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An analysis of pendulum motion in the presence of quadratic and linear drag

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Abstract
Mathematical and physical aspects in the analysis of pendulum motion in the presence of quadratic and linear damping are presented. New results are obtained from the analysis which are of research and educational interest. Closed-form expressions for the velocity vs large angle of swing are derived in the case of quadratic damping for both forward and backward swings. In the case of linear damping, there is no closed-form expression for the velocity vs large angle of swing, but an implicit relationship between the velocity and small angle of swing is derived and discussed. The derived formulas are useful for the determination of the amplitudes of the pendulum swings corresponding to given initial velocities and for the evaluation of the dissipated energy at any stage of the pendulum motion. Instructive project-based activities are suggested, which are suitable for collaborative learning and undergraduate student research.

Keywords: drag force, Lagrangian, linear damping, pendulum motion, quadratic damping, spherical pendulum, spring pendulum

1. Introduction

Pendulums have been used since Galileo Galilei's time for various scientific and engineering applications. They have been used to design mechanical clocks (Huygens pendulum), to experimentally determine the acceleration of gravity and demonstrate Earth's rotation (Foucault pendulum), as parts of seismometer instruments to record earthquake motion, in metronomes to help maintain the speed of music, in viscometry and rheometry to experimentally determine viscoelastic properties of fluids (torsional pendulums), in vibration absorption and

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attenuation devices to protect engines, buildings, and bridges from damage (centrifugal pendulums, friction pendulums), in devices for structural health monitoring, in materials testing to determine the crushing strengths of rocks (absorbing and striking pendulums) or the toughness of the material (Charpy's impact test of notched bars). Other well-known examples are pendulums used in recreation and amusement (park swings, swinging trapeze), wrecking ball pendulums used to demolish old buildings, and wooden ballistic pendulums used to measure the velocity of a fired bullet.

In this paper, we present the analysis of pendulum motion under quadratic and linear damping. The quadratic damping is based on the assumption that the magnitude of the drag force, opposite in direction to the velocity vector, is proportional to the square of the velocity, $F_\mathrm{d} = cv^2$, where c is the damping resistance coefficient. For the motion of a spherical ball of diameter d in a fluid of mass density ρ, the resistance coefficient is $c = (1/8)c_\mathrm{d}\rho\pi d^2$, where $c_\mathrm{d} = 0.47$ is the so-called drag coefficient. This type of drag, known as Newton's drag, applies in the range of Reynolds number $10^3 < \mathrm{Re} = vd/\nu < 3 \times 10^5$, where ν is the kinematic viscosity of the fluid. In the model of linear damping the magnitude of the drag force is assumed to be proportional to the velocity, $F_\mathrm{d} = c_\mathrm{L}v$, where the resistance coefficient c_L is, for a spherical ball, given by $c_\mathrm{L} = 3\pi\eta d$, with $\eta = \rho\nu$ being the dynamic viscosity of the fluid. This type of drag is known as Stokes' drag, which applies for slow (creepy) flows, in the range of small Reynolds number ($\mathrm{Re} < 1$). For the intermediate range of Reynolds number ($1 < \mathrm{Re} < 1000$), the drag force is a more complicated nonlinear function of velocity, which can be obtained by fitting experimental data [1].

There is a large amount of literature devoted to pendulum motion in the presence of various types of drag. The effects of damping on pendulum motion in the context of the precise experimental determination of the local gravitational acceleration were examined in [2]. Different models of pendulum damping and their comparison with the behavior of real pendulums were studied in [3]. The estimation of damping parameters in planar motion of pendulums have been reported in [4]. An analytical approximation of the solution to the differential equation describing a linearly damped pendulum undergoing large-angle swings is presented in [5]. Asymptotic stability of pendulums under quadratic damping with a time-varying damping coefficient has also been investigated [6]. The free and forced oscillations of a torsion spring pendulum damped by viscous and dry friction were investigated in [7]. The nonlinear behavior of a kinematically excited spring pendulum, in which its suspension point moves in an elliptic path, has been considered in [8]. The resonance dynamics of an elastic pendulum (swinging spring) in three dimensions has been studied in [9]. A historical survey of different approaches used in the study of pendulums with air resistance was given in [10]. The analysis of damped pendulum motion in relation to undergraduate engineering and physics education was discussed from both theoretical and experimental points of view in [11–15]. Classical mechanics and physics textbooks such as [16–18] also address various aspects of pendulum motion. The linear and quadratic damping models have also been frequently used in the study of projectile motion in the presence of ambient drag [19–24].

The contents of the present paper are as follows. The plane pendulum undergoing large swings is considered in section 2. A closed-form expression for the velocity vs angle of swing, $v = v(\theta)$, is derived for both the forward and backward swings. The non-dimensional parameter kl, where $k = c/m$ and l is the length of the pendulum, accounts for the interplay of the geometry, inertia, and damping. The minimum initial velocity required for a full pendulum revolution is determined as a function of kl. Pendulum motion under linear damping is considered in section 3. In contrast to quadratic damping, in this case there is no explicit closed-form expression for the velocity $v = v(\theta)$. A new derivation is presented in the case of small angle of swing which provides an implicit closed-form expression of the type $f(v, \theta) = 0$, as well as the parametric expressions for v and θ in terms of an introduced time-like parameter. Energy considerations are also discussed. The analysis of an elastic (spring) pendulum and a spherical pendulum under quadratic damping is briefly discussed in the closing section 4. Throughout the presentation, effort is made to include pedagogically appealing exercises which can be used in teaching undergraduate courses of applied mechanics and physics.

2. Pendulum motion in the presence of quadratic damping

Figure 1(a) shows a pendulum with a small spherical ball of mass m attached to an inextensible and massless cable of length l. The other end of the cable is fixed. The acceleration of gravity is g. The initial position of the pendulum is defined by the angle θ_0, at which the ball is given an initial velocity v_0. At an arbitrary position defined by the angle θ, the forces acting on the ball are its weight mg, the drag force cv^2, opposite in direction to the current velocity v and assumed to be quadratic in the velocity, and the force of the cable S. By Newton's second law applied in the direction orthogonal to the cable, and in the direction of the cable, we have, during the forward swing,

$$\frac{\mathrm{d}v}{\mathrm{d}t} + kv^2 = -g\sin\theta, \qquad \frac{v^2}{l} = \frac{S}{m} - g\cos\theta, \tag{2.1}$$

where $k = c/m$ is the parameter with the dimension 1/length. Since $v = l\mathrm{d}\theta/\mathrm{d}t$, the first equation in (2.1) becomes a second-order quasi-linear differential equation for the angle θ,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + kl\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + \frac{g}{l}\sin\theta = 0. \tag{2.2}$$

Equation (2.2) does not have a closed-form solution, but it can be readily solved by numerical means. The force in the cable then follows from the second equation in (2.1), and it must remain positive (tensile) during pendulum swings. For the backward swing, the plus sign in front of the kl term in (2.2) is replaced with the minus sign.

Student exercise 1. Solve (2.2) numerically for $kl = 0.25$ and initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = \sqrt{gk}$. Determine the extents of the first forward and backward pendulum swings and plot the corresponding variations of the cable force with θ. [Hint: reduce the second order ODE (2.2) to two first order ODEs $\mathrm{d}\theta/\mathrm{d}t = \omega$ and $\mathrm{d}\omega/\mathrm{d}t = -kl\,\mathrm{sign}(\omega)\omega^2 - (g/l)\sin\theta$, and apply the MATLAB function ode45.]

2.1. Closed-form expression for $v = v(\theta)$

Although there is no closed-form solution to equation (2.2), there is a closed-form expression for the velocity in terms of the angle of swing, which is derived in this section. Because the velocity of the ball during forward swing is defined by $v = l\mathrm{d}\theta/\mathrm{d}t$, its time rate of change can be expressed as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v}{l}\frac{\mathrm{d}v}{\mathrm{d}\theta} = \frac{1}{2l}\frac{\mathrm{d}v^2}{\mathrm{d}\theta}, \tag{2.3}$$

and the first equation in (2.1) becomes [12, 13]

$$\frac{\mathrm{d}v^2}{\mathrm{d}\theta} + 2klv^2 = -2gl\sin\theta. \tag{2.4}$$

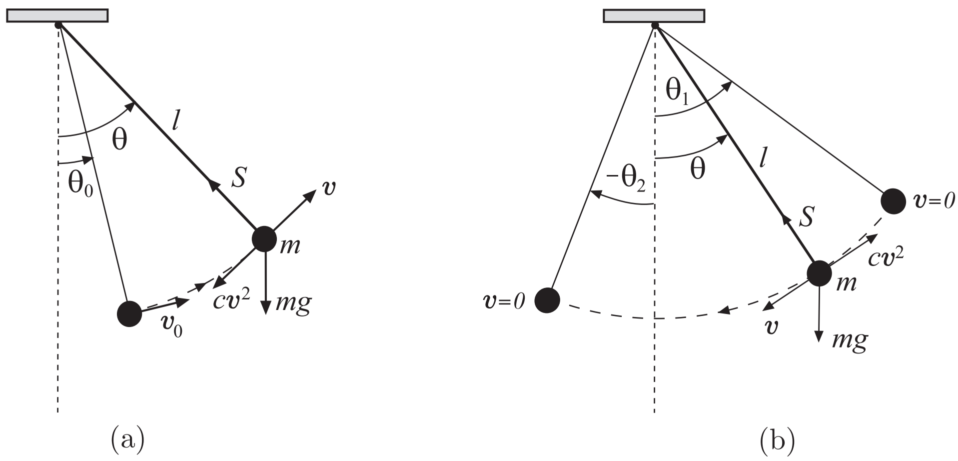


Figure 1. (a) A pendulum with a spherical ball of mass m attached to a thin inextensible cable of length l. The other end of the cable is fixed. The initial position of the pendulum is defined by the angle θ_0, at which the ball is given an initial velocity v_0. In an arbitrary configuration during forward swing of the pendulum, defined by angle θ, the forces acting on the ball are its weight mg, the drag force cv^2, and the force in the cable S. (b) The motion of pendulum during its reverse swing $\theta_2 \leqslant \theta \leqslant \theta_1$. Shown are the weight mg, the drag force cv^2 and the cable force S in the configuration specified by the angle θ. The velocity of the ball is equal to zero for $\theta = \theta_1$ and $\theta = \theta_2$.

The general solution of the linear differential equation (2.4) for v^2 can be easily derived by multiplying (2.4) with the integrating factor $e^{2kl\theta}$, which gives, upon integration,

$$v^2 = Ce^{-2kl\theta} + \frac{2gl}{1+4k^2l^2}(\cos\theta - 2kl\sin\theta). \tag{2.5}$$

Alternatively, (2.5) can be obtained as the sum of the complementary and particular solution to (2.4), the latter being sought in the form of a linear combination of $\sin\theta$ and $\cos\theta$. The integration constant C in (2.5) is found by applying the initial condition $v(\theta_0) = v_0$, which gives

$$C = v_0^2 e^{2kl\theta_0} - \frac{2gl}{1+4k^2l^2}(\cos\theta_0 - 2kl\sin\theta_0)\,e^{2kl\theta_0}. \tag{2.6}$$

Consequently, the velocity expression (2.5) becomes

$$v^2 = v_0^2 e^{-2kl(\theta-\theta_0)} + \frac{2gl}{1+4k^2l^2}\Big[\cos\theta - 2kl\sin\theta - (\cos\theta_0 - 2kl\sin\theta_0)e^{-2kl(\theta-\theta_0)}\Big]. \tag{2.7}$$

If $kl = 0$ (no air resistance), (2.7) reduces to $v^2 = v_0^2 - 2gl(\cos\theta_0 - \cos\theta)$, which also follows directly from the conservation of mechanical energy.

Student exercise 2. Derive equation (2.4) by using the work principle $\mathrm{d}K + \mathrm{d}U = \mathrm{d}W_{\mathrm{diss}}$, where K and U are the kinetic and potential energy of the ball, and $\mathrm{d}W_{\mathrm{diss}}$ is the increment of dissipative work from the drag force opposing the motion. [Hint: $K = (1/2)mv^2$, $U = mgl(1-\cos\theta)$, $\mathrm{d}W_{\mathrm{diss}} = -(cv^2)l\mathrm{d}\theta$.]

The swings of the pendulum in the range $|\theta| \leqslant \pi/2$ are of particular interest. The initial velocity must then be $v_0 < v_{0,\pi/2}$, where $v_{0,\pi/2}$ is determined from the condition $v(\theta = \pi/2) = 0$. This gives

$$v_{0,\pi/2}^2 = \frac{2gl}{1+4k^2l^2}\left[\cos\,\theta_0 - 2kl\,\sin\,\theta_0 + 2kle^{kl(\pi-2\theta_0)}\right]. \tag{2.8}$$

For any $v_0 < v_{0,\pi/2}$, the force in the cable $S = mg\cos\theta + mv^2/l$ is positive, because $\cos\theta > 0$. If $\theta_0 = 0$, (2.8) simplifies to

$$v_{0,\pi/2}^2 = \frac{2gl}{1+4k^2l^2}\left(1 + 2kle^{kl\pi}\right). \tag{2.9}$$

2.2. Angle of maximum swing

The angle of maximum swing (θ_1) in the range of pendulum swings $|\theta| \leqslant \pi/2$ is obtained from the condition $v(\theta_1) = 0$, which, by (2.7), is

$$(\cos\,\theta_1 - 2kl\,\sin\,\theta_1)e^{2kl(\theta_1-\theta_0)} + (1+4k^2l^2)\,\frac{v_0^2}{2gl} - (\cos\,\theta_0 - 2kl\,\sin\,\theta_0) = 0. \tag{2.10}$$

For a given $v_0/(2gl)^{1/2}$, and a given value of kl, this nonlinear equation for θ_1 can be solved numerically by using, for example, the MATLAB function *fzero*. The results for the variation of θ_1/π with $v_0/(2gl)^{1/2}$ for several values of kl are shown in figure 2 (portions of the curves corresponding to $\theta_1 \leqslant \pi/2$). The initial angle is taken to be $\theta_0 = 0$, and the initial velocity is in the range $0 \leqslant v_0 \leqslant v_{0,\pi/2}$, where $v_{0,\pi/2}$ is specified by (2.9). Note also that for $|\theta| \leqslant \pi/2$, the force in the cable $S = mg\cos\theta + mv^2/l$ is necessarily positive (tensile).

2.3. Pendulum swings beyond $\theta = \pi/2$

Pendulum motion beyond $\theta = \pi/2$ is possible provided that the force in the cable remains tensile, i.e.

$$S = \frac{mv^2}{l} + mg\,\cos\,\theta \geqslant 0, \tag{2.11}$$

where v^2 is given by (2.7). The force becomes equal to zero ($S = 0$) at the angle $\theta = \theta_1 \geqslant \pi/2$, defined by $v^2(\theta_1) = -gl\cos\theta_1$. Upon using (2.7), this gives the relationship between θ_1 and v_0,

$$\begin{aligned} v_0^2 = &\frac{2gl}{1+4k^2l^2}\left[\cos\,\theta_0 - 2kl\,\sin\,\theta_0 - (\cos\,\theta_1 - 2kl\,\sin\,\theta_1)e^{2kl(\theta_1-\theta_0)}\right] \\ &- gl\,\cos\,\theta_1 e^{2kl(\theta_1-\theta_0)}. \end{aligned} \tag{2.12}$$

If the ball is launched with the initial velocity determined from (2.12), the ball will swing from $\theta = \theta_0$ to $\theta = \theta_1$, but will subsequently, in the absence of the cable force (slack cable), begin its projectile motion in the field of gravity with the initial velocity $v = \sqrt{-gl\,\cos\,\theta_1}$, where $\cos\theta_1 < 0$ in the range $\pi/2 < \theta_1 < 3\pi/2$.

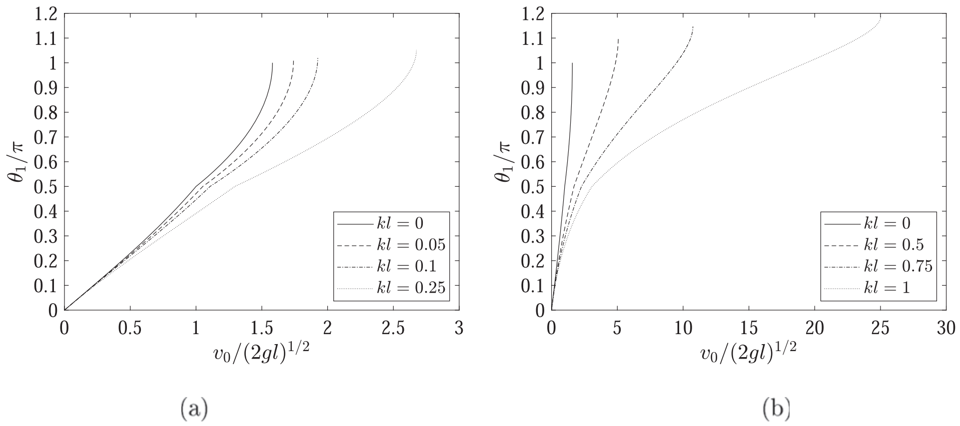


Figure 2. The variation of the angle of maximum swing θ_1 (scaled by π) with $cv_0/(2gl)^{1/2}$ for the case $\theta_0 = 0$ and for the selected values of kl shown in the figure legends of parts (a) and (b). For $v_0 \leqslant v_{0,\pi/2}$, the velocity $v(\theta_1) = 0$ and $\theta_1 \leqslant \pi/2$. For $v_{0,\pi/2} < v_0 \leqslant v_{0,\pi}$, the cable force $S(\theta_1) = 0$, the angle $\pi/2 < \theta_1 \leqslant \pi$, and $v(\theta_1) > 0$. For $v_{0,\pi} < v_0 \leqslant v_{0,2\pi}$, the cable force $S(\theta_1) = 0$, the angle $\pi < \theta_1 \leqslant \theta_1^{\max}$, and $v(\theta_1) > 0$. If the pendulum reaches the angle $\theta_1^{\max}$, it will extend its swing to complete a full revolution, and continue to swing beyond $\theta = 2\pi$ for an amount that depends on the value of the residual velocity $v(2\pi)$. The expression for $v_{0,\pi/2}$ is given by (2.9), and for $v_{0,\pi}$ by (2.13). The value $v_{0,2\pi}$ is the value of initial velocity required for the pendulum to reach $\theta_1^{\max}$ (and thus 2π).

The required (minimum) initial velocity to reach the angle $\theta_1 = \pi$ is defined by

$$v_{0,\pi}^2 = \frac{2gl}{1+4k^2l^2}\left[\cos\,\theta_0 - 2kl\,\sin\,\theta_0 + e^{2kl(\pi-\theta_0)}\right] + gle^{2kl(\pi-\theta_0)}. \tag{2.13}$$

The velocity of the ball at that angle is $v(\pi) = \sqrt{gl}$, independently of k. The variation of the angle of maximum swing θ_1/π with $v_0/(2gl)^{1/2}$ in the range $v_{0,\pi/2} \leqslant v_0 \leqslant v_{0,\pi}$, where $S(\theta_1) = 0$, is shown in figure 2 (portions of the curves corresponding to $\pi/2 \leqslant \theta_1 \leqslant \pi$). The initial angle is taken to be $\theta_0 = 0$ with used values of kl shown in the figure captions.

If $kl = 0$, (2.12) simplifies to

$$v_0^2 = 2gl\left(\cos\,\theta_0 - \frac{3}{2}\cos\,\theta_1\right). \tag{2.14}$$

In this case, from (2.13), the (minimum) initial velocity required to reach the angle $\theta_1 = \pi$ from $\theta_0 = 0$ is $v_0 = \sqrt{5gl}$, while $v(\pi) = \sqrt{gl}$. For any initial velocity v_0 greater or equal to that specified by (2.14) with $\theta_1 = \pi$, the ball will circle indefinitely (in the same counterclockwise direction) around the hanging point of the pendulum cable.

2.4. Pendulum swings beyond $\theta = \pi$

If the initial velocity v_0, corresponding to a given value of kl, is greater than $v_{0,\pi}$ specified by (2.13), the cable force $S(\theta = \pi)$ is positive and the pendulum motion extends beyond $\theta = \pi$, until S becomes equal to zero in the range $\pi < \theta < 3\pi/2$ (or the velocity becomes equal to zero, if v_0 is sufficiently large for the pendulum to swing beyond 2π, but less than $5\pi/2$). For each kl and an assumed $v_0 > v_{0,\pi}$, one can determine numerically the corresponding value of $\pi < \theta_1 < 3\pi/2$ at which $S(\theta_1) = 0$. The numerical results for the variation of θ_1 with v_0 are included in figure 2 (portions of the curves corresponding to $\theta_1 > \pi$). For example, for $kl = 0.25$, it is found that $\theta_1^{\max} = 1.051\pi$, corresponding to $v_0 = 2.675\sqrt{2gl}$, while for $kl = 0.5$ the values are $\theta_1^{\max} = 1.102\pi$ and $v_0 = 5.073\sqrt{2gl}$. The corresponding velocities $v(\theta_1^{\max})$ in these two cases are $0.7026\sqrt{2gl} = 0.2645 v_{0\,\pi}$ and $0.6889\sqrt{2gl} = 0.1417 v_{0\,\pi}$, respectively, where $v_{0,\pi}$ is defined by (2.13).

Once the pendulum reaches the configuration defined by $\theta_1^{\max}$, it has enough velocity (energy) to extend its swing and complete a full revolution (because for $\theta_1^{\max} < \theta < 2\pi$, both $S > 0$ and $v > 0$), and to swing beyond $\theta = 2\pi$ by an amount that depends on the residual velocity $v(2\pi)$ at the end of the first revolution. The motion beyond $\theta = 2\pi$ can be analyzed by considering the velocity $v(2\pi)$ as the initial velocity for the beginning of that stage of motion. (Note that within the fourth quadrant $3\pi/2 \leqslant \theta \leqslant 2\pi$, $v^2(\theta)$ and $S(\theta)$ as given by (2.7) and (2.11) are both positive definite, thus if the pendulum enters the third quadrant it will necessarily swing through it and exit from it).

Figure 3 shows the variations of $v = v(\theta)$ and $S = S(\theta)$ in the case when v_0 is just large enough ($v_0 = v_{0,2\pi}$) for the pendulum to reach the angle $\theta_1^{\max}$ for each value of kl, and subsequently to complete a full revolution of 2π. Clearly, from figure 3 the cable force $S(\theta_1^{\max}) = 0$ for each kl, while $S(\theta) > 0$ for all other values of $0 \leqslant \theta \leqslant 2\pi$.

Student exercise 3. If $kl = 0.3$, determine the minimum initial velocity $v_0(\theta = 0)$ for which the pendulum will complete a full revolution, from $\theta = 0$ to $\theta = 2\pi$. At what angle θ is the cable force equal to zero? Plot the corresponding variations $v = v(\theta)$ and $S = S(\theta)$ for $0 \leqslant \theta \leqslant 2\pi$. How far will the pendulum extend its swing beyond the angle $\theta = 2\pi$? [Partial answer: $v_0 = 3.0124\sqrt{2gl}$, $\theta_1^{\max} = 1.0622\pi$.]

2.5. *Reverse swing*

The analysis in this and subsequent sections will be restricted to pendulum swings in the range $|\theta| \leqslant \pi/2$. Upon reaching the angle of maximum swing θ_1, at which the velocity of the ball is equal to zero, the pendulum begins its reverse swing (figure 1(b)). The corresponding equation of motion is

$$\frac{\mathrm{d}v^2}{\mathrm{d}\theta} - 2klv^2 = -2gl\sin\theta. \tag{2.15}$$

This equation is identical to equation (2.4), provided that k in equation (2.4) is replaced with $-k$. Thus, the general solution of (2.15) can be recognized from (2.5) to be

$$v^2 = Ce^{2kl\theta} + \frac{2gl}{1+4k^2l^2}(\cos\theta + 2kl\sin\theta). \tag{2.16}$$

The integration constant C is found by applying the initial condition $v(\theta = \theta_1) = 0$, which gives

$$C = -\frac{2gl}{1+4k^2l^2}(\cos\theta_1 + 2kl\sin\theta_1)\,e^{-2kl\theta_1}. \tag{2.17}$$

Thus, the velocity expression (2.16) becomes

$$v^2 = \frac{2gl}{1+4k^2l^2}\left[\cos\theta + 2kl\sin\theta - (\cos\theta_1 + 2kl\sin\theta_1)e^{2kl(\theta-\theta_1)}\right]. \tag{2.18}$$

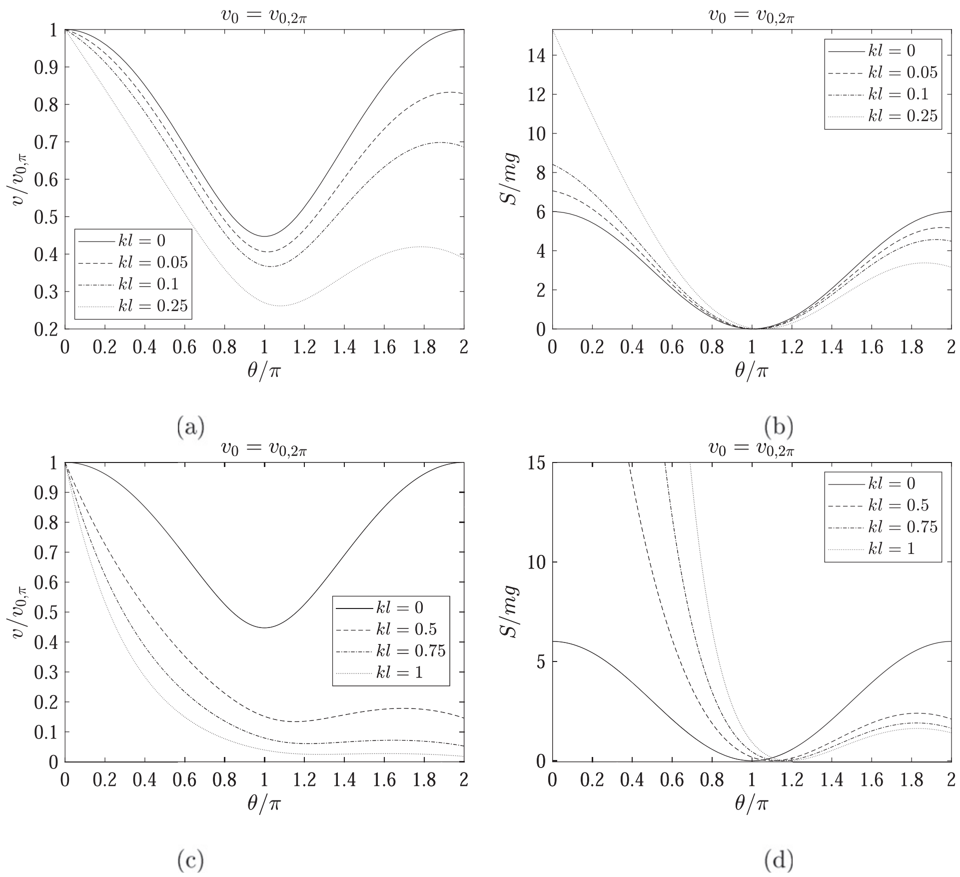


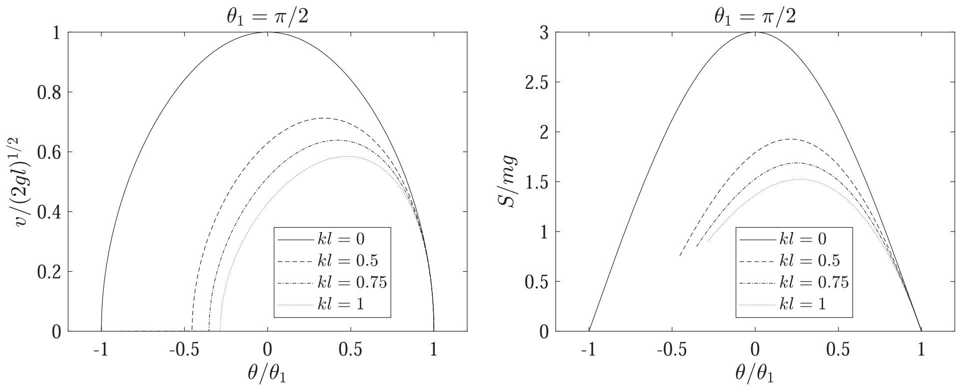
Figure 3. (a), (c) The variation of the velocity $v = v(\theta)$ (scaled by $v_{0,\pi}$) in the range $0 \leqslant \theta \leqslant 2\pi$ in the case of initial velocity $v_0 = v_{0,2\pi}$, where $v_{0,2\pi}$ is the smallest initial velocity for the pendulum to swing from $\theta = 0$ to $\theta = 2\pi$. (b), (d) The corresponding variation of the cable force $S = S(\theta)$. For each value of kl, the cable force $S(\theta_1^{\max}) = 0$, while $v(\theta_1^{\max}) > 0$. The large values of the cable force for large values of kl in the early stage of motion, associated with large velocity and correspondingly large centripetal force, are cut off in part (d) to better depict the variation of S with θ near $S = 0$.

2.6. Angle of maximum reverse swing

The angle of maximum reverse swing (θ_2) is obtained from the condition $v(\theta_2) = 0$. From (2.18), this condition is

$$\cos\,\theta_2 + 2kl\,\sin\,\theta_2 - (\cos\,\theta_1 + 2kl\,\sin\,\theta_1)e^{2kl(\theta_2-\theta_1)} = 0. \tag{2.19}$$

For any value of $0 < \theta_1 \leqslant \pi/2$, the nonlinear equation (2.19) can be solved for θ_2 numerically by using the MATLAB function *fzero*. The variation of the velocity of the ball $v(\theta)$ in the angle range $\theta_2 \leqslant \theta \leqslant \theta_1 = \pi/2$, for the same values of kl as used in figure 2, is shown in figure 4(a). The variation of the corresponding force in the cable $S(\theta)/mg$ is shown in figure 4(b). This force is determined from $S = mg\cos\theta + mv^2/l$, where v^2 is defined by (2.18). Since the velocity of the ball decreases with the increase of the parameter kl, the maximum value of S also decreases with the increase of kl.



(a) (b)

Figure 4. (a) The variation of the velocity of the ball $v(\theta)/\sqrt{2gl}$ in the entire angle range of the reverse swing from $\theta = \theta_1 = \pi/2$ to $\theta = \theta_2$ for the indicated values of kl. The value of θ_2 depends on kl and is determined from (2.19). (b) The corresponding variation of the cable force $S(\theta)/mg$. Except for $kl = 0$, the force at the end of the reverse swing $S(\theta_2) \neq 0$.

2.7. Subsequent swings

The expressions for the velocity of the ball during subsequent forward and reverse swings can be recognized immediately from the derived expressions for the first forward and the first reverse swing. Indeed, from (2.18) it follows that the square of the velocity for the reverse swing from $\theta = \theta_{2n-1}$ to $\theta = \theta_{2n}$ $(n = 1, 2, 3, \ldots)$ is

$$v^2 = \frac{2gl}{1 + 4k^2l^2}\left[\cos\theta + 2kl\sin\theta - (\cos\theta_{2n-1} + 2kl\sin\theta_{2n-1})e^{2kl(\theta-\theta_{2n-1})}\right], \tag{2.20}$$

while for the forward swing from $\theta = \theta_{2n}$ to $\theta = \theta_{2n+1}$, the square of the velocity is, from (2.7),

$$v^2 = \frac{2gl}{1 + 4k^2l^2}\left[\cos\theta - 2kl\sin\theta - (\cos\theta_{2n} - 2kl\sin\theta_{2n})e^{-2kl(\theta-\theta_{2n})}\right]. \tag{2.21}$$

2.8. Time variation $\theta = \theta(t)$

The time variation $\theta = \theta(t)$ in the case of quadratic damping cannot be determined analytically, even in the case of small angles of swing, but can be obtained numerically by consecutive numerical integration of

$$\int^t \mathrm{d}t = \pm l \int^\theta \frac{\mathrm{d}\theta}{v(\theta)}, \tag{2.22}$$

with the explicit expressions for $v = v(\theta)$ determined in the previous sections for both forward and reverse swings. The time and angle intervals of each swing, needed to specify the

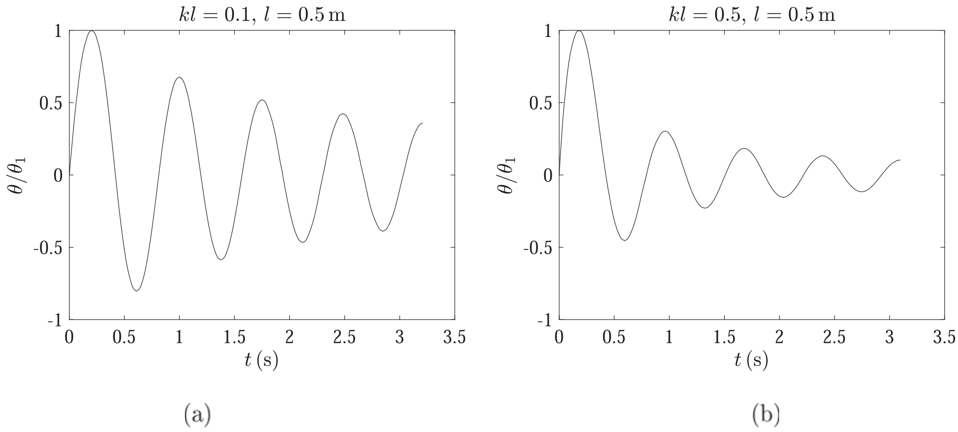


Figure 5. The time variation of θ for two selected values of *kl* during several consecutive forward and reverse swings of the pendulum whose length is $l = 0.5$ m. The pendulum motion is launched from the configuration $\theta = 0$ with the initial velocity adjusted so that the maximum angle in the first forward swing is $\theta_1 = \pi/2$ for both values of *kl*.

bounds of the integrals in (2.22), have been determined numerically in previous sections. The plus sign in (2.22) corresponds to forward swings and the minus sign to reverse swings. The results of numerical integration using the MATLAB function *integral* for several consecutive swings and two selected values of *kl* are shown in figure 5. The length of the pendulum is taken to be $l = 0.5$ m. In each case the time duration of the reverse swing is longer than the time duration of the preceding forward swing, due to continuously decreasing magnitude of the velocity caused by damping. The pronounced effect of the parameter *kl* on the rate of decrease of the maximum angle of swing with the increase of the number of swings can be quantified by comparing parts (a) and (b) of figure 5.

Student exercise 4. The motion of a toboggan of mass *m* along a circular track of radius *R* in the presence of dry (Coulomb) friction is described by equations analogous to equations that describe the pendulum motion in the presence of quadratic drag. The coefficient of kinetic friction μ appears in the equations instead of the parameter *kl*, and the normal reaction force *N* from the track replaces the cable force *S*. Derive and solve the differential equation of the toboggan motion for the velocity $v = v(\theta)$, with the initial condition $v(\theta_0) = v_0$. [Hint: The governing differential equations of the toboggan's motion are: $m\mathrm{d}v/\mathrm{d}t = -\mu N - mg\sin\theta$ and $N = (mv^2/R) + mg\cos\theta$.]

2.9. Time variation in the absence of damping

As is well known, in the absence of damping there is an explicit $t = t(\theta)$ expression in terms of the elliptic integral of the first kind [16–18]. In this case, by energy considerations or from (2.7), the velocity $v = v(\theta)$ expression during the forward swing from $\theta = \theta_0$, with initial velocity v_0, is

$$v = \sqrt{v_0^2 - 2gl(\cos\,\theta_0 - \cos\,\theta)}. \tag{2.23}$$

Using conveniently the angle $\theta/2$, (2.23) can be rewritten as

$$v = \frac{2\sqrt{gl}}{K_0}\sqrt{1 - K_0^2 \sin^2\frac{\theta}{2}}, \quad \frac{1}{K_0^2} = \frac{v_0^2}{4gl} + \sin^2\frac{\theta_0}{2}. \tag{2.24}$$

The explicit time vs angle relationship then follows by integration of the expression $\mathrm{d}t = l\mathrm{d}\theta/v$, i.e.

$$t = K_0\sqrt{l/g}\int_{\theta_0/2}^{\theta/2} \frac{\mathrm{d}(\vartheta/2)}{\sqrt{1 - K_0^2\sin^2(\vartheta/2)}}, \quad \theta \leqslant \theta_{\max}. \tag{2.25}$$

The expression for the angle $\theta_{\max}$ depends on the range of the forward swing. If $\theta_{\max} \leqslant \pi/2$, the condition to determine $\theta_{\max}$ is the vanishing of the velocity, $v(\theta_{\max}) = 0$. From (2.24), this gives $\theta_{\max} = 2\sin^{-1}(1/K_0)$. In order that $\theta_{\max} \leqslant \pi/2$, the initial velocity must be such that $v_0^2 \leqslant 2gl\cos\theta_0$. If $2gl\cos\theta_0 \leqslant v_0^2 \leqslant gl(3 + 2\cos\theta_0)$, then $\pi/2 \leqslant \theta_{\max} \leqslant \pi$, where $\theta_{\max}$ is determined from the condition of the vanishing cable force, $S(\theta_{\max}) = 0$. From (2.7) and (2.11), this gives

$$\theta_{\max} = 2\sin^{-1}\sqrt{\frac{1}{6} + \frac{2}{3K_0^2}}. \tag{2.26}$$

The time to reach a particular angle $\theta \leqslant \theta_{\max}$ can be obtained from (2.25) with the help of the MATLAB function ellipticF. For example, if the ball is given the initial velocity $v_0 = \sqrt{2gl}$ at $\theta_0 = 0$, it will reach the angle $\theta = \pi/2$ in time $t = 1.854\sqrt{l/g}$. On the other hand, if the ball is given the initial velocity $v_0 = \sqrt{5gl}$ at $\theta_0 = 0$, it will reach the angle $\theta = \pi/2$ in time $t = 0.766\sqrt{l/g}$ and the angle $\theta = \pi$ in time $t = 2.019\sqrt{l/g}$. The derivation of the exact and approximate expressions for the large-swing amplitude-dependent pendulum period is also of great importance, which has been elaborated upon in this journal by a number of investigators [25–28].

Student exercise 5. If a pendulum is released from the position θ_0 with zero initial speed, derive the expression for the pendulum period T, i.e. show that

$$T = \frac{2T_0}{\pi}\int_0^{\phi_0}\frac{\mathrm{d}\phi}{\sqrt{\sin^2\phi_0 - \sin^2\phi}} = \frac{2T_0}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\psi}{\sqrt{1 - \sin^2\phi_0\sin^2\psi}},$$

where the last integral represents the complete elliptic integral of the first kind, $K(sin^2\,\phi_0)$, and the small-angle period is $T_0 = 2\pi\sqrt{l/g}$. [Hint: Note that $\sin\phi = \sin\phi_0\sin\psi$.]

3. Pendulum motion in the presence of linear damping

Slow swings of certain pendulums in a viscous fluid could be analyzed by assuming the drag force to be linear in velocity, provided that the Reynolds number remains less than 1. In contrast to quadratic damping, however, in the case of linear damping there is no explicit closed-form expression for the $v = v(\theta)$ relationship for an arbitrary amplitude of oscillations, although there is a well-known closed-form expression for $\theta = \theta(t)$ in the case of small-amplitude oscillations. This is discussed in this section. We first give the classical derivation of the expression $\theta = \theta(t)$, and then present an alternative, less known and possibly novel derivation of the solution to the considered problem.

The differential equation of motion in the case of a pendulum with linear damping and small swing angles ($\sin\theta \approx \theta$) is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + k_\mathrm{L}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega_\mathrm{n}^2\theta = 0, \quad k_\mathrm{L} = \frac{c_\mathrm{L}}{m}, \quad \omega_\mathrm{n}^2 = \frac{g}{l}, \tag{3.1}$$

subjected to the initial condition $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = \dot{\theta}_0$, where the superimposed dot denotes the time derivative, and ω_n is the natural frequency. By assuming the solution to (3.1) in the form $\theta = \exp(pt)$, it readily follows that, in the case of oscillatory motion ($\omega_\mathrm{n} > k_\mathrm{L}/2$),

$$p_{1,2} = -\frac{k_\mathrm{L}}{2} \pm i\omega_\mathrm{d}, \quad \omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}, \quad \zeta = \frac{k_\mathrm{L}}{2\omega_\mathrm{n}}, \tag{3.2}$$

where $i = \sqrt{-1}$ and ω_d is the damped natural frequency, dependent on the damping ratio ζ. Consequently,

$$\theta = e^{-k_\mathrm{L}t/2}\left[\theta_0 \cos\omega_\mathrm{d}t + \frac{1}{2\omega_\mathrm{d}}\left(k_\mathrm{L}\theta_0 + 2\dot{\theta}_0\right)\sin\omega_\mathrm{d}t\right], \tag{3.3}$$

and

$$\dot{\theta} = -\frac{k_\mathrm{L}}{2}\theta - \omega_\mathrm{d}e^{-k_\mathrm{L}t/2}\left[\theta_0 \sin\omega_\mathrm{d}t - \frac{1}{2\omega_\mathrm{d}}\left(k_\mathrm{L}\theta_0 + 2\dot{\theta}_0\right)\cos\omega_\mathrm{d}t\right]. \tag{3.4}$$

If $\theta_0 = 0$, the response (θ and $\dot{\theta}$) is proportional to $\dot{\theta}_0$; if $\dot{\theta}_0 = 0$, the response is proportional to θ_0.

3.1. Relation $v = v(\theta)$

An alternative approach to describe the pendulum motion in the presence of linear damping during the first forward swing is as follows. Instead of solving the second-order differential equation (3.1) for the angle of swing $\theta = \theta(t)$, we consider the first-order, albeit quasi-linear, differential equation for the velocity $v = v(\theta)$, similarly as in the analysis of quadratic damping presented in section 2.1. Since $(l\mathrm{d}\theta/\mathrm{d}t)^2 = v\mathrm{d}v/\mathrm{d}\theta$, the differential equation (3.1) can be rewritten as

$$v\frac{\mathrm{d}v}{\mathrm{d}\theta} + k_\mathrm{L}lv + (\omega_\mathrm{n}l)^2\theta = 0, \tag{3.5}$$

with the initial condition $v(\theta_0) = v_0$. In contrast to quadratic damping, there is no explicit closed-form solution to (3.5) for $v = v(\theta)$. However, the following derivation provides an implicit closed-form expression of the type $f(v,\theta) = 0$, as well as the parametric expressions for v and θ, from which the velocity can be determined for any given angle of swing. To that goal, we introduce a new time scale $\tau = \theta/\dot{\theta}$, in terms of which the velocity is

$$v = \frac{l\mathrm{d}\theta}{\mathrm{d}t} = \frac{l\theta}{\tau}. \tag{3.6}$$

The substitution of (3.6) and

$$\frac{\mathrm{d}v}{\mathrm{d}\theta} = l\left(\frac{1}{\tau} - \frac{\theta}{\tau^2}\frac{\mathrm{d}\tau}{\mathrm{d}\theta}\right) \tag{3.7}$$

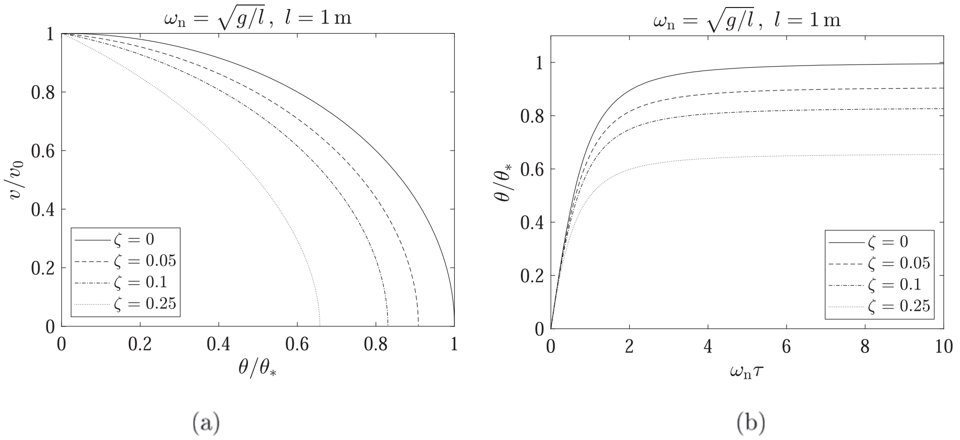


Figure 6. (a) The variation of the velocity $v = v(\theta)$ during the first swing in the case $\theta_0 = 0$ and $\dot{\theta}_0 = v_0/l$ for several values of the damping ratio $\zeta = k_L/2\omega_n$. The pendulum length is $l = 1$ m. The velocity is scaled with the initial velocity v_0, and the angle θ with the angle $\theta_* = v_0/\sqrt{gl}$, which is the angle at the end of the first swing in the absence of damping ($k_L = 0$). (b) The corresponding variation of $\theta = \theta(\tau)$, where $\tau = l\theta/v$.

into (3.5) gives

$$\frac{d\theta}{\theta} = \left(\frac{1}{\tau} - \frac{k_L\tau + \omega_n^2\tau}{1 + k_L\tau + \omega_n^2\tau^2}\right) d\tau. \tag{3.8}$$

For $2\omega_n > k_L$, (3.8) can be integrated by using tables of integrals [29] to obtain

$$\frac{\theta}{\theta_0} = \frac{\tau}{\tau_0}\left(\frac{1 + k_L\tau_0 + \omega_n^2\tau_0^2}{1 + k_L\tau + \omega_n^2\tau^2}\right)^{1/2} \exp\left[-\frac{k_L\gamma(\tau)}{2\omega_d}\right], \tag{3.9}$$

where $\tau_0 = l\theta_0/v_0$, and

$$\gamma(\tau) = \tan^{-1}\frac{k_L + 2\omega_n^2\tau}{2\omega_d} - \tan^{-1}\frac{k_L + 2\omega_n^2\tau_0}{2\omega_d}. \tag{3.10}$$

The expression (3.9) is a single-valued expression for $\theta = \theta(\tau)$ during the first forward swing. For any θ in this range, one can apply the MATLAB function *fzero* to obtain the corresponding τ, and thus the velocity $v = l\theta/\tau$. In this sense, we have determined the desired relationship $v = v(\theta)$. The plot for several values of $\zeta = k_L/2\omega_n$ and $l = 1$ m is shown in figure 6. The subsequent swings can be analyzed similarly.

The relationship between the physical time t and the time parameter τ can be easily obtained from the definition of τ, given in (3.6), and expression (3.8), i.e.

$$dt = \tau\,\frac{d\theta}{\theta} = \frac{d\tau}{1 + k_L\tau + \omega_n^2\tau^2}. \tag{3.11}$$

Upon integration, this gives $t = \gamma(\tau)/\omega_d$, where $\gamma(\tau)$ is defined by (3.10). The time to reach the end of the first forward swing is obtained from

$$t_* = \lim_{\tau\to\infty} t = \frac{1}{\omega_d}\tan^{-1}\frac{2\omega_d}{k_L + 2\omega_n^2\tau_0}. \tag{3.12}$$

With the expression (3.9) for $\theta = \theta(\tau)$ derived, the velocity $v = v(\tau)$ follows from $v = l\theta/\tau$. This expression and expression (3.9) represent the parametric expressions $v = v(\tau)$ and $\theta = \theta(\tau)$, in terms of the parameter τ, which specify the relationship $v = v(\theta)$.

Student exercise 6. Solve the differential equation (3.5) by introducing as a parameter the auxiliary velocity u, related to the actual velocity v by $v = \theta u$. Compare the obtained expression for $\theta = \theta(u)$ with expression (3.9) for $\theta = \theta(\tau)$. [Hint: the differential equation relating u and θ turns out to be

$$\frac{\mathrm{d}\theta}{\theta} = -\frac{u\,\mathrm{d}u}{u^2 + k_\mathrm{L} l u + \omega_\mathrm{n}^2 l^2}\,.]$$

3.2. Energy-angle of swing relation

Another derivation of the implicit-type relationship between the velocity and angle of swing can be obtained from the consideration of energy. Toward that end, equation (3.5) is recast as

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = -mk_\mathrm{L} l v, \quad E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}mgl\theta^2, \tag{3.13}$$

where K is the kinetic energy, and $U = mgl(1 - \cos\theta) \approx (1/2)mgl\theta^2$ is the potential energy relative to the datum $\theta = 0$. Thus, since v can be expressed in terms of E and θ as

$$v = \sqrt{\frac{2}{m}}(E - U)^{1/2} = \sqrt{\frac{2}{m}}\left(E - \frac{1}{2}mgl\theta^2\right)^{1/2}, \tag{3.14}$$

the differential equation (3.13) becomes

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = -b\left(E - \frac{1}{2}mgl\theta^2\right)^{1/2}, \quad b = k_\mathrm{L} l\sqrt{2m}. \tag{3.15}$$

This equation can be solved by introducing the function $\kappa = \kappa(\theta)$ such that

$$K = E - \frac{1}{2}mgl\theta^2 = \left(\frac{\theta}{\kappa}\right)^2, \tag{3.16}$$

which transforms (3.15) into

$$\frac{\mathrm{d}\theta}{\theta} = -b\left[\frac{1}{\kappa} - \frac{(b/2) + (m/2)gl\kappa}{1 + (b/2)\kappa + (m/2)gl\kappa^2}\right]\mathrm{d}\kappa. \tag{3.17}$$

For $\omega_\mathrm{n} > k_\mathrm{L}/2$, (3.17) can be integrated to obtain

$$\frac{\theta}{\theta_0} = \frac{\kappa}{\kappa_0}\left(\frac{1 + b\kappa_0/2 + mgl\kappa_0^2/2}{1 + b\kappa/2 + mgl\kappa^2/2}\right)^{1/2}\exp\left[-\frac{k_\mathrm{L}\beta(\kappa)}{2\omega_\mathrm{d}}\right], \tag{3.18}$$

where

$$\beta(\kappa) = \tan^{-1}\frac{k_\mathrm{L} + \sqrt{2m}\,g\kappa}{2\omega_\mathrm{d}} - \tan^{-1}\frac{k_\mathrm{L} + \sqrt{2m}\,g\kappa_0}{2\omega_\mathrm{d}}, \tag{3.19}$$

and $\theta_0/\kappa_0 = K_0^{1/2} = \sqrt{m/2}\,v_0$. Expression (3.18) is equivalent to expression (3.9), as can be verified by using the relationship between the parameters τ and κ, given by $\tau = \kappa l\sqrt{m/2}$. A comprehensive study of the energy-displacement relations in other oscillators with linear and quadratic damping can be found in [12, 30].

4. Conclusions and discussion

We have presented in this paper an analysis of damped pendulum motion, in which the ambient drag was assumed to be either quadratic (Newton's type drag), or linear in velocity (Stokes' type drag). In the case of quadratic damping, a closed-form expression for the velocity v vs angle of swing θ is derived for both the forward and backward swings. A single non-dimensional parameter kl, where $k = c/m$ is the ratio of the damping coefficient and the hanging mass, and l is the length of the pendulum, accounts for the combined effect of geometry, inertia, and damping on the motion of the pendulum. The minimum initial velocity required for a full pendulum revolution is determined as a function of kl. In contrast to quadratic damping, in the case of linear damping there is no explicit closed-form expression for the velocity vs angle of swing relationship, but in the case of small-angle oscillations there is a closed-form implicit relationship of the type $f(v, \theta) = 0$. The parametric expressions for v and θ are derived in terms of an introduced time-like parameter, from which the velocity can be readily determined by numerical means. The presented combined analytical–numerical solution is appealing for design considerations, complementing an entirely numerical solution to the governing differential equations of motion, which can be obtained directly from the outset of the analysis.

In the cases of more involved pendulums, it is necessary to solve the governing differential equations of motion numerically. For example, the motion of a spring pendulum in the presence of quadratic damping (figure 7(a)) is described by two coupled second-order differential equations, expressed in polar coordinates (r, θ) as

$$ma_r = mg\cos\theta - cv\dot{r} - k_{\mathrm{s}}(r - l_0), \qquad ma_\theta = -mg\sin\theta - cvr\dot{\theta}, \tag{4.1}$$

where $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ are the radial and circumferential acceleration components. The spring constant of a linearly elastic spring to which the ball of mass m is attached is k_{s} and the quadratic damping coefficient is c. The magnitude of the velocity is $v = (v_r^2 + v_\theta^2)^{1/2}$, where $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$. The unstretched length of the spring is l_0, so that the force in the spring of current length r is $S = k_{\mathrm{s}}(r - l_0)$. Details of the numerical solution of (4.1) are not presented here, but obtaining such a solution would be an instructive student assignment. Further analysis of the spring pendulum, including the analysis of the sensitivity of motion to initial conditions and the resulting chaotic motion [31, 32] is also beyond the scope of the present paper, but it could constitute an appealing undergraduate research project, assigned to groups of students or as an independent study in the junior or senior year. The same applies to double, triple, and multipendulums, as well as coupled pendulums.

The motion of a spherical pendulum (figure 7(b)) also requires numerical treatment. The governing differential equations of motion in this case can be most conveniently derived by using the Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \tag{4.2}$$

where the Lagrangian L is defined by

$$L = K - U = \frac{1}{2}m(v_\theta^2 + v_\phi^2) - mgl(1 - \cos\theta), \quad v_\theta = l\dot{\theta}, \quad v_\phi = l\sin\theta\,\dot{\phi}. \tag{4.3}$$

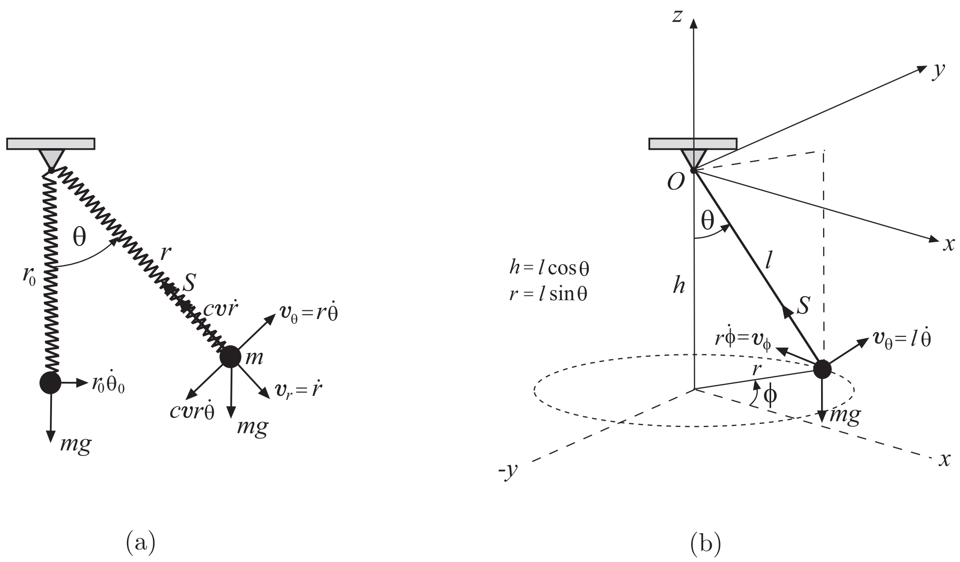


Figure 7. (a) A spring pendulum with a spherical ball of mass m attached to elastic spring of unstretched length l_0 and stiffness k_s. The mass is given an initial horizontal velocity $r_0\dot{\theta}_0$, where $r_0 = l_0 + \Delta_0$ and $\Delta_0 = mg/k_s$. In an arbitrary configuration defined by the angle θ, the forces acting on the ball are its weight mg, the drag force cv^2, whose polar components are $-cv\dot{r}$ and $-cvr\dot{\theta}$, and the force in the cable $S = k_s(r - l_0)$. (b) A spherical pendulum with a mass m attached to the cable of length l. In the initial configuration ($\theta_0 < \pi/2, \phi_0 = 0$) the mass is given an initial velocity $v_\phi^0 = l \sin\theta_0\dot{\phi}_0$, where $\dot{\phi}(0) = \sqrt{g/(l\cos\theta_0)}$. In the presence of damping, the mass moves along the spherical surface of radius l with the center at O, approaching at large times the equilibrium position $(0, 0, -l)$. The velocity components in an arbitrary configuration of the pendulum, defined by angles θ and ϕ, are $v_\theta = l\dot{\theta}$ and $v_\phi = l \sin\theta\dot{\phi}$. The force in the cable is S.

The generalized dissipative forces Q_i can be recognized from the expression for the increment of the dissipative work $\mathrm{d}W_{\mathrm{diss}} = Q_i \mathrm{d}q_i$. By choosing the generalized coordinates to be $q_1 = \theta$ and $q_2 = \phi$, and by assuming quadratic damping with the damping coefficient c, the dissipative generalized forces are $Q_\theta = -cv_\theta v$ and $Q_\phi = -cv_\phi v$, where $v = (v_\theta^2 + v_\phi^2)^{1/2}$. The equations of motion consequently become

$$\begin{aligned} &\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 + k\dot{\theta}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)^{1/2} + \frac{g}{l}\sin\theta = 0, \\ &\sin\theta\,\ddot{\phi} + 2\cos\theta\,\dot{\theta}\dot{\phi} + k\sin\theta\,\dot{\phi}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)^{1/2} = 0, \end{aligned} \tag{4.4}$$

where $k = c/m$. The trajectory is obtained by solving (4.4) numerically, under given initial conditions. The force in the cable is then determined from $m\ddot{z} = S\cos\theta - mg$, where $z = -l\cos\theta$. Performing the numerical solution of (4.4) can be nicely incorporated as part of a project assignment in an upper division mechanics or design course, particularly in conjunction with an experimental setup of a spherical pendulum and the measurement of the parameters of its motion. If damping is absent, the differential equation (4.4) can be decoupled and solved in closed-form in terms of special functions [33]. The incorporation of some of this analysis could serve well to strengthen the analytical portion of the assigned project assignment. The involvement of undergraduate students in project-based activities, collaborative learning, and research has become an important part of undergraduate education, with numerous positive effects on students' academic engagement and achievements [34, 35].

Conflict of interest

The authors declare no conflict of interest.

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