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Dual Eshelby stress tensors and related integrals in micropolar elasticity with body forces and couples

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The Eshelby stress tensor of micropolar elasticity with body forces and body couples, and the corresponding J_k, L_k and M integrals are derived. These are used to determine the energy release rates and configurational forces associated with particular modes of defect motion. The dual Eshelby stress tensor and dual $\hat{J}_k$, $\hat{L}_k$ and $\hat{M}$ integrals are then introduced. The duality properties $J_k + \hat{J}_k = 0$, $L_k + \hat{L}_k = 0$ and $M + \hat{M} = 0$ are established and used to construct alternative expressions for the configurational forces on moving defects. The three-dimensional results are specialized to the plain strain case and compared with earlier results obtained in the absence of body forces and body couples, which are of interest for two-dimensional dislocation and fracture mechanics problems.

1. Introduction

A great amount of research was devoted to the study of conservation integrals in classical and micropolar elasticity, thermoelasticity, piezoelectricity, and finite-strain elasticity. This research initiated with the Eshelby's (1951, 1956) work on the energy momentum tensor and configurational forces on moving material defects, and the subsequent contributions by Günther (1962), Knowles and Sternberg (1972), Budiansky and Rice (1973), and Eshelby (1975), who related the conservation integrals to Noether's theorem on invariant variational principles (Noether, 1918). A comprehensive survey of the advancements in the field can be found in reviews by Olver (1984), Rice (1985), and Maugin (1995), and books by Maugin (1993, 2011), Gurtin (2000a) and Kienzler and Herrmann (2001). The energy momentum tensor, also known as the Eshelby stress tensor, and configurational forces on defects in couple stress and micropolar elasticity were studied by Kluge (1969), Dai (1986), Jaric (1986), Vukobrat (1989), Pucci and Saccomandi (1990), Lubarda and Markenscoff (2000, 2003), and Lazar and Maugin (2007), among others.

The classical conservation integrals are expressed in terms of spatial gradients of displacements and are related to the release rates of the potential energy associated with a defect motion. The dual conservation integrals are related to the release rates of the complementary potential energy and are expressed in terms of spatial gradients of stresses. The study of dual integrals originated from Bui's (1973, 1974) introduction of a dual integral to Rice's (1968) J integral of plane fracture mechanics. An independent study of dual conservation integrals was presented by Carlsson (1974). The subsequent work includes the contributions by Sun (1985), Moran and Shih (1987), Li (1988), Bui (1994), Trimarco and Maugin (1995), Li and Gupta (2006), and Bui (2007). Lubarda and Markenscoff (2007a,b) derived the complementary energy momentum tensor and dual integrals of classical and micropolar elasticity (without body forces and couples), and related them to the release rates of the potential and complementary potential energy associated with particular modes of defect motion.

The analysis of configurational forces in the presence of body forces is different, because the stress tensor and the energy momentum tensor (Eshelby stress tensor) in this case are not divergence free tensors, which precludes the existence of the J, L, and M conservation laws (Eshelby, 1970; Cherepanov, 1979; Kishimoto et al., 1980; Atluri, 1982; Honein and Herrmann, 1997; Kirchner, 1999; Herrmann and Kienzler, 2001; Lubarda, 2008). Lazar and Kirchner (2007) studied the Eshelby stress tensor and related integrals of micropolar elasticity in the presence of body forces and couples, as well as distributed dislocations and disclinations, but without addressing the dual Eshelby stress tensor and the corresponding dual integrals. On the other hand, Lubarda

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and Markenscoff (2008) addressed the dual integrals with body forces, but only in the framework of the classical nonpolar elasticity. The objective of this paper is thus to derive the dual Eshelby stress tensors and the corresponding dual integrals of micropolar elasticity in the presence of body forces and body couples, and to use them to evaluate the energy release rates and configurational forces associated with different types of defect motion.

Body couples commonly appear in a solid body due to its exposure to an external field, e.g., within a polarized dielectric solid in an electric field, where they are defined by a cross product of the polarization vector and the force due to electric field, or in a polarizable and magnetizable medium in the presence of electromagnetic field, where they are defined by a cross product of the magnetization vector and the external magnetic field (Tiersten, 1971; Pao and Yeh, 1973; Pao and Hutter, 1975; Verma and Singh, 1984). Body couples can also be generated by an inhomogeneous external body-force field, e.g., an inhomogeneous mass distribution in the presence of gravity (Almong and Brenner, 1999). In general, for microstructured continua they arise as the average of all moments exerted by surroundings on microconstituents comprising a continuum particle. Body couples are also important for kinetic studies based on intermolecular potentials which account for non-central force interactions. Furthermore, they can appear as part of the mathematical procedure to solve various elasticity problems (Boschi, 1973), notably the Eshelby inclusion problem in micropolar elasticity, where, in addition to fictitious body forces, the fictitious body couples are distributed within the volume of the inclusion, associated with the couple-stress-free compatible micro-strain (Hsieh, 1982). Fictitious body forces and body couples can also be associated with micropolar elastic multipoles, which are the sources of micropolar elastic singularities, and which can be utilized to quantitatively describe the behavior of lattice defects (Hsieh et al., 1980).

2. Basic equations of micropolar elasticity

Deformation of a micropolar continuum is described by the displacement vector and an independent rotation vector, because it is assumed that an infinitesimal material element can experience a microrotation without undergoing a macrodisplacement. An infinitesimal surface element transmits a force and a couple vector, which give rise to nonsymmetric stress and couple-stress tensors. The stress tensor is related to nonsymmetric strain tensor, and the couple-stress is related to the curvature tensor, defined as the gradient of the rotation vector. This model of continuum mechanics was originally introduced by Voigt (1887) and the brothers Cosserat (1909), and then further developed by Günther (1958), Grioli (1960), Aero and Kuvshinskii (1960), Toupin (1962), Mindlin (1964), Eringen and Suhubi (1964), Eringen (1968), Stojanović (1970), and Nowacki (1986). Additional contributions can be found in the review article by Dhaliwal and Singh (1987), and Jasiuk and Ostoja-Starzewski (1995), and in the books by Brulin and Hsieh (1982), and Eringen (1999).

The physical motivation to extend the nonpolar to micropolar elasticity was that the former could not predict the size effect experimentally observed in problems with a geometrical length scale that is comparable to the microstructural material length, such as the grain size in a polycrystalline or granular material. For example, the apparent strength of some materials with stress concentrators such as holes and notches is higher for smaller grain size; for a given volume fraction of dispersed hard particles, the strengthening of metals is greater for smaller particles; the bending and torsional strengths are higher for very thin beams and wires; the singular nature of the crack tip fields is affected by the couple stresses (Mindlin, 1963; Muki and Sterberg, 1965; Sternberg and Muki, 1967; Kaloni and Ariman, 1967; Fleck et al., 1994; Xia and Hutchinson, 1996). The nonpolar theory was also in disagreement with experiments involving high-frequency ultra-short wave propagation, in which the wave length was comparable to material's microstructural length (Mindlin, 1964; Brulin and Hsieh, 1982). The research in micropolar and related non-local and strain-gradient theories of elastic and inelastic response has intensified during the past two decades, because of an increasing interest to describe the deformation mechanisms at micro and nanostructural level (Fleck and Hutchinson, 1997, 2001; Valiev et al., 2000; Gurtin, 2000b; Chen and Wang, 2001; Lazar and Maugin, 2005; Asaro and Suresh, 2005; Meyers et al., 2006; Dao et al., 2007; Kuroda and Tvergaard, 2008), inelastic localization and instability phenomena (Zbib and Aifantis, 1989; De Borst and Van der Giessen, 1998; Niordson and Tvergaard, 2005; Asaro and Lubarda, 2006), and micromechanics of dislocations, inclusions, and fractal media (Lubarda, 2003a,b; Yavari et al., 2002; Lazar and Maugin, 2005; Li and Ostoja-Starzewski, 2011).

In a micropolar continuum, the surface forces T_i are in equilibrium with the nonsymmetric Cauchy stress t_{ij}, and the surface couples M_i are in equilibrium with the nonsymmetric couple-stress m_{ij}, such that $T_i = n_j t_{ji}$ and $M_i = n_j m_{ji}$, where n_j are the rectangular components of the unit vector orthogonal to the surface element under consideration. The integral conditions of equilibrium are

$$\int_S T_i \mathrm{d}S + \int_V b_i \mathrm{d}V = 0,$$
$$\int_S \left(M_i + e_{ijk}x_jT_k\right)\mathrm{d}S + \int_V \left(\mu_i + e_{ijk}x_jb_k\right)\mathrm{d}V = 0, \tag{1}$$

where b_i are the body forces (per unit volume), μ_i are the body couples, e_{ijk} are the components of the permutation tensor, and x_i are the rectangular coordinates with respect to the selected coordinate origin. The corresponding differential equations of equilibrium are

$$t_{ji,j} + b_i = 0, \quad m_{ji,j} + \mu_i = -e_{ijk}t_{jk}. \tag{2}$$

The comma designates the partial derivative with respect to the spatial coordinate.

For infinitesimal elastic deformations of micropolar continuum, the specific strain energy W (per unit volume) is a function of the nonsymmetric strain tensor γ_{ij} and the curvature tensor κ_{ij}, which are defined by

$$\gamma_{ij} = u_{j,i} - e_{ijk}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \tag{3}$$

where u_i are the components of the macroscopic displacement, and φ_i of the microscopic rotation vector. The constitutive relations of infinitesimal micropolar elasticity are then

$$t_{ij} = \frac{\partial W}{\partial \gamma_{ij}}, \quad m_{ij} = \frac{\partial W}{\partial \kappa_{ij}}, \tag{4}$$

so that $\dot{W} = t_{ij}\dot{\gamma}_{ij} + m_{ij}\dot{\kappa}_{ij}$.

If the strain energy is a quadratic function of the strain and curvature components,

$$W = \frac{1}{2}C_{ijkl}\gamma_{ij}\gamma_{kl} + \frac{1}{2}K_{ijkl}\kappa_{ij}\kappa_{kl}, \tag{5}$$

the constitutive expressions (4) are the linear relations

$$t_{ij} = C_{ijkl}\gamma_{kl}, \quad m_{ij} = K_{ijkl}\kappa_{kl}. \tag{6}$$

Since the strain and curvature tensors are not symmetric, the micropolar elastic moduli tensors obey only the reciprocal symmetries $C_{ijkl} = C_{klij}$ and $K_{ijkl} = K_{klij}$. In the case of isotropy, and in the notation of Nowacki (1986), these moduli are specified by

$$\begin{aligned} C_{ijkl} &= (\mu + \overline{\mu})\delta_{ik}\delta_{jl} + (\mu - \overline{\mu})\delta_{il}\delta_{jk} + \lambda\delta_{ij}\delta_{kl}, \\ K_{ijkl} &= (\alpha + \overline{\alpha})\delta_{ik}\delta_{jl} + (\alpha - \overline{\alpha})\delta_{il}\delta_{jk} + \beta\delta_{ij}\delta_{kl}, \end{aligned} \tag{7}$$

where μ, $\overline{\mu}$, λ and α, $\overline{\alpha}$, β are the Lamé-type elastic constants of micropolar elasticity. They are defined so that the symmetric and anti-symmetric parts of the stress and couple-stress tensors are

$$\begin{aligned} t_{(ij)} &= 2\mu\gamma_{(ij)} + \lambda\gamma_{kk}\delta_{ij}, \quad t_{<ij>} = 2\overline{\mu}\gamma_{<ij>}, \\ m_{(ij)} &= 2\alpha\kappa_{(ij)} + \beta\kappa_{kk}\delta_{ij}, \quad m_{<ij>} = 2\overline{\alpha}\kappa_{<ij>}. \end{aligned} \tag{8}$$

The microstructural length scale (l) is implicitly embedded in (8), because the two types of micropolar elastic moduli are dimensionally related by $(\alpha, \beta, \overline{\alpha}) \sim l^2(\mu, \lambda, \overline{\mu})$.

3. The Eshelby stress tensor

The spatial gradient of the strain energy function is

$$W_{,k} = \frac{\partial W}{\partial \gamma_{ij}}\gamma_{ij,k} + \frac{\partial W}{\partial \kappa_{ij}}\kappa_{ij,k} = t_{ij}\gamma_{ij,k} + m_{ij}\kappa_{ij,k}. \tag{9}$$

By using the expressions (3) for the strain and curvature tensors, (9) can be rewritten as

$$W_{,j}\delta_{jk} - t_{ji}u_{i,jk} - m_{ji}\varphi_{i,jk} + t_{ji}e_{jil}\varphi_{l,k} = 0. \tag{10}$$

In view of the equilibrium Eq. (2), this becomes

$$\left[(W - b_iu_i - \mu_i\varphi_i)\delta_{jk} - t_{ji}u_{i,k} - m_{ji}\varphi_{i,k}\right]_{,j} = -\left(b_{j,k}u_j + \mu_{j,k}\varphi_j\right). \tag{11}$$

Eq. (11) defines the Eshelby stress tensor of linear micropolar elasticity, in the presence of body forces and couples,

$$P_{jk} = (W - b_iu_i - \mu_i\varphi_i)\delta_{jk} - t_{ji}u_{i,k} - m_{ji}\varphi_{i,k}, \tag{12}$$

such that

$$P_{jk,j} = -\left(b_{j,k}u_j + \mu_{j,k}\varphi_j\right). \tag{13}$$

As shown in Section 7, the Eshelby stress tensor (12) is related to the release rate of the potential energy associated with defect motion within a micropolar medium, in the presence of body forces and couples. In the absence of body forces and couples, (12) reduces to the divergence free expression of Lubarda and Markenscoff (2003). In the absence of micropolar effects, but in the presence of body forces, (12) and (13) reduce to the results of Lubarda (2008). With neither micropolar effects nor body forces, (12) and (13) reproduce the celebrated results of Eshelby (1951, 1956).

4. *J* integrals in the presence of body forces and couples

The J_k integrals can be defined in terms of the Eshelby stress tensor P_{jk} by

$$J_k = \int_S P_{jk}n_j \mathrm{d}S, \tag{14}$$

where S is the bounding surface of the volume V, which does not contain any singularity of defect. Thus, by applying the Gauss divergence theorem to (14), and by incorporating (13), the J_k integrals are equal to

$$J_k = -\int_V \left(b_{j,k}u_j + \mu_{j,k}\varphi_j\right)\mathrm{d}V. \tag{15}$$

In general, the right-hand side of (15) is not equal to zero, so that $J_k \neq 0$. Therefore, the presence of spatially variable body forces or couples precludes the existence of the $J_k = 0$ conservation law.[1] However, if the body forces and couples are spatially uniform ($b_{i,k} = 0$ and $\mu_{i,k} = 0$), there is a conservation law

$$J_k = \int_S \left[(W - b_iu_i - \mu_i\varphi_i)n_k - T_iu_{i,k} - M_i\varphi_{i,k}\right]\mathrm{d}S = 0, \tag{16}$$

for any surface S that does not enclose a singularity or defect.

The micropolar version of the conservation law $J_k = 0$ in the absence of body forces and couples was earlier derived by Dai (1986) and Jaric (1986) in the case of elastostatics, and by Fletcher (1975) and Vukobrat (1989) in the case of elastodynamics. A derivation based on Noether's theorem on invariant variational principles was given by Pucci and Saccomandi (1990) and, in a more general context, by Lubarda and Markenscoff (2000, 2003). An extension of the latter analysis to account for the body forces and couples was presented by Lazar and Kirchner (2007). Earlier, the body force term was included in the structure of the J integral to study the progressive failure of over-consolidated clay by Palmer and Rice (1973), and the free-boundary flows in fluid mechanics by Ben Amar and Rice (2002). The inclusion of the body force term in the structure of the Eshelby stress tensor is also reminiscent of the structure of the energy momentum tensor in the dynamic fracture mechanics (Freund, 1990).

5. *M* integral in the presence of body forces and couples

If the strain energy is a homogeneous function of degree 2 in both the strain and curvature components,[2] from (4)–(6) it follows that

$$W = \frac{1}{2}\left(t_{jk}\gamma_{jk} + m_{jk}\kappa_{jk}\right). \tag{17}$$

Furthermore, it can be verified by inspection that the Eshelby stress tensor (12) satisfies the equation

$$\left(P_{jk}x_k\right)_{,j} - P_{kk} = -\left(u_jb_{j,k} + \varphi_j\mu_{j,k}\right)x_k, \tag{18}$$

where the trace of the Eshelby stress tensor is

$$P_{kk} = 3(W - b_ku_k - \mu_k\varphi_k) - t_{jk}u_{k,j} - m_{jk}\varphi_{k,j}. \tag{19}$$

By incorporating (17), this can be rewritten as

$$P_{kk} = \frac{1}{2}\left(t_{jk}u_k + m_{jk}\varphi_k\right)_{,j} - \frac{5}{2}(b_ku_k + \mu_k\varphi_k) - e_{ijk}t_{ij}\varphi_k. \tag{20}$$

The substitution of (20) into (18) yields

$$\begin{aligned} \left(P_{jk}x_k - \frac{1}{2}t_{jk}u_k - \frac{1}{2}m_{jk}\varphi_k\right)_{,j} &= -\frac{5}{2}(b_ku_k + \mu_k\varphi_k) - e_{ijk}t_{ij}\varphi_k \\ &\quad - \left(u_jb_{j,k} + \varphi_j\mu_{j,k}\right)x_k. \end{aligned} \tag{21}$$

[1] If the energy momentum tensor of nonpolar elasticity is defined by $\overline{P}_{jk} = W\delta_{jk} - \sigma_{ji}u_{i,k}$, there follows another nonconserved integral, $\overline{J}_k = \int_S \overline{P}_{jk}n_j\mathrm{d}S = \int_V b_iu_{i,k}\mathrm{d}V$, which was used in the work of Huang et al. (2002) and Liang et al. (2003).

[2] The analysis can be easily extended to encompass the case when the strain energy is a homogeneous function of degree different from 2; Lubarda and Markenscoff (2007b).

An equivalent, but more convenient form of (21) is

$$D_{j,j} = -m_{jk}\kappa_{jk} - \frac{5}{2}b_k u_k - \frac{3}{2}\mu_k\varphi_k - \left(u_j b_{j,k} + \varphi_j \mu_{j,k}\right)x_k, \tag{22}$$

where

$$D_j = P_{jk}x_k - \frac{1}{2}t_{jk}u_k - \frac{3}{2}m_{jk}\varphi_k, \tag{23}$$

which is referred to as a dilatation or scaling vector (Lazar and Kirchner, 2007).

The M integral of micropolar elasticity is defined by

$$M = \int_V D_j n_j \mathrm{d}S = \int_S \left(P_{jk}x_k - \frac{1}{2}t_{jk}u_k - \frac{3}{2}m_{jk}\varphi_k\right)n_j \mathrm{d}S. \tag{24}$$

The application of the Gauss divergence theorem to (24) within a defect-free region reveals that the M integral of micropolar elasticity is not equal to zero but to the volume integral of the right-hand side of (22), i.e.,

$$M = -\int_V \left[m_{jk}\kappa_{jk} + \frac{5}{2}b_k u_k + \frac{3}{2}\mu_k\varphi_k + \left(u_j b_{j,k} + \varphi_j\mu_{j,k}\right)x_k\right]\mathrm{d}V. \tag{25}$$

In the absence of body forces and couples, this reduces to

$$M = -\int_V m_{jk}\kappa_{jk}\mathrm{d}V. \tag{26}$$

There is no $M = 0$ conservation law in micropolar elasticity, because there is a material length scale in the structure of the corresponding constitutive equations, so that the total strain energy is not infinitesimally invariant under a self-similar scale change (Lubarda and Markenscoff, 2003). In the absence of micropolar effects and body forces, there is a conservation law

$$M = \int_S \left(P_{jk}x_k - \frac{1}{2}t_{jk}u_k\right)n_j \mathrm{d}S, = 0 \tag{27}$$

for any closed surface that does not embrace a singularity or defect, as originally shown by Günther (1962), Knowles and Sternberg (1972), and Budiansky and Rice (1973).

6. *L* integrals in the presence of body forces and couples

An appealing construction of the L_k integrals of isotropic micropolar elasticity is based on the identity

$$e_{kij}\left(t_{il}\gamma_{jl} + t_{li}\gamma_{lj} + m_{il}\kappa_{jl} + m_{li}\kappa_{lj}\right) = 0. \tag{28}$$

This identity holds because for linear isotropic elasticity the tensors $(t_{il}\gamma_{jl} + t_{li}\gamma_{lj})$ and $(m_{il}\kappa_{jl} + m_{li}\kappa_{lj})$ are symmetric in (i,j), which can be verified by the substitution of the constitutive expressions (6). The identity also holds in the case of material nonlinearity, as demonstrated by Lubarda and Markenscoff (2003). In view of the strain and curvature expressions, (28) can be rewritten as

$$e_{kij}\left(t_{il}u_{l,j} + t_{li}u_{j,l} + m_{il}\varphi_{l,j} + m_{li}\varphi_{j,l} - e_{irs}t_{rs}\varphi_j\right) = 0. \tag{29}$$

By using the Eshelby stress tensor (12), this is equivalent to

$$e_{kij}\left(P_{ji} + t_{li}u_{j,l} + m_{li}\varphi_{j,l} - e_{irs}t_{rs}\varphi_j\right) = 0, \tag{30}$$

because $e_{kij}P_{ji} = e_{kij}(t_{il}u_{l,j}+m_{il}\varphi_{l,j})$.

Introducing the second-order tensor, referred to as the angular energy momentum tensor,

$$H_{kl} = e_{kij}\left(P_{li}x_j + t_{li}u_j + m_{li}\varphi_j\right), \tag{31}$$

and having in mind (2) and (13), it follows that

$$H_{kl,l} = -e_{kij}\left[b_i u_j + \mu_i\varphi_j + (b_{l,i}u_l + \mu_{l,i}\varphi_l)x_j\right]. \tag{32}$$

Thus, by defining the integrals

$$L_k = \int_S H_{kl}n_l \mathrm{d}S = e_{kij}\int_S \left(P_{li}x_j + t_{li}u_j + m_{li}\varphi_j\right)n_l \mathrm{d}S, \tag{33}$$

the application of the Gauss divergence theorem gives

$$L_k = -e_{kij}\int_V \left[b_i u_j + \mu_i\varphi_j + (b_{l,i}u_l + \mu_{l,i}\varphi_l)x_j\right]\mathrm{d}V, \tag{34}$$

for any closed surface S enclosing a volume V without singularities or defects. In the absence of body forces and couples, (34) yields the conservation law $L_k = 0$, originally derived by Lubarda and Markenscoff (2000, 2003) by using the Noether's theorem. The plane-strain version of the results is presented in the Appendix of the paper.

7. The energy release rates and configurational forces

The J_k, L_k, and M integrals, evaluated over the free surface of a defect, are related to the potential energy release rates and configurational forces associated with specific modes of defect's motion. By extending the nonpolar analysis of Budiansky and Rice (1973), and micropolar analysis without body forces of Lubarda and Markenscoff (2007b), consider the body of volume V loaded by surface tractions $T_i = \overline{T}_i$ over the portion S_T of its external surface S, and surface couples $M_i = \overline{M}_i$ over the portion S_M. The displacements $u_i = \overline{u}_i$ are prescribed over S_u and the rotations $\varphi_i = \overline{\varphi}_i$ over S_φ. Within the body there is a defect (cavity) with the bounding surface S_0, free of surface tractions or couples. The potential energy of such body and the loading system is

$$\Pi = \int_V (W - b_i u_i - \mu_i\varphi_i)\mathrm{d}V - \int_{S_T} \overline{T}_i u_i \mathrm{d}S - \int_{S_M} \overline{M}_i\varphi_i \mathrm{d}S. \tag{35}$$

If the boundary conditions on S are held fixed, the rate of change of the potential energy associated with a spatial variation of the surface of cavity, caused by its velocity field $\dot{u}_i^0$, is

$$\dot{\Pi} = \int_V (\dot{W} - b_i\dot{u}_i - \mu_i\dot{\varphi}_i)\mathrm{d}V - \int_{S_0} (W - b_i u_i - \mu_i\varphi_i)\dot{u}_j^0 n_j \mathrm{d}S - \int_{S_T} \overline{T}_i\dot{u}_i \mathrm{d}S - \int_{S_M} \overline{M}_i\dot{\varphi}_i \mathrm{d}S, \tag{36}$$

where $\dot{u}_i$ and $\dot{\varphi}_i$ are the kinematic fields within V due to the imposed velocity $\dot{u}_i^0$ and $\dot{\varphi}_i^0 = 0$ over S_0. Body forces and couples are assumed to be unaffected by the cavity motion (dead body forces and couples). The surface integral over S_0 follows from the Reynolds transport theorem, where n_i is the unit normal to S_0, directed into the material surrounding the cavity. Assuming that $\dot{u}_i$ and $\dot{\varphi}_i$ are kinematically admissible fields, the rate of the strain energy is

$$\dot{W} = t_{ij}\dot{\gamma}_{ij} + m_{ij}\dot{\kappa}_{ij}, \quad \dot{\gamma}_{ij} = \dot{u}_{j,i} - e_{ijk}\dot{\varphi}_k, \quad \dot{\kappa}_{ij} = \dot{\varphi}_{j,i}, \tag{37}$$

i.e., by using the equilibrium conditions (2),

$$\dot{W} = \left(t_{ij}\dot{u}_j + m_{ij}\dot{\varphi}_i\right)_{,i} + b_i\dot{u}_i + \mu_i\dot{\varphi}_i. \tag{38}$$

Since the surface of the cavity is not loaded, by means of the Gauss divergence theorem, the volume integral of (38) becomes

$$\int_V \dot{W}\mathrm{d}V = \int_{S_T} \overline{T}_j\dot{u}_j\mathrm{d}S + \int_{S_M} \overline{M}_i\dot{\varphi}_i\mathrm{d}S + \int_V (b_i\dot{u}_i + \mu_i\dot{\varphi}_i)\mathrm{d}V. \tag{39}$$

The substitution of (39) into (36) then yields

$$\dot{\Pi} = -\int_{S_0} (W - b_iu_i - \mu_i\varphi_i)\dot{u}_j^0 n_j\mathrm{d}S. \tag{40}$$

7.1. Configurational forces

The rate of energy release due to spatial variation of S_0, specified by a prescribed velocity field $\dot{u}_i^0$, is $f = -\dot{\Pi}$, which represents the configurational force on a cavity or defect. Since $(W - b_iu_i - \mu_i\varphi_i)n_j = P_{ij}n_i$ over the free surface S_0, (40) implies that

$$f = -\dot{\Pi} = \int_{S_0} P_{ji}\dot{u}_i^0 n_j\mathrm{d}S. \tag{41}$$

If the cavity translates with a unit velocity in the k-direction, $\dot{u}_i^0$ can be replaced by δ_{ik}, and (41) gives the rate of energy release per unit cavity translation in the k-direction,

$$f_k = \int_{S_0} P_{jk}n_j\mathrm{d}s = J_k(S_0). \tag{42}$$

By applying the Gauss divergence theorem to the surface $S_0 + S$ bounding a region V between S_0 and any closed surface S around the cavity, and in view of (15), the configurational force f_k can also be expressed as

$$f_k = J_k(S) + \int_V \left(b_{j,k}u_j + \mu_{j,k}\varphi_j\right)\mathrm{d}V. \tag{43}$$

If the body forces and couples are spatially uniform, $f_k = J_k(S_0) = J_k(S)$.

If the cavity is given a unit angular velocity around the k-axis, $\dot{u}_i^0$ in (41) can be replaced by $-e_{kil}x_l$, and since $e_{kil}P_{ji}x_ln_j = H_{kl}n_l$ over the free surface S_0, where H_{kl} is the angular energy momentum tensor of Eq. (31), the configurational force (41) becomes

$$f_k = -\int_{S_0} H_{kl}n_l\mathrm{d}S = -L_k(S_0). \tag{44}$$

When expressed in terms of the surface integral over any other surface S around the cavity, from (34) and (44), it follows that the configuration force can also be expressed as

$$f_k = -L_k(S) - e_{kij}\int_V \left[b_iu_j + \mu_i\varphi_j + (b_{l,i}u_l + \mu_{l,i}\varphi_l)x_j\right]\mathrm{d}V. \tag{45}$$

If the absence of body forces and couples, $f_k = -L_k(S_0) = -L_k(S)$, as originally shown by Lubarda and Markenscoff (2007b).

Finally, if the cavity transforms such that $\dot{u}_i^0 = x_i$, (41) yields

$$f = \int_{S_0} D_jn_j\mathrm{d}S = M(S_0), \tag{46}$$

because $P_{ji}x_in_j = D_jn_j$ over S_0, where D_j is the dilatation vector from Eq. (23). In view of (25), the configurational force (46) is also equal to

$$f = M(S) + \int_V \left[m_{jk}\kappa_{jk} + \frac{5}{2}b_ku_k + \frac{3}{2}\mu_k\varphi_k + \left(u_jb_{j,k} + \varphi_j\mu_{j,k}\right)x_k\right]\mathrm{d}V, \tag{47}$$

where V is the volume between S_0 and S. In the absence of polar effects and body forces, $f = M(S_0) = M(S)$, for any closed surface S surrounding the cavity (Budiansky and Rice, 1973).

8. Dual Eshelby stress tensor and related dual integrals

The complementary strain energy function $\hat{W} = \hat{W}(t_{ij}, m_{ij})$ is related to the strain energy function $W = W(\gamma_{ij}, \kappa_{ij})$ by

$$W + \hat{W} = t_{ij}\gamma_{ij} + m_{jk}\kappa_{jk}. \tag{48}$$

A dual Eshelby stress tensor of linear micropolar elasticity, in the presence of body forces and couples, is then defined by

$$\hat{P}_{jk} = \hat{W}\delta_{jk} - u_it_{ji,k} - \varphi_im_{ji,k}, \tag{49}$$

such that

$$\hat{P}_{jk,j} = u_jb_{j,k} + \varphi_j\mu_{j,k}. \tag{50}$$

The sum of the Eshelby stress tensor (12) and its dual (49) is

$$P_{jk} + \hat{P}_{jk} = (W + \hat{W} - b_ku_k - \mu_k\varphi_k)\delta_{jk} - (t_{ji}u_i + m_{ji}\varphi_i)_{,k}. \tag{51}$$

In view of (13) and (50), this sum is divergence free, i.e.,

$$\left(P_{jk} + \hat{P}_{jk}\right)_{,j} = 0. \tag{52}$$

Furthermore, the traces of the two Eshelby stress tensors are

$$P_{kk} = W - 3(b_ku_k + \mu_k\varphi_k) - e_{ijk}t_{ij}\varphi_k,$$
$$\hat{P}_{kk} = 3\hat{W} + b_ku_k + \mu_k\varphi_k + e_{ijk}t_{ij}\varphi_k. \tag{53}$$

8.1. Dual $\hat{J}$ integrals

The dual $\hat{J}_k$ integrals are defined in terms of the dual Eshelby stress tensor by

$$\hat{J}_k = \int_S \hat{P}_{jk}n_j\mathrm{d}S, \tag{54}$$

where S is the bounding surface of the volume V, which does not include any singularity of defect. Thus, by applying the Gauss divergence theorem, and by incorporating (50), it follows that

$$\hat{J}_k = \int_V (u_ib_{i,k} + \varphi_i\mu_{i,k})\mathrm{d}V. \tag{55}$$

The right-hand side of (55) is opposite to the right-hand side of (15), so that the duality holds

$$J_k + \hat{J}_k = 0. \tag{56}$$

While the J_k integrals in (15) are expressed in terms of spatial gradients of displacement and rotation, the $\hat{J}_k$ integrals in (55) are expressed in terms of spatial gradients of stress and couple stress. In the absence of micropolar effects and body forces, (15) and (55) yield the conservation laws $J_k = 0$ and $\hat{J}_k = 0$, for any surface that does not enclose a singularity or defect. The first of these is originally due to Eshelby (1951, 1956), and the second due to Bui (1973, 1974).

8.2. Dual $\hat{M}$ integrals

The dual Eshelby stress tensor (49) satisfies the equation

$$\left(\hat{P}_{jk}x_k\right)_{,j} - \hat{P}_{kk} = \left(u_jb_{j,k} + \varphi_j\mu_{j,k}\right)x_k. \tag{57}$$

Since the trace of the dual Eshelby stress tensor is

$$\hat{P}_{kk} = \frac{3}{2}\Big(t_{jk}u_k + m_{jk}\varphi_k\Big)_{,j} + \frac{5}{2}(b_k u_k + \mu_k\varphi_k) + e_{ijk}t_{ij}\varphi_k, \tag{58}$$

the substitution of (58) into (57) yields

$$\hat{D}_{j,j} = m_{jk}\kappa_{jk} + \frac{5}{2}b_k u_k + \frac{3}{2}\mu_k\varphi_k + \Big(u_j b_{j,k} + \varphi_j\mu_{j,k}\Big)x_k, \tag{59}$$

where

$$\hat{D}_j = \hat{P}_{jk}x_k - \frac{3}{2}t_{jk}u_k - \frac{1}{2}m_{jk}\varphi_k \tag{60}$$

is a dual dilatation vector; cf. (23). The duality is such that

$$\big(D_j + \hat{D}_j\big)_{,j} = 0, \tag{61}$$

as obtained by adding (22) and (59).

A dual $\hat{M}$ integral of micropolar elasticity is defined by

$$\hat{M} = \int_V \hat{D}_j n_j \mathrm{d}S = \int_S \left(\hat{P}_{jk}x_k - \frac{3}{2}t_{jk}u_k - \frac{1}{2}m_{jk}\varphi_k\right) n_j \mathrm{d}S. \tag{62}$$

The application of the Gauss divergence theorem to (62), within a defect-free region, shows that a dual $\hat{M}$ integral of micropolar elasticity is not equal to zero but to the volume integral of the right-hand side of (59), i.e.,

$$\hat{M} = \int_V \left[m_{jk}\kappa_{jk} + \frac{5}{2}b_k u_k + \frac{3}{2}\mu_k\varphi_k + \Big(u_j b_{j,k} + \varphi_j\mu_{j,k}\Big)x_k\right]\mathrm{d}V. \tag{63}$$

Since the sum $(D_j + \hat{D}_j)$ is divergence free, the duality property holds

$$M + \hat{M} = 0. \tag{64}$$

Alternatively, this duality follows from (22) and (63), because their right-hand sides are opposite to each other. In the absence of micropolar effects and body forces (Sun, 1985; Lubarda and Markenscoff, 2007a), there is a dual conservation law

$$\hat{M} = \int_S \left(\hat{P}_{jk}x_k - \frac{3}{2}u_k t_{jk}\right)n_j \mathrm{d}S = 0. \tag{65}$$

8.3. Dual $\hat{L}$ integrals

In analogy with the derivation of the L_k integrals from Section 6, consider the identity

$$e_{kij}\Big(u_{i,l}t_{lj} + u_{l,i}t_{jl} + \varphi_{i,l}m_{lj} + \varphi_{l,i}m_{jl} - \varphi_i e_{jrs}t_{rs}\Big) = 0. \tag{66}$$

By using the expression for the dual Eshelby stress tensor (49), this can be rewritten as

$$e_{kij}\Big(\hat{P}_{ji} + u_{i,l}t_{lj} + \varphi_{i,l}m_{lj} - \varphi_i e_{jrs}t_{rs} + u_{l,i}t_{jl} + u_l t_{jl,i} + \varphi_{l,i}m_{jl} + \varphi_l m_{jl,i}\Big) = 0. \tag{67}$$

Introducing the dual angular energy momentum tensor,

$$\hat{H}_{kij}\Big[\hat{P}_{li}x_j + u_i t_{lj} + \varphi_i m_{lj} + \delta_{il}\big(u_r t_{jr} + \varphi_r m_{jr}\big)\Big], \tag{68}$$

and having in mind (50) and (67), it can be shown that

$$\hat{H}_{kl,l} = e_{kij}\big[u_j b_i + \varphi_j\mu_i + (u_l b_{l,i} + \varphi_l\mu_{l,i})x_j\big]. \tag{69}$$

The sum of the two dual angular energy momentum tensors is divergence free, i.e.,

$$\big(H_{kl} + \hat{H}_{kl}\big)_{,l} = 0, \tag{70}$$

which is obtained by adding (32) and (69). Thus, by defining the integrals

$$\hat{L}_k = \int_S \hat{H}_{kl}n_l \mathrm{d}S = e_{kij}\int_S \Big[\hat{P}_{li}x_j + u_i t_{lj} + \varphi_i m_{lj} + \delta_{il}\big(u_r t_{jr} + \varphi_r m_{jr}\big)\Big]n_l \mathrm{d}S, \tag{71}$$

and by using (69), the Gauss divergence theorem yields

$$\hat{L}_k = e_{kij}\int_V \big[u_j b_i + \varphi_j\mu_i + (u_l b_{l,i} + \varphi_l\mu_{l,i})x_j\big]\mathrm{d}V, \tag{72}$$

for any closed surface S enclosing a volume V without a singularity or defect. The integrals L_k and $\hat{L}_k$ are dual in the sense that

$$L_k + \hat{L}_k = 0, \tag{73}$$

which follows by adding (34) and (72). In the absence of body forces and couples, (72) gives the conservation law $\hat{L}_k = 0$, originally derived by Lubarda and Markenscoff (2007b).

8.4. Configurational forces

The complementary potential energy $\hat{\Pi}$ is defined by

$$\hat{\Pi} = \int_V \hat{W}\mathrm{d}V - \int_{S_u} \bar{u}_i T_i \mathrm{d}S - \int_{S_\varphi} \bar{\varphi}_i M_i \mathrm{d}S. \tag{74}$$

It is related to the potential energy Π of Eq. (35) by the duality relation $\Pi + \hat{\Pi} = 0$. Indeed, since the surface of cavity S_0 is not loaded,

$$\Pi + \hat{\Pi} = \int_V \big(W + \hat{W} - b_i u_i - \mu_i\varphi_i\big)\mathrm{d}V - \int_S \big(T_i u_i + M_i\varphi_i\big)\mathrm{d}S = 0, \tag{75}$$

which follows from $W + \hat{W} = t_{ij}\gamma_{ij} + m_{ij}\kappa_{ij}$ by using the equilibrium conditions (2), the geometric relationships (3), and the Gauss divergence theorem.

The rate of change of the complementary potential energy associated with a spatial variation of the surface of cavity, caused by its velocity field $\dot{u}_i^0$, is

$$\dot{\hat{\Pi}} = \int_V \dot{\hat{W}}\mathrm{d}V - \int_{S_0} \hat{W}\dot{u}_i^0 n_i \mathrm{d}S - \int_{S_u} \bar{u}_i \dot{T}_i \mathrm{d}S - \int_{S_\varphi} \bar{\varphi}_i \dot{M}_i \mathrm{d}S. \tag{76}$$

Here, $\dot{T}_i$ and $\dot{M}_i$ are the loading rates on S_u and S_φ, where $\bar{u}_i$ and $\bar{\varphi}_i$ are prescribed, due to imposed infinitesimal motion of the surface of cavity. By the same analysis as in Lubarda and Markenscoff (2007b), it can be shown that (76) reduces to

$$\dot{\hat{\Pi}} = -\int_{S_0} \hat{P}_{ij}n_i \dot{u}_j^0 \mathrm{d}S. \tag{77}$$

In Section 7 it was shown that the configurational force associated with the defect motion is $f = -\dot{\Pi}$. Since the complementary potential energy is related to the potential energy by $\Pi + \hat{\Pi} = 0$, if follows that $\dot{\hat{\Pi}} = -\dot{\Pi}$. Consequently, in addition to being the negative of the potential energy release rate, the configurational force is also equal to the release rate of the complementary potential energy associated a defect motion, i.e.,

$$f = \dot{\hat{\Pi}} = -\int_{S_0} \hat{P}_{ij} n_i \dot{u}_j^0 \mathrm{d}S. \tag{78}$$

By selecting $\dot{u}_j^0$ to correspond to translation, rotation and dilatation, it follows that the configurational force for these three types of defect motion is

$$f_k = \begin{cases} -\hat{J}_k(S_0), & \text{translation,} \\ \hat{L}_k(S_0), & \text{rotation,} \end{cases} \tag{79}$$

and

$$f = -\hat{M}(S_0), \quad \text{dilatation.} \tag{80}$$

Since $J_k(S_0) + \hat{J}_k(S_0) = 0$, and since $J_k(S) + \hat{J}_k(S) = 0$ for any closed surface S which does not enclose a defect, it also follows that $J_k(S_1) + \hat{J}_k(S_1) = 0$ for any surface S_1 enclosing a defect. Similarly, $L_k(S_1) + \hat{L}_k(S_1) = 0$ and $M(S_1) + \hat{M}(S_1) = 0$.

9. Conclusion

The Eshelby stress tensor of micropolar elasticity with body forces and body couples, and the corresponding J_k, L_k and M integrals are derived. The dual Eshelby stress tensor and dual $\hat{J}_k$, $\hat{L}_k$ and $\hat{M}$ integrals are also introduced. It is shown that the sums of the dual energy momentum tensors, dual angular momentum tensors, and dual dilatation vectors are divergence free, which yields the duality properties $J_k + \hat{J}_k = 0$, $L_k + \hat{L}_k = 0$, and $M + \hat{M} = 0$. The configurational forces associated with the translation, rotation, and dilatation of the defect can therefore be determined from $f_k = J_k(S_0) = -\hat{J}_k(S_0)$, $f_k = -L_k(S_0) = \hat{L}_k(S_0)$, and $f_k = M(S_0) = -\hat{M}(S_0)$, respectively, with the integrals evaluated over the free surface of a defect. The three-dimensional results are specialized in the Appendix to the plain strain case and compared with results obtained in the absence of body forces and body couples, which were used by Lubarda and Markenscoff (2007a,b) to evaluate the energetic forces on an edge dislocation and a crack tip in a long slab of nonpolar and micropolar materials. Lubarda (2008) applied the J integral in the presence of body forces to evaluate the Peach–Koehler force on a dislocation residing within a large block of the material, determining its equilibrium position under different boundary conditions, which are of interest for geomechanics. The potential applications also include the fracture mechanics problems of piezoelectric materials, e.g., piezoelectric and ferroelectric actuators (Suo et al., 1992; Loge and Suo, 1996), micromechanics of human bone with included interactions between Haversian osteons and the cement substance (Park and Lakes, 1986), granular and nanograin crystalline materials (Ieşan, 1981; Asaro and Suresh, 2005; Meyers et al., 2006), and other interacting particle systems (Yavari and Marsden, 2009; Kim et al., 2010). The presented analysis can further be extended to micromorphic materials (Eringen, 2003; Georgiadis and Grentzelou, 2006; Lazar and Anastassiadis, 2006; Lazar, 2007; Agiasofitou and Lazar, 2009; Galeş, 2012), microstretch elasticity (Lazar and Anastassiadis, 2006), and piezoelectromagnetic materials (Kiral and Eringen, 1990; O'Handley, 2000; Kronmüller and Parkin, 2007; Gao and Zhou, 2009).

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Appendix. Plane-strain micropolar elasticity with body forces and couples

In the case of plane strain parallel to (x_1, x_2) plane, the components u_α, b_α, $t_{\alpha\beta}$, t_{33}, φ_3, $m_{\alpha 3}$ are in general different from zero, while other kinematic and kinetic components are equal to zero. The Greek indices take the values (1,2). The corresponding inplane components of the Eshelby stress tensor are

$$P_{\alpha\beta} = (W - b_\gamma u_\gamma - \mu_3\varphi_3)\delta_{\alpha\beta} - t_{\alpha\gamma}u_{\gamma,\beta} - m_{\alpha 3}\varphi_{3,\beta}. \tag{A.1}$$

The nonvanishing out-of-plane component is given by $P_{33} = W - b_\gamma u_\gamma - \mu_3\varphi_3$. The J_β integrals are defined over the contour C within the (x_1, x_2) plane, such that

$$J_\beta = \int_C P_{\alpha\beta} n_\alpha \mathrm{d}C = -\int_A \left(b_{\alpha,\beta}u_\alpha + \mu_{3,\beta}\varphi_3\right)\mathrm{d}A, \tag{A.2}$$

where A is the area enclosed by C. In the context of plain strain couple-stress theory without body forces, the J_β integrals were used by Atkinson and Leppington (1974, 1977), Jaric (1986), and Xia and Hutchinson (1996) to study the stress field around the crack tip. For example, the J_1 integral for an infinitely long rectangular slab, made of a micropolar material and weakened by a semi-infinite crack, is $J_1 = K(\mu u^2)/h$, where the top and bottom side of the slab ($x_2 = \pm h$) are given the opposite uniform displacement $\pm u$ in the x_1-direction (for the plane strain, Atkinson and Leppington, 1974), and x_3-direction (for the antiplane strain, Lubarda and Markenscoff, 2007b), and

$$K = \left[1 - \frac{\bar{\mu}}{\mu + \bar{\mu}}\frac{\tanh(kh)}{kh}\right]^{-1}, \quad k^2 = \frac{4\mu\bar{\mu}}{(\mu + \bar{\mu})(\alpha + \bar{\alpha})}. \tag{A.3}$$

The micropolar elastic constants $\mu, \bar{\mu}, \alpha$, and $\bar{\alpha}$ appear in the representation of the elastic moduli tensors (7).

The L_3 integral is

$$\begin{aligned} L_3 &= \int_C H_{3\gamma} n_\gamma \mathrm{d}C = e_{3\alpha\beta}\int_C \left(P_{\gamma\alpha}x_\beta + t_{\gamma\alpha}u_\beta\right)n_\gamma \mathrm{d}C \\ &= -e_{3\alpha\beta}\int_A \left[b_\alpha u_\beta + (b_{\gamma,\alpha}u_\gamma + \mu_{3,\alpha}\varphi_3)x_\beta\right]\mathrm{d}A, \end{aligned} \tag{A.4}$$

where $H_{3\gamma} = e_{3\alpha\beta}(P_{\gamma\alpha}x_\beta + t_{\gamma\alpha}u_\beta)$ is the angular energy momentum. Finally, the M integral of plane-strain micropolar elasticity is

$$\begin{aligned} M &= \int_C D_\alpha n_\alpha \mathrm{d}C = \int_C \left(P_{\alpha\beta}x_\beta - m_{\alpha 3}\varphi_3\right)n_\alpha \mathrm{d}C \\ &= -\int_A \left[m_{\alpha 3}\kappa_{\alpha 3} + 2b_\alpha u_\alpha + \mu_3\varphi_3 + \left(u_\alpha b_{\alpha,\beta} + \varphi_3\mu_{3,\beta}\right)x_\beta\right]\mathrm{d}A, \end{aligned} \tag{A.5}$$

where $D_\alpha = P_{\alpha\beta}x_\beta - m_{\alpha 3}\varphi_3$ is the dilatation vector. An alternative nonconserved M integral of plain strain couple-stress elasticity, without body forces and couples, was proposed by Atkinson and Leppington (1977); see also Lubarda and Markenscoff (2000, 2003).

The inplane components of the dual Eshelby stress tensor are

$$\hat{P}_{\alpha\beta} = \hat{W}\delta_{\alpha\beta} - u_\gamma t_{\alpha\gamma,\beta} - \varphi_3 m_{\alpha 3,\beta}, \tag{A.6}$$

where $W + \hat{W} = t_{\alpha\beta}\gamma_{\alpha\beta} + m_{\alpha 3}\varphi_{3,\alpha}$ and $\gamma_{\alpha\beta} = u_{\beta,\alpha} - e_{3\alpha\beta}\varphi_3$. The out-of-plane component of the dual Eshelby stress tensor is $\hat{P}_{33} = \hat{W}$. The dual $\hat{J}_\beta$ integrals are

$$\hat{J}_\beta = \int_C \hat{P}_{\alpha\beta} n_\alpha \mathrm{d}C = \int_A \left(u_\alpha b_{\alpha,\beta} + \varphi_3 \mu_{3,\beta}\right) \mathrm{d}A. \tag{A.7}$$

The dual $\hat{L}_3$ integral is

$$\begin{aligned}\hat{L}_3 &= \int_C \hat{H}_{3\gamma} n_\gamma \mathrm{d}C = e_{3\alpha\beta} \int_C \Big[\hat{P}_{\gamma\alpha} x_\beta + u_\alpha t_{\gamma\beta} \\ &\qquad + \delta_{\alpha\gamma}\left(u_\delta t_{\beta\delta} + \varphi_3 m_{\beta 3}\right)\Big] n_\gamma \mathrm{d}C \\ &= e_{3\alpha\beta} \int_A \Big[u_\beta b_\alpha + \left(u_\gamma b_{\gamma,\alpha} + \varphi_3 \mu_{3,\alpha}\right) x_\beta\Big] \mathrm{d}A, \end{aligned} \tag{A.8}$$

where $\hat{H}_{3\gamma} = e_{3\alpha\beta}[\hat{P}_{\gamma\alpha} x_\beta + u_\alpha t_{\gamma\beta} + \delta_{\alpha\gamma}(u_\delta t_{\beta\delta} + \varphi_3 m_{\beta 3})]$ is the dual angular energy momentum. Finally, the dual $\hat{M}$ integral is

$$\begin{aligned}\hat{M} &= \int_C \hat{D}_\alpha n_\alpha \mathrm{d}C = \int_C \left(\hat{P}_{\alpha\beta} x_\beta - u_\beta t_{\alpha\beta}\right) n_\alpha \mathrm{d}C \\ &= \int_A \Big[m_{\alpha 3} \kappa_{\alpha 3} + 2u_\alpha b_\alpha + \varphi_3 \mu_3 + \left(u_\alpha b_{\alpha,\beta} + \varphi_3 \mu_{3,\beta}\right) x_\beta\Big] \mathrm{d}A, \end{aligned} \tag{A.9}$$

with $\hat{D}_\alpha = \hat{P}_{\alpha\beta} x_\beta - u_\beta t_{\alpha\beta}$ denoting the dual dilatation vector.

It is noted that the sums $(P_{\alpha\beta} + \hat{P}_{\alpha\beta})$, $(H_{3\alpha} + \hat{H}_{3\alpha})$, and $(D_\alpha + \hat{D}_\alpha)$ are divergence free, i.e., $(P_{\alpha\beta} + \hat{P}_{\alpha\beta})_{,\alpha} = 0$, $(H_{3\alpha} + \hat{H}_{3\alpha})_{,\alpha} = 0$, and $(D_\alpha + \hat{D}_\alpha)_{,\alpha} = 0$, so that in the plane-strain problems the duality holds $J_\beta + \hat{J}_\beta = 0$ $(\beta = 1,2)$, $L_3 + \hat{L}_3 = 0$ and $M + \hat{M} = 0$, as anticipated from the three-dimensional results of Section 8. If there is a defect within the body, whose contour in the (x_1, x_2) plane is C_0, the configurational force corresponding to its translation, rotation and dilation is, respectively, $f_\beta = J_\beta(C_0) = -\hat{J}_\beta(C_0)$, $f_3 = -L_3(C_0) = \hat{L}_3(C_0)$, and $f = M(C_0) = -\hat{M}(C_0)$.

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