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\end{aligned} \tag{A.3}$$

In view of the relationship of the type

$$\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} = \delta_{ij} - n_i n_j \tag{A.4}$$

where $n_i = \delta_{i3}$, Equation (A.3) can be rewritten as

$$\begin{aligned}\int_0^{2\pi} I^{6m}_{ijkl}\,d\theta = \frac{\pi}{4}[&\delta_{i1}\delta_{k1}(\delta_{jl} - n_j n_l) + \delta_{j1}\delta_{k1}(\delta_{il} - n_i n_l) \\ &+ \delta_{i1}\delta_{l1}(\delta_{jk} - n_j n_k) + \delta_{j1}\delta_{l1}(\delta_{ik} - n_i n_k) \\ &+ \delta_{i2}\delta_{j2}(\delta_{kl} - n_k n_l) + \delta_{k2}\delta_{l2}(\delta_{ij} - n_i n_j) \\ &+ \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2} - \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\end{aligned} \tag{A.5}$$

An alternative rearrangement of Equation (A.2) similarly gives

$$\begin{aligned}\int_0^{2\pi} I^{6m}_{ijkl}\,d\theta = \frac{\pi}{4}[&\delta_{i1}\delta_{j1}(\delta_{kl} - n_k n_l) + \delta_{k1}\delta_{l1}(\delta_{ij} - n_i n_j) \\ &+ \delta_{i2}\delta_{k2}(\delta_{jl} - n_j n_l) + \delta_{j2}\delta_{k2}(\delta_{il} - n_i n_l) \\ &+ \delta_{i2}\delta_{l2}(\delta_{jk} - n_j n_k) + \delta_{j2}\delta_{l2}(\delta_{ik} - n_i n_k) \\ &- \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2} + \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\end{aligned} \tag{A.6}$$

Summation of Equations (A.5) and (A.6) finally gives

$$\begin{aligned}\int_0^{2\pi} I^{6m}_{ijkl}\,d\theta = \frac{\pi}{4}[&(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l) \\ &+ (\delta_{jk} - n_j n_k)(\delta_{il} - n_i n_l) + (\delta_{ki} - n_k n_i)(\delta_{jl} - n_j n_l)\end{aligned} \tag{A.7}$$

which is Equation (37) of the text.

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