Acta Mech 231, 155–178 (2020)
https://doi.org/10.1007/s00707-019-02539-z

ORIGINAL PAPER

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On the Kelvin, Boussinesq, and Mindlin problems

Received: 19 June 2019 / Revised: 14 September 2019 / Published online: 9 October 2019

Abstract Three fundamental three-dimensional axisymmetric problems of elasticity are revisited: the Kelvin problem of a concentrated force in an infinite space, the Boussinesq problem of a vertical concentrated force at the boundary of a half-space, and the Mindlin problem of a vertical concentrated force in the interior of a half-space. New elements of the derivation of the solution to each of these problems are included to make the presentation more appealing for the coverage of the topic in graduate courses of solid mechanics. Two approaches are employed and compared, one based on the Galerkin method and Love's potential function, and the other based on the Papkovich–Neuber displacement representation and Boussinesq's potential functions. A historical perspective and a referral to more recent contributions in the field are also given.

1 Introduction

We review in this paper the determination of the stress and displacement fields in three fundamental three-dimensional axisymmetric elasticity problems: the Kelvin problem of a concentrated force in the interior of an infinite space, the Boussinesq problem of a concentrated force orthogonal to the boundary of a half-space, and the Mindlin problem of a concentrated force in the interior of a half-space. Different procedures are used to construct the solutions to each problem, which are related to each other, yet are conceptually appealing and instructive on their own. Three derivations of the solution of the Boussinesq problem, and two derivations of the solution of the Mindlin problem are presented. The simplification in the latter derivation is achieved by requiring at the early stage of the analysis that the Mindlin problem reduces to the Boussinesq problem in an appropriate limit. The presentation is made in as simple a way as possible, so as to make the exposition appealing for the coverage of this important topic in an introductory graduate course of solid mechanics. Thus, the referral to the more advanced framework of the theory of elasticity, based on the use of Green's formula or integral transform techniques, is omitted. Two approaches are adopted instead, one based on the Galerkin method and Love's potential function, and the other based on the Papkovich–Neuber displacement representation and the Boussinesq's potential functions. In the Papkovich–Neuber approach, two harmonic functions play a prominent role, $f = \ln(\rho + z)$ and $g = 1/\rho \equiv \partial f/\partial z$, where $\rho^2 = r^2 + z^2$ and (r, θ, z) are the cylindrical coordinates. In the approach based on the use of Love's potential, it is two biharmonic functions that play a prominent role, $\varphi = zf = z\ln(\rho + z)$ and $\psi = \rho^2 g = \rho$.

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2 Governing equations of axisymmetric elasticity

The Cauchy equations of equilibrium for axisymmetric three-dimensional problems with body forces b_r and b_z (e.g., [1,2]) are

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+b_r=0\,,\quad \frac{\partial\sigma_{zr}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{zr}}{r}+b_z=0\,. \tag{2.1}$$

The stresses are related to strains by Hooke's law,

$$\sigma_{rr}=2\mu\epsilon_{rr}+\lambda e\,,\quad \sigma_{\theta\theta}=2\mu\epsilon_{\theta\theta}+\lambda e\,,\quad \sigma_{zz}=2\mu\epsilon_{zz}+\lambda e\,,\quad \sigma_{rz}=2\mu\epsilon_{rz}\,, \tag{2.2}$$

where λ and μ are Lamé's constants, and $e=\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz}$ is the volumetric strain. The strains are related to displacements by

$$\epsilon_{rr}=\frac{\partial u_r}{\partial r}\,,\quad \epsilon_{\theta\theta}=\frac{u_r}{r}\,,\quad \epsilon_{zz}=\frac{\partial u_z}{\partial z}\,,\quad \epsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right). \tag{2.3}$$

The three compatibility equations for strains [3,4] are

$$\begin{aligned}
&\frac{\partial^2\epsilon_{\theta\theta}}{\partial z^2}+\frac{1}{r}\,\frac{\partial\epsilon_{zz}}{\partial r}-\frac{2}{r}\,\frac{\partial\epsilon_{zr}}{\partial z}=0\,,\\
&\frac{\partial^2\epsilon_{rr}}{\partial z^2}+\frac{\partial^2\epsilon_{zz}}{\partial r^2}-2\,\frac{\partial^2\epsilon_{zr}}{\partial r\partial z}=0\,,\\
&r\,\frac{\partial\epsilon_{\theta\theta}}{\partial r}+\epsilon_{\theta\theta}-\epsilon_{rr}=0\,.
\end{aligned} \tag{2.4}$$

The corresponding compatibility equations in terms of stresses are

$$\begin{aligned}
&\nabla^2\sigma_{rr}-\frac{2}{r^2}\,(\sigma_{rr}-\sigma_{\theta\theta})+\frac{1}{1+\nu}\,\frac{\partial^2\sigma}{\partial r^2}=0\,,\\
&\nabla^2\sigma_{\theta\theta}+\frac{2}{r^2}\,(\sigma_{rr}-\sigma_{\theta\theta})+\frac{1}{1+\nu}\,\frac{1}{r}\,\frac{\partial\sigma}{\partial r}=0\,,\\
&\frac{\partial\sigma_{zr}}{\partial z}-\frac{\partial\sigma_{zz}}{\partial r}+\frac{1}{1+\nu}\,\frac{\partial\sigma}{\partial r}=0\,,
\end{aligned} \tag{2.5}$$

where $\sigma=\sigma_{rr}+\sigma_{\theta\theta}+\sigma_{zz}$ and $\sigma=(2\mu+3\lambda)e$.

The Navier equations of elasticity for three-dimensional axisymmetric problems read

$$\begin{aligned}
&\mu\left(\nabla^2u_r-\frac{u_r}{r^2}\right)+(\lambda+\mu)\,\frac{\partial e}{\partial r}+b_r=0\,,\\
&\mu\nabla^2u_z+(\lambda+\mu)\,\frac{\partial e}{\partial z}+b_z=0\,,
\end{aligned} \tag{2.6}$$

which are subjected to boundary conditions of each specific problem. The utilized Laplacian operator is

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\,. \tag{2.7}$$

3 Galerkin vector and Love's potential

The general three-dimensional Navier equations of elasticity, written in vector form, are

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\,\nabla(\nabla\cdot\mathbf{u}) + \mathbf{b} = 0\,. \tag{3.1}$$

If the displacement vector $\mathbf{u}$ is expressed in terms of the vector function $\boldsymbol{\Omega}$ as

$$2\mu\mathbf{u} = 2(1-\nu)\nabla^2\boldsymbol{\Omega} - \nabla(\nabla\cdot\boldsymbol{\Omega})\,, \tag{3.2}$$

it readily follows that (3.1) is identically satisfied, provided that $\boldsymbol{\Omega}$ satisfies a non-homogeneous biharmonic equation

$$\nabla^4\boldsymbol{\Omega} = -\frac{\mathbf{b}}{1-\nu} \tag{3.3}$$

where ν is Poisson's ratio of the material. The vector function $\boldsymbol{\Omega}$ is known as the Galerkin vector [5].

If only the z-component of the Galerkin vector is assumed to be nonzero, so that $\boldsymbol{\Omega} = \{0,0,\Omega\}$ and, correspondingly, $\mathbf{b} = \{0,0,b\}$, and if the problem is axisymmetric, $\Omega = \Omega(r,z)$, then (3.2) gives

$$2\mu u_r = -\frac{\partial^2\Omega}{\partial r\partial z}, \quad 2\mu u_z = 2(1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\,. \tag{3.4}$$

It can be readily verified that (3.4) identically satisfies the axisymmetric Navier equations (2.6), provided that

$$\nabla^4\Omega = -\frac{b}{1-\nu}\,. \tag{3.5}$$

The potential function Ω was used by Love [6] in his study of axisymmetrically loaded solids of revolution. The corresponding stress components can be expressed in terms of Ω as

$$\begin{aligned}
\sigma_{rr} &= \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{\partial^2\Omega}{\partial r^2}\right), \quad \sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right),\\
\sigma_{zz} &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\right], \quad \sigma_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\right].
\end{aligned} \tag{3.6}$$

It can be easily verified that these stress expressions identically satisfy the first of the Cauchy equilibrium equations in (2.1) when $b_r = 0$, while the second equilibrium equation requires that Ω satisfies (3.5). The function Ω is commonly referred to as Love's potential. In the absence of body forces ($b_r = b_z = 0$), Love's potential satisfies the biharmonic equation $\nabla^4\Omega = 0$.

The sum of three normal stresses is

$$\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = (1+\nu)\,\frac{\partial}{\partial z}\left(\nabla^2\Omega\right), \quad \nabla^2(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = 0\,. \tag{3.7}$$

Thus, if Ω turns out in a specific problem to be harmonic ($\nabla^2\Omega = 0$), the stress state in that problem is deviatoric. Also, in the case when Ω is harmonic, displacement and stress components can be derived from the harmonic potential $\omega = -\partial\Omega/\partial z$, because then (3.4) and (3.6) simplify to

$$2\mu u_r = \frac{\partial\omega}{\partial r}, \quad 2\mu u_z = \frac{\partial\omega}{\partial z}\,, \tag{3.8}$$

and

$$\sigma_{rr} = \frac{\partial^2\omega}{\partial r^2}\,, \quad \sigma_{\theta\theta} = \frac{1}{r}\,\frac{\partial\omega}{\partial r}\,, \quad \sigma_{zz} = \frac{\partial^2\omega}{\partial z^2}\,, \quad \sigma_{rz} = \frac{\partial^2\omega}{\partial r\partial z}\,. \tag{3.9}$$

The function ω is referred to as Lamé's potential, because Lamé used it in his study of the pressurized hollow cylinders and spheres.

4 Papkovich–Neuber representation and Boussinesq's potentials

Papkovich [7] and Neuber [8] independently constructed an appealing expression for the displacement vector $\mathbf{u}$ which satisfies the general three-dimensional Navier equations (3.1). In this expression, the displacement vector $\mathbf{u}$ is given in terms of one vector potential ($\mathbf{B}$) and one scalar potential (β), such that [9–11]

$$2\mu\mathbf{u} = \mathbf{B} - \nabla\left[\frac{1}{4(1-\nu)}\,\mathbf{B}\cdot\boldsymbol{\rho} + \beta\right] \tag{4.1}$$

where $\mathbf{B}$ and β are the solutions to Poisson's equations

$$\nabla^2\mathbf{B} = -2\mathbf{b}\,, \quad \nabla^2\beta = \frac{\boldsymbol{\rho}\cdot\mathbf{b}}{2(1-\nu)}\,. \tag{4.2}$$

The spherical radius vector $\boldsymbol{\rho} = r\mathbf{e}_r + z\mathbf{e}_z$, where (r, θ, z) are cylindrical coordinates (see Fig. 1a), and $\mathbf{e}_r$ and $\mathbf{e}_z$ are the unit vectors in the r and z directions. In general, since there are three displacement components, only two of the three components of the vector potential $\mathbf{B}$ are independent. The scalar potential β in (4.2) can be scaled by $4(1-\nu)$, as is often done in the literature [12–16], in which case the coefficient $1/[4(1-\nu)]$ can be pulled out in front of the ∇ operator, while the second Poisson's equation in (4.2) becomes $\nabla^2\beta = 2\boldsymbol{\rho}\cdot\mathbf{b}$. See also [17] for yet another scaling of potentials $\mathbf{B}$ and β. A scalar potential ϕ, related to β in (4.1) by $\phi = -4(1-\nu)\beta$, has been used in [18].

Certain axisymmetric problems can be solved by assuming that the vector potential $\mathbf{B}$ has only a z-component, $\mathbf{B} = \{0, 0, B\}$, with $B = B(r, z)$. If the body force is $\mathbf{b} = \{0, 0, b\}$, then the potential functions B and β are the solutions to Poisson's equations

$$\nabla^2 B = -2b\,, \quad \nabla^2\beta = \frac{zb}{2(1-\nu)}\,. \tag{4.3}$$

The corresponding nonvanishing displacement components can then be expressed, from (4.1), as

$$2\mu u_r = -\frac{\partial A}{\partial r}\,, \quad 2\mu u_z = B - \frac{\partial A}{\partial z} \tag{4.4}$$

where, conveniently, we have introduced the function

$$A = \beta + \frac{zB}{4(1-\nu)}\,. \tag{4.5}$$

The functions B and β are known as Boussinesq's potentials. In the absence of body force, they are harmonic functions ($\nabla^2 B = 0$ and $\nabla^2\beta = 0$). The function A itself is biharmonic, because zB is biharmonic.

The stresses are obtained from Hooke's law (2.2) by using (4.4) and the strain–displacement relations (2.3). The resulting expressions are

$$\sigma_{rr} = -\frac{\partial^2 A}{\partial r^2} + \frac{\nu}{2(1-\nu)}\,\frac{\partial B}{\partial z}, \tag{4.6.1}$$

$$\sigma_{\theta\theta} = -\frac{1}{r}\,\frac{\partial A}{\partial r} + \frac{\nu}{2(1-\nu)}\,\frac{\partial B}{\partial z}\,, \tag{4.6.2}$$

$$\sigma_{zz} = -\frac{\partial^2 A}{\partial z^2} + \frac{2-\nu}{2(1-\nu)}\,\frac{\partial B}{\partial z}\,, \tag{4.6.3}$$

$$\sigma_{zr} = \frac{1}{2}\,\frac{\partial B}{\partial r} - \frac{\partial^2 A}{\partial r\,\partial z}\,. \tag{4.6.4}$$

It is noted that

$$\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = \frac{1+\nu}{2(1-\nu)}\,\frac{\partial B}{\partial z}\,, \tag{4.7}$$

and

$$\nabla^2 A = \nabla^2\beta + \frac{\nabla^2(zB)}{4(1-\nu)} = \frac{1}{2(1-\nu)}\,\frac{\partial B}{\partial z}\,, \tag{4.8}$$

which holds with or without the presence of the body force b.

4.1 Boussinesq's potentials versus Love's potential

By comparing the displacement expressions (3.4) and (4.4), it follows that Boussinesq's potentials B and β can be expressed in terms of Love's potential Ω by

$$B = 2(1-\nu)\nabla^2\Omega\,, \quad \beta = \frac{\partial\Omega}{\partial z} - \frac{1}{2}\,z\nabla^2\Omega\,. \tag{4.9}$$

Also, we have

$$A = \frac{\partial\Omega}{\partial z} \quad \Rightarrow \quad \Omega = \int A\,\mathrm{d}z\,, \tag{4.10}$$

which can be used to determine the expression for Ω, if the expressions for B and β have already been determined.

4.2 Special cases

If the plane $z = 0$ is free from shear traction, from Eq. (4.6.4) and expression (4.5), we can write

$$\sigma_{zr}(r, z=0) = \left[\frac{\partial}{\partial r}\left(\frac{1}{2}\,B - \frac{\partial A}{\partial z}\right)\right]_{z=0} = \frac{\partial}{\partial r}\left[\frac{1-2\nu}{4(1-\nu)}\,B - \frac{\partial\beta}{\partial z} - \frac{z}{4(1-\nu)}\,\frac{\partial B}{\partial z}\right]_{z=0} = 0. \tag{4.11}$$

Some problems in which $\sigma_{zr}(r, z=0) = 0$ can be solved by assuming that

$$\frac{1-2\nu}{4(1-\nu)}\,B - \frac{\partial\beta}{\partial z} = 0 \quad \Rightarrow \quad B = \frac{4(1-\nu)}{1-2\nu}\,\frac{\partial\beta}{\partial z} \tag{4.12}$$

everywhere in the medium, because then the condition $\sigma_{zr}(r, z=0) = 0$ in (4.11) is automatically satisfied. For this type of problems (e.g., Boussinesq problem in Sect. 7), the shear stress is specified by

$$\sigma_{zr} = -\frac{z}{1-2\nu}\,\frac{\partial^3\beta}{\partial r\,\partial z^2}\,. \tag{4.13}$$

The corresponding displacement components are obtained by substituting (4.12) into (4.4), which gives

$$2\mu u_r = -\frac{\partial A}{\partial r}\,, \quad 2\mu u_z = \frac{\partial}{\partial z}\left[\frac{4(1-\nu)}{1-2\nu}\,\beta - A\right] \tag{4.14}$$

where

$$A = \beta + \frac{z}{1-2\nu}\,\frac{\partial\beta}{\partial z}\,. \tag{4.15}$$

The normal stresses follow from (4.6), (4.12), and (4.15) and are given by

$$\begin{aligned}
\sigma_{rr} &= -\frac{1}{1-2\nu}\left[2\nu\,\frac{1}{r}\,\frac{\partial\beta}{\partial r} + \frac{\partial}{\partial z}\left(z\,\frac{\partial^2\beta}{\partial r^2}\right)\right],\\
\sigma_{\theta\theta} &= -\frac{1}{1-2\nu}\left[2\nu\,\frac{\partial^2\beta}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial z}\left(z\,\frac{\partial\beta}{\partial r}\right)\right],\\
\sigma_{zz} &= \frac{1}{1-2\nu}\left(\frac{\partial^2\beta}{\partial z^2} - z\,\frac{\partial^3\beta}{\partial z^3}\right).
\end{aligned} \tag{4.16}$$

Therefore, if (4.12) holds, all displacement and stress components are expressed in terms of the single function $\beta = \beta(r, z)$. See also a related analysis in [17,19].

In a few problems (e.g, a center of dilatation, a doublet of dilatation/compression centers, a line of dilatation centers), Love's potential turns out to be harmonic ($\nabla^2\Omega = 0$). In this case, from (4.9) and (4.10), it follows that $B = 0$ and $\beta = A = \partial\Omega/\partial z$, while the displacement and stress expressions (4.4) and (4.5) simplify to (3.8) and (3.9) of Sect. 3, with $\omega = -\beta$ (Lamé's potential).

4.3 Tabular summary of potentials for axisymmetric 3D elasticity with $b_z = b$

To ease the subsequent developments and the comparison of the derived results, we tabulate in this Section the definitions of the utilized potentials for the three-dimensional axisymmetric elasticity problems in the presence of body force $b_z = b$, and the corresponding displacement and stress expressions.

Love's potential: $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mathbf{r}, \mathbf{z})$

$$
\begin{gathered}
\nabla^4\Omega = -\frac{b}{1-\nu} \\
2\mu u_r = -\frac{\partial^2\Omega}{\partial r\partial z}, \quad 2\mu u_z = 2(1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2} \\
\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{\partial^2\Omega}{\partial r^2}\right), \quad \sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right) \\
\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\right], \quad \sigma_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\right]
\end{gathered}
\tag{4.17}
$$

Boussinesq's potentials: $\mathbf{B} = \mathbf{B}(\mathbf{r}, \mathbf{z})$ and $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{r}, \mathbf{z})$

$$
\begin{gathered}
\nabla^2 B = -2b, \quad \nabla^2\beta = -\frac{zb}{2(1-\nu)} \\
2\mu u_r = -\frac{\partial A}{\partial r}, \quad 2\mu u_z = B - \frac{\partial A}{\partial z} \\
\sigma_{rr} = -\frac{\partial^2 A}{\partial r^2} + \frac{\nu}{2(1-\nu)}\frac{\partial B}{\partial z} \\
\sigma_{\theta\theta} = -\frac{1}{r}\frac{\partial A}{\partial r} + \frac{\nu}{2(1-\nu)}\frac{\partial B}{\partial z} \\
\sigma_{zz} = -\frac{\partial^2 A}{\partial z^2} + \frac{2-\nu}{2(1-\nu)}\frac{\partial B}{\partial z} \\
\sigma_{zr} = \frac{1}{2}\frac{\partial B}{\partial r} - \frac{\partial^2 A}{\partial r\partial z}, \quad A = \beta + \frac{zB}{4(1-\nu)}
\end{gathered}
\tag{4.18}
$$

5 Kelvin problem: concentrated force in an infinite space

Figure 1a shows the concentrated force P in the interior of an infinitely extended solid. The resulting stress and displacement fields were originally determined by Lord Kelvin [20]. The problem is of great importance because its solution provides the influence functions (Green's functions) for the three-dimensional Navier equations of elasticity and the stress and displacement fields in an infinite medium under other types of loading [21].

By placing the coordinate origin at the point of application of the force, Love's potential Ω can be assumed to be of the form

$$
\Omega = c\rho, \quad \left(\nabla^2\Omega = 2c/\rho, \quad \nabla^4\Omega = 0\right), \tag{5.1}
$$

where c is a constant to be determined in the sequel, and $\rho^2 = r^2 + z^2$. By substitution of (5.1) into (3.6), the stress components are found to be

$$
\begin{gathered}
\sigma_{rr} = c\left[(1-2\nu)\frac{z}{\rho^3} - 3\frac{r^2 z}{\rho^5}\right], \quad \sigma_{\theta\theta} = c(1-2\nu)\frac{z}{\rho^3}, \\
\sigma_{zz} = -c\left[(1-2\nu)\frac{z}{\rho^3} + 3\frac{z^3}{\rho^5}\right], \quad \sigma_{zr} = -c\left[(1-2\nu)\frac{r}{\rho^3} + 3\frac{rz^2}{\rho^5}\right].
\end{gathered}
\tag{5.2}
$$

The corresponding displacement components are, from (3.4),

$$
u_r = \frac{c}{2\mu}\frac{rz}{\rho^3}, \quad u_z = \frac{c}{2\mu}\left[4(1-\nu)\frac{1}{\rho} - \frac{r^2}{\rho^3}\right]. \tag{5.3}
$$

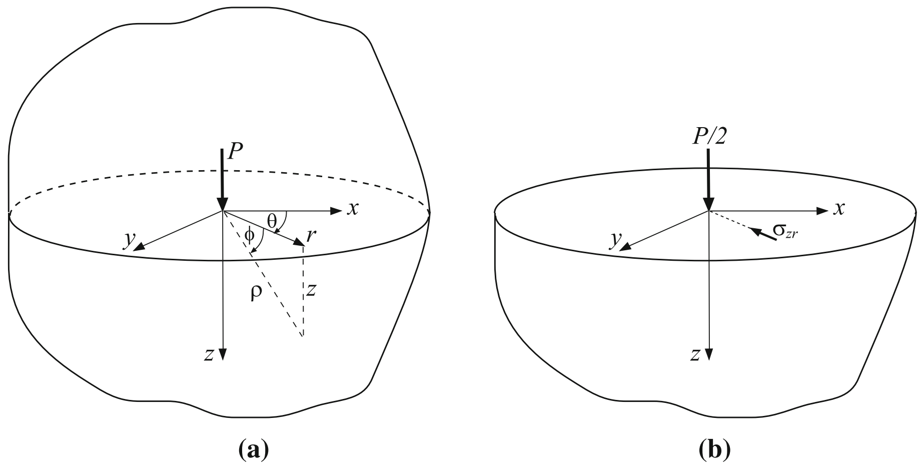


Fig. 1 **a** A concentrated force P along the z axis in the interior of an infinitely extended solid. Shown are the rectangular (x, y, z) and cylindrical (r, θ, z) coordinates. The spherical radius $\rho = (r^2 + z^2)^{1/2}$ is at an angle ϕ from the cylindrical radius r. **b** A lower portion of the infinite solid from part (**a**). The boundary $z = 0$ is under the concentrated force $P/2$ and the shear stress distribution $\sigma_{zr} \sim 1/r^2$

The constant c can be determined by imposing the integral equilibrium condition

$$\int_0^\infty \sigma_{zz}(r, z > 0) 2\pi r \, \mathrm{d}r = -\frac{P}{2}, \tag{5.4}$$

which follows because the two halves of the space ($z \geq 0$ and $z \leq 0$) each carry one-half of the total force P. The substitution of the expression for $\sigma_{zz}(r, z)$ from (5.2) into (5.4) and integration gives

$$c = \frac{P}{8\pi(1-\nu)}. \tag{5.5}$$

The traction components in the mid-plane $z = 0$ (Fig. 1b) are

$$\sigma_{zr}(r, z = 0) = -\frac{(1-2\nu)P}{8\pi(1-\nu)} \frac{1}{r^2} = -\frac{c_*}{2} \frac{1}{r^2}, \quad \sigma_{zz}(r \neq 0, z = 0) = 0, \tag{5.6}$$

where

$$c_* = \frac{(1-2\nu)P}{4\pi(1-\nu)} = 2(1-2\nu)c. \tag{5.7}$$

5.1 Treating **P** as a body force

If P is considered to be a concentrated body force, then by using the Dirac delta function one can write $b = P\delta(z)$, and the governing equation (3.5) for Love's potential becomes

$$\nabla^4 \Omega = -\frac{P}{1-\nu} \delta(z). \tag{5.8}$$

The fundamental (particular) solution to this non-homogeneous biharmonic equation is

$$\Omega = \frac{P}{1-\nu} \frac{\rho}{8\pi}. \tag{5.9}$$

This follows because the fundamental solution to Poisson's equation $\nabla^2 g = -\delta(\rho)$ is $g = 1/(4\pi\rho)$, e.g., [22,23]. Thus, by writing $g = \nabla^2 f$, so that $\nabla^4 f = \nabla^2 g = -\delta(\rho)$, it follows by integration that $f = \rho/(8\pi)$. Indeed,

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial f}{\partial \rho} = \frac{1}{4\pi\rho} \quad \Rightarrow \quad f = \frac{\rho}{8\pi}. \tag{5.10}$$

In the derivation, we have assumed a spherical symmetry of $f = f(\rho)$, based on the rotation invariance of the Laplacian operator. Expression (5.9), with $P = 1$, defines the influence function (Green's function) for the biharmonic equation (3.5).

5.2 Solution of the Kelvin problem by using Boussinesq's potentials

The Boussinesq's potentials for the Kelvin problem can be deduced by substituting (5.1) into (4.9), and are given by

$$B = \frac{P}{2\pi}\,\frac{1}{\rho}\,, \quad \beta = 0\,. \tag{5.11}$$

The expression for B also follows as a fundamental solution to Poisson's equation for B listed in (4.3), if the body force is taken to be $b = P\delta(z)$, i.e.,

$$\nabla^2 B = -2P\delta(z) \quad \Rightarrow \quad B = 2P\,\frac{1}{4\pi\rho}\,. \tag{5.12}$$

If the coordinate origin is not at the point of the application of the force P, but at a distance h above it, then

$$\Omega = c\rho_1\,, \quad B = 4(1-\nu)c\,\frac{1}{\rho_1}\,, \quad \beta = -c\,\frac{h}{\rho_1} \tag{5.13}$$

where $\rho_1^2 = r^2 + (z-h)^2$ and $c = P/[8\pi(1-\nu)]$. Expressions (5.13) will be used in Sect. 8.1 to determine Boussinesq's potentials for the Mindlin problem of the concentrated force in the interior of a half-space.

6 Doublet of forces, center of dilatation, doublet of dilatation–compression centers, and a line of centers of dilatation

In this Section, we review the solutions for the elastic fields in an infinite solid produced by a doublet of forces, a center of dilatation, a doublet of dilatation–compression centers, and a line of centers of dilatation. These elastic fields are used in the derivation of the solutions to the Boussinesq's and Mindlin problems, considered in the subsequent Sects. 7 and 8.

6.1 Doublet of forces

Figure 2a shows a doublet of opposite forces P along the z axis at a small distance d from each other [1,6]. The corresponding potential function can be generated from

$$\Omega(r,z) = \Omega^{\mathrm{K}}(r, z-d/2) - \Omega^{\mathrm{K}}(r, z+d/2) = -\frac{\partial\Omega^{\mathrm{K}}}{\partial z}\,d\,, \quad \Omega^{\mathrm{K}}(r,z) = c\rho \tag{6.1}$$

where Ω^{K} is the potential function of the Kelvin problem. This gives

$$\Omega(r,z) = -cd\,\frac{z}{\rho}\,, \quad c = \frac{P}{8\pi(1-\nu)}\,. \tag{6.2}$$

The product cd is referred to as the strength of the doublet. The corresponding stresses are obtained by substituting (6.2) into (3.6),

$$\begin{aligned}
\sigma_{rr} &= cd\left[(1+\nu)\,\frac{2}{\rho^3} - 6\nu\,\frac{z^2}{\rho^5} - 15\,\frac{r^2z^2}{\rho^7}\right],\\
\sigma_{\theta\theta} &= (1-2\nu)cd\left(\frac{2}{\rho^3} - 3\,\frac{r^2}{\rho^5}\right),\\
\sigma_{zz} &= cd\left[(1-2\nu)\,\frac{1}{\rho^3} + 6(1+\nu)\,\frac{z^2}{\rho^5} - 15\,\frac{z^4}{\rho^7}\right],\\
\sigma_{zr} &= 3cd\left[(1+2\nu)\,\frac{rz}{\rho^5} - 5\,\frac{rz^3}{\rho^7}\right].
\end{aligned} \tag{6.3}$$

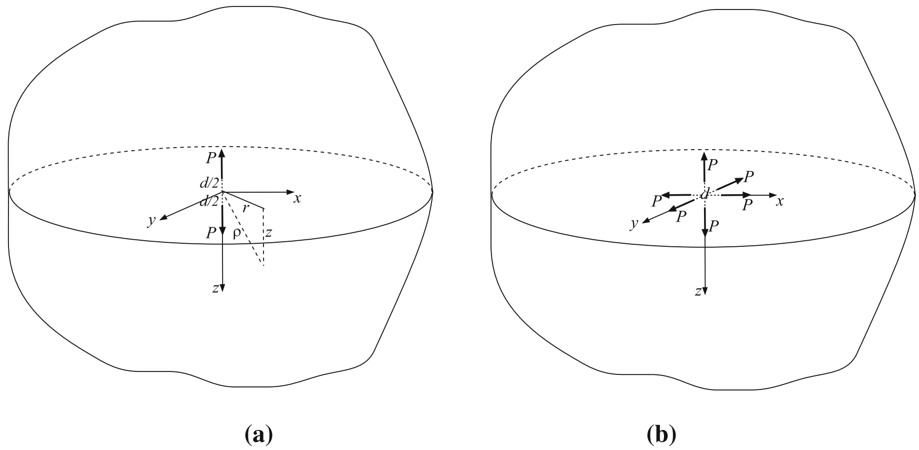


Fig. 2 **a** A doublet of opposite forces P at a small distance d along the z axis in the interior of an infinitely extended solid. **b** A center of dilatation in the interior of an infinitely extended solid made by three orthogonal doublets of forces P at a small distance d from each other

The displacement expressions are obtained by substituting (6.2) into (3.4),

$$
\begin{aligned}
u_r &= -\frac{cd}{2\mu}\,\frac{r}{\rho^3}\left(1-3\,\frac{z^2}{\rho^2}\right) = \frac{cd}{2\mu}\,\frac{r}{\rho^3}\left(2-3\,\frac{r^2}{\rho^2}\right),\\
u_z &= \frac{cd}{2\mu}\,\frac{z}{\rho^3}\left(1-4\nu+3\,\frac{z^2}{\rho^2}\right) = \frac{cd}{2\mu}\,\frac{z}{\rho^3}\left[4(1-\nu)-3\,\frac{r^2}{\rho^2}\right].
\end{aligned}
\tag{6.4}
$$

The Boussinesq's potentials for a doublet of forces are readily found from (4.9) to be

$$
B = 4(1-\nu)cd\,\frac{z}{\rho^3}\,, \quad \beta = -cd\,\frac{1}{\rho}\,. \tag{6.5}
$$

6.2 Center of dilatation: three orthogonal doublets of forces

Figure 2b shows a center of dilatation made by three orthogonal doublets of opposite forces P separated at some small distance d from each other [1,6]. By linearity and symmetry, we conclude from (6.5) that the corresponding potentials $\mathbf{B}$ and β are of the form

$$
\mathbf{B} = 4(1-\nu)cd\,\frac{\boldsymbol{\rho}}{\rho^3}\,, \quad \beta = -3cd\,\frac{1}{\rho}\,. \tag{6.6}
$$

By substituting (6.6) into the Papkovich–Neuber representation (4.1), the following displacement expression is obtained:

$$
2\mu\mathbf{u} = c_* d\,\frac{\boldsymbol{\rho}}{\rho^3}\,, \quad c_* = 2(1-2\nu)c\,. \tag{6.7}
$$

The displacement expression in (6.7) can be rewritten as

$$
2\mu\mathbf{u} = -k_*\,\nabla\left(\frac{1}{\rho}\right), \quad k_* = c_* d\,. \tag{6.8}
$$

If this is compared with (4.1), we recognize an alternative representation of the Papkovich–Neuber potentials, which is

$$
\mathbf{B} = \mathbf{0}\,, \quad \beta = k_*\,\frac{1}{\rho}\,. \tag{6.9}
$$

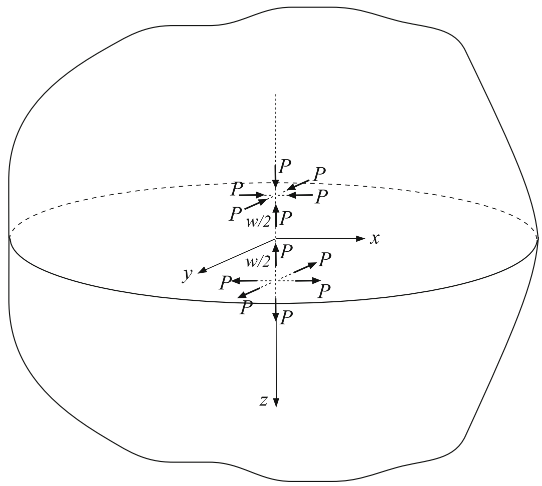


Fig. 3 A doublet of dilatation/compression centers in the interior of an infinitely extended solid at a small distance w above each other

The product $k_* = c_* d$ is referred to as the strength of the dilatation center. Thus, in this case the potential β alone serves as the stress potential, such that

$$\sigma_{rr} = -\frac{\partial^2 \beta}{\partial r^2}\,, \quad \sigma_{\theta\theta} = -\frac{1}{r}\,\frac{\partial \beta}{\partial r}\,, \quad \sigma_{zz} = -\frac{\partial^2 \beta}{\partial z^2}\,, \quad \sigma_{rz} = -\frac{\partial^2 \beta}{\partial r \partial z} \tag{6.10}$$

which follows from (4.6), because $A = \beta$ and $\nabla^2 \beta = 0$. The stresses are, therefore,

$$\begin{aligned} \sigma_{rr} &= k_* \left(\frac{1}{\rho^3} - 3\,\frac{r^2}{\rho^5} \right), \quad \sigma_{\theta\theta} = k_* \frac{1}{\rho^3}\,, \\ \sigma_{zz} &= k_* \left(\frac{1}{\rho^3} - 3\,\frac{z^2}{\rho^5} \right), \quad \sigma_{zr} = -3k_*\,\frac{rz}{\rho^5}\,. \end{aligned} \tag{6.11}$$

This is a purely deviatoric state of stress, because $\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = 0$. Thus, interestingly, there is no volumetric strain at any point in the infinite medium caused by the dilatation center.

Since $A = \beta$, we obtain from (4.10) by integration the expression for Love's potential,

$$\Omega = k_* \ln(\rho + z)\,. \tag{6.12}$$

Thus, in the case of a center of dilatation, the potential Ω is a harmonic function ($\nabla^2 \Omega = 0$).

6.3 Doublet of dilatation–compression centers

Figure 3 shows a doublet made by a dilatation center at a point $(0, w/2)$ and an opposite compression center, at a small distance w above the dilatation center [24]. The corresponding Love's potential can be generated from

$$\Omega(r, z) = \Omega^{\mathrm{DC}}(r, z - w/2) - \Omega^{\mathrm{DC}}(r, z + w/2) = -\frac{\partial \Omega^{\mathrm{DC}}}{\partial z}\, w \tag{6.13}$$

where $\Omega^{\mathrm{DC}}(r, z) = k_* \ln(\rho + z)$ denotes the potential function of a single dilatation center, as defined by (6.12). This gives

$$\Omega(r, z) = -k\,\frac{1}{\rho}\,, \quad k = k_* w\,. \tag{6.14}$$

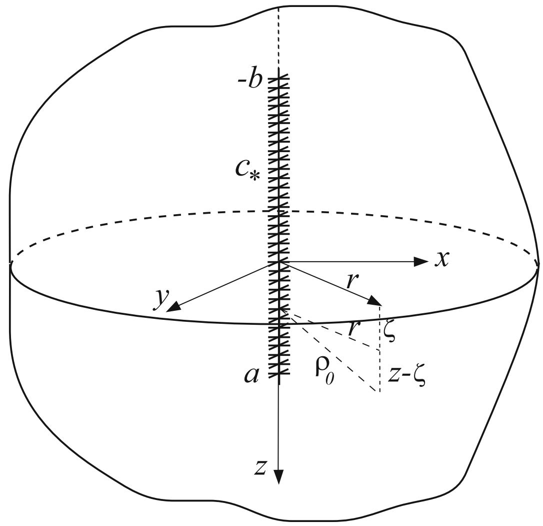


Fig. 4 A line of centers of dilatation of intensity c_* (per unit length), uniformly distributed in an infinitely extended solid along the z axis from $z = -b$ to $z = a$. An arbitrary center of dilatation from the line is specified by the coordinate ζ, while (r, z) are the coordinates of an arbitrary point of the infinite solid, which is at the distance ρ_0 from the considered center of dilatation

The product $k = k_* w$ is the strength of the doublet of dilatation–compression centers. The corresponding stresses are

$$\sigma_{rr} = 3k\left(\frac{z}{\rho^5} - 5\,\frac{zr^2}{\rho^7}\right), \quad \sigma_{\theta\theta} = 3k\,\frac{z}{\rho^5}\,,$$
$$\sigma_{zz} = 3k\left(3\,\frac{z}{\rho^5} - 5\,\frac{z^3}{\rho^7}\right), \quad \sigma_{zr} = 3k\left(\frac{r}{\rho^5} - 5\,\frac{rz^2}{\rho^7}\right). \tag{6.15}$$

The stress state is deviatoric, because $\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = 0$. Over the plane $z = 0$, the normal stress σ_{zz} vanishes, while the shear stress is

$$\sigma_{zr}(r, z = 0) = 3k\,\frac{1}{r^4}\,. \tag{6.16}$$

Since Ω in (6.14) is harmonic ($\nabla^2\Omega = 0$), it follows from (4.9) that the Boussinesq's potentials are given by

$$B = 0\,, \quad \beta = k\,\frac{z}{\rho^3}\,. \tag{6.17}$$

6.4 A line of centers of dilatation

The lines of centers of dilatation or compression are useful building blocks in the construction of solutions to other three-dimensional elasticity problems. Figure 4 shows a line of centers of dilatation of specific strength (intensity) c_* per unit length. The centers of dilatation are uniformly distributed along the z axis from $z = -b$ to $z = a$. The potential function for a center of dilatation of intensity P at the location $z = \zeta$ is, by (6.12),

$$\Omega_0(\zeta) = c_* \ln(\rho_0 + z - \zeta)\,, \quad \rho_0 = \sqrt{r^2 + (z-\zeta)^2}\,. \tag{6.18}$$

The potential function for the entire line of centers of dilatation is obtained by integration,

$$\Omega = \int_{-b}^{a} \Omega_0(\zeta)\,\mathrm{d}\zeta = c_* \int_{-b}^{a} \ln\left[\sqrt{r^2 + (z-\zeta)^2} + z - \zeta\right]\mathrm{d}\zeta\,, \tag{6.19}$$

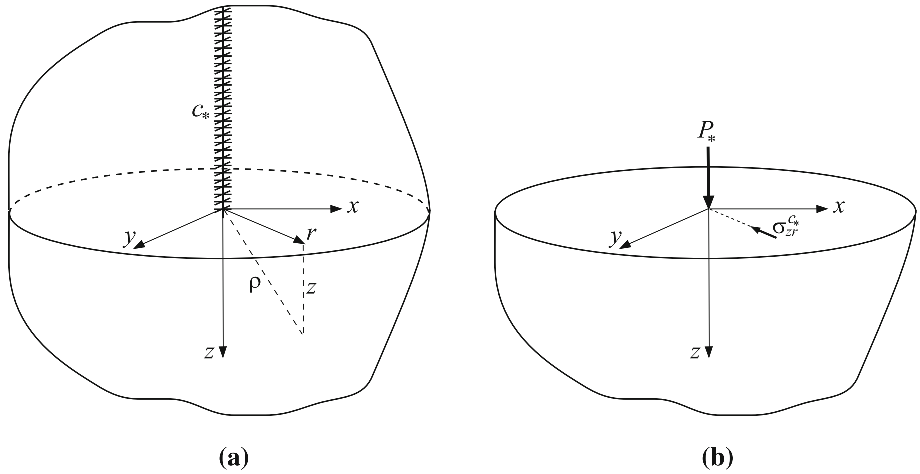


Fig. 5 **a** The line of centers of dilatation of intensity c_* (per unit length), uniformly distributed in an infinitely extended solid along the z axis from $z = -\infty$ to $z = 0$. **b** The lower portion of the solid from part (**a**). The boundary $z = 0$ is under the concentrated force P_*, specified by (6.26), and the shear stress distribution $\sigma_{zr}^{c_*}$, specified by (6.27)

i.e., upon introducing the integration variable $w = z - \zeta$,

$$\Omega = c_* \int_{z-a}^{z+b} \ln\left[\sqrt{r^2+w^2}+w\right] \mathrm{d}w = c_* \left[w \ln\left(\sqrt{r^2+w^2}+w\right) - \sqrt{r^2+w^2}\right]_{z-a}^{z+b} . \tag{6.20}$$

This gives

$$\begin{aligned} \frac{\Omega}{c_*} &= z \ln \frac{\sqrt{r^2+(z+b)^2}+(z+b)}{r} + z \ln \frac{\sqrt{r^2+(z-a)^2}-(z-a)}{r} \\ &+ b \ln\left[\sqrt{r^2+(z+b)^2}+(z+b)\right] + a \ln\left[\sqrt{r^2+(z-a)^2}+(z-a)\right] \\ &+ \sqrt{r^2+(z-a)^2} - \sqrt{r^2+(z+b)^2} . \end{aligned} \tag{6.21}$$

If $a = 0$ and $b \to \infty$, we obtain the line of centers of dilatations along the negative z axis (Fig. 5a), and, apart from an immaterial constant term, (6.21) simplifies to [1,25]

$$\Omega = c_* \left[\rho - z \ln(\rho + z)\right] , \quad \rho = \sqrt{r^2+z^2} . \tag{6.22}$$

Note that this form of Ω is harmonic, i.e., $\nabla^2 \Omega = 0$. The corresponding stresses are, from (3.6),

$$\begin{aligned} &\sigma_{rr} = c_* \left(\frac{z}{\rho^3} + \frac{z}{r^2\rho} - \frac{1}{r^2} \right) , \quad \sigma_{\theta\theta} = c_* \left(-\frac{z}{r^2\rho} + \frac{1}{r^2} \right) , \\ &\sigma_{zz} = -c_* \frac{z}{\rho^3} , \quad \sigma_{zr} = -c_* \frac{r}{\rho^3} . \end{aligned} \tag{6.23}$$

As expected the stress state is purely deviatoric, because $\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = 0$.

The presented solution for this line of continuously distributed centers of dilatation will be used in Sect. 7.2 to derive the solution to the Boussinesq problem of the concentrated force orthogonal to the boundary of a half-space, and in Sect. 8 for the Mindlin problem of a vertical concentrated force in the interior of a half-space. By substituting expression (6.22) for Love's potential Ω into (4.9), we obtain the Boussinesq's potentials for the line of dislocation centers along the negative z axis,

$$B = 0 , \quad \beta = -c_* \ln(\rho + z) , \quad A = \beta . \tag{6.24}$$

From (6.23), it follows that $\sigma(r \neq 0, z = 0) = 0$, and

$$\int_0^\infty \sigma_{zz}(r, z \geq 0) 2\pi r \, \mathrm{d}r = -2\pi c_* z \int_0^\infty \frac{r \, \mathrm{d}r}{(r^2+z^2)^{3/2}} = -2\pi c_* . \tag{6.25}$$

Thus, if we consider a free-body diagram of the lower half ($z \geq 0$) of the entire space with the line of centers of dilatation along the negative z axis, the boundary of the so-obtained half-space ($z = 0$, Fig. 5b) is under a compressive point force of magnitude

$$P_* = 2\pi c_* \,, \tag{6.26}$$

and the shear stress distribution is

$$\sigma_{zr}(r, z = 0) = -c_* \frac{1}{r^2} \,. \tag{6.27}$$

Expressions (6.26) and (6.27) will be used in Sect. 7.2 to derive the solution to the Boussinesq problem of a vertical concentrated force at the boundary of a half-space by an appropriate superposition method.

If $a \to \infty$ and $b \to \infty$, we obtain the line of centers of dilatations all along the entire z axis. In this case, (6.21) gives

$$\Omega = -2c_* z \ln r \,. \tag{6.28}$$

The corresponding stresses are

$$\sigma_{rr} = -\frac{2c_*}{r^2} \,, \quad \sigma_{\theta\theta} = \frac{2c_*}{r^2} \,, \quad \sigma_{zz} = \sigma_{zr} = 0 \,, \tag{6.29}$$

while the displacements are $u_r = (c_*/\mu)r^{-1}$ and $u_z = 0$.

It turns out that the solution for a pressurized circular cylindrical hole in an infinite medium (or in a half-space, because $\sigma_{zz} = \sigma_{zr} = 0$) can be deduced from the derived solution for an infinitely long line of centers of dilatation by replacing the constant $2c_*$ with another constant, say k. This constant can be determined from the boundary condition $\sigma_{rr}(r = a) = -p$, where p is the applied pressure over the surface of the hole of radius a. This gives $k = pa^2$, and thus, $\sigma_{rr} = -pa^2/r^2$. The corresponding hoop stress is tensile and equal to $\sigma_{\theta\theta} = pa^2/r^2$. Both stresses vanish as $r \to \infty$.

7 Boussinesq problem: vertical concentrated force at the boundary of a half-space

Figure 6 shows a half-space $z \geq 0$ under the concentrated compressive force P orthogonal to the free surface $z = 0$. This problem is of engineering importance in the analysis of contact and foundation problems [26–30]. The determination of the corresponding stress and displacement fields was originally made by Boussinesq [31]. The problem with the concentrated force tangential to the boundary of a half-space was solved by Cerruti [32], but since this is not an axially symmetric problem, it is not discussed in this review (see, for example, [18]).

Love's potential function for the three-dimensional Boussinesq problem is assumed, by inspection, to be of the form

$$\Omega = c_1\rho + c_2 z \ln(\rho + z) \,, \tag{7.1}$$

where c_1 and c_2 are constants to be determined. This form of Ω is biharmonic, because $\nabla^2\Omega$ is harmonic, i.e.,

$$\nabla^2\Omega = 2(c_1 + c_2)\frac{1}{\rho} \,, \quad \nabla^4\Omega = 0 \,. \tag{7.2}$$

By substituting (7.1) into (3.6), the stress components are found to be

$$\begin{aligned}
\sigma_{rr} &= [(1-2\nu)c_1 - 2\nu c_2]\frac{z}{\rho^3} - 3(c_1+c_2)\frac{r^2 z}{\rho^5} + c_2\frac{\rho - z}{r^2\rho} \,, \\
\sigma_{\theta\theta} &= (1-2\nu)(c_1+c_2)\frac{z}{\rho^3} - c_2\frac{\rho - z}{r^2\rho} \,, \\
\sigma_{zz} &= -[(1-2\nu)c_1 - 2\nu c_2]\frac{z}{\rho^3} - 3(c_1+c_2)\frac{z^3}{\rho^5} \,, \\
\sigma_{zr} &= -[(1-2\nu)c_1 - 2\nu c_2]\frac{r}{\rho^3} - 3(c_1+c_2)\frac{rz^2}{\rho^5} \,.
\end{aligned} \tag{7.3}$$

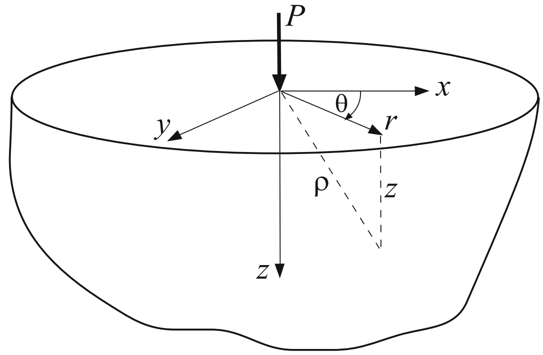


Fig. 6 A compressive concentrated force P orthogonal to the boundary $z = 0$ of a half-space

To determine the constants c_1 and c_2, the conditions are imposed that the shear stress σ_{zr} over the boundary of the half-space ($z = 0$, $\rho = r$) must vanish and that the integral of the normal stress σ_{zz} must be equal to $-P$. The normal stress σ_{zz} in (7.3) clearly vanishes for $z = 0$, except at the point of the application of the force, where it is singular.

The vanishing shear stress condition gives

$$(1-2\nu)c_1 - 2\nu c_2 = 0\,. \tag{7.4}$$

The integral condition for the normal stress is

$$\int_0^\infty \sigma_{zz}(r, z \geq 0) 2\pi r \, \mathrm{d}r = -P\,. \tag{7.5}$$

Upon substituting the expression for σ_{zz} from (7.3) into (7.5) and integrating, we obtain

$$2(1-\nu)c_1 + (1-2\nu)c_2 = \frac{P}{2\pi}\,. \tag{7.6}$$

By solving (7.4) and (7.6) for c_1 and c_2, it follows that

$$c_1 = \nu\,\frac{P}{\pi}\,, \quad c_2 = (1-2\nu)\,\frac{P}{2\pi}\,, \quad c_1 + c_2 = \frac{P}{2\pi}\,. \tag{7.7}$$

With the so-determined constants c_1 and c_2, the potential function Ω in (7.1) becomes

$$\Omega = \frac{P}{2\pi}\left[2\nu\rho + (1-2\nu)z\ln(\rho+z)\right]. \tag{7.8}$$

The corresponding stress components, from (3.6), are (e.g., [1,2])

$$\begin{aligned}
&\sigma_{rr} = \frac{P}{2\pi}\left[(1-2\nu)\,\frac{\rho-z}{r^2\rho} - 3\,\frac{r^2 z}{\rho^5}\right], \quad \sigma_{\theta\theta} = \frac{P}{2\pi}\,(1-2\nu)\left(\frac{z}{\rho^3} - \frac{\rho-z}{r^2\rho}\right),\\
&\sigma_{zz} = -\frac{3P}{2\pi}\,\frac{z^3}{\rho^5}\,, \quad \sigma_{zr} = -\frac{3P}{2\pi}\,\frac{rz^2}{\rho^5}\,.
\end{aligned} \tag{7.9}$$

The displacement components are obtained by substituting (7.8) into (3.4). The results are

$$u_r = \frac{P}{4\pi\mu}\left[\frac{rz}{\rho^3} - (1-2\nu)\,\frac{\rho-z}{r\rho}\right], \quad u_z = \frac{P}{4\pi\mu}\left[2(1-\nu)\,\frac{1}{\rho} + \frac{z^2}{\rho^3}\right]. \tag{7.10}$$

The two-dimensional version of the problem was solved by Flamant [33]. The plane strain version of Flamant's solution can be deduced by integration using Boussinesq's solution as the Green's function, while the plane stress solution follows from the plane strain solution by the usual change of elastic constants $\nu \to \nu/(1+\nu)$ and $\mu \to \mu$.

7.1 Solution by using Boussinesq's potentials

Having the expression (7.8) for Ω, the expressions for Boussinesq's potentials B and β can be deduced directly from (4.9), but it is appealing to establish them independently of Ω. To do so, one could proceed by assuming from the outset that the harmonic functions B and β are of the form

$$B = \frac{k_1}{\rho}\,, \quad \beta = k_2 \ln(\rho + z)\,, \tag{7.11}$$

and then determine the constants k_1 and k_2 from the boundary and the integral equilibrium conditions

$$\sigma_{zr}(r, z = 0) = 0\,, \quad \int_0^\infty \sigma_{zz}(r, z \geq 0) 2\pi r \, \mathrm{d}r = -P\,. \tag{7.12}$$

This would give

$$k_1 = \frac{2(1-\nu)P}{\pi}\,, \quad k_2 = \frac{(1-2\nu)P}{2\pi}\,. \tag{7.13}$$

However, we choose to take an alternative route, by using the results from Sect. 4.2. Since the boundary $z = 0$ is free from shear traction, $\sigma_{zr}(r, z = 0) = 0$, we adopt the assumption (4.12), i.e.,

$$B = \frac{4(1-\nu)}{1-2\nu} \frac{\partial \beta}{\partial z}\,, \tag{7.14}$$

and express all displacement and stress components in terms of the single harmonic function $\beta = \beta(r, z)$. If this function is taken to be

$$\beta = k \ln(\rho + z)\,, \tag{7.15}$$

the normal stress σ_{zz}, from (4.16), becomes

$$\sigma_{zz} = -\frac{3k}{1-2\nu} \frac{z^3}{\rho^5}\,. \tag{7.16}$$

By substituting (7.16) into the integral equilibrium condition in (7.12), it follows that

$$k = \frac{(1-2\nu)P}{2\pi}\,, \tag{7.17}$$

which completes the solution. See also the derivations in [34,35].

7.2 Boussinesq problem as superposition of Kelvin problem and a line of CD

Another approach to derive the solution to the Boussinesq problem is to use the superposition of the solutions to the Kelvin problem and the problem of a line of centers of dilatation (CD) in an infinite medium (e.g., [1,25]). In this Section, we review this derivation by taking a slightly different path from that followed in [1,25].

Figure 7a shows an infinite medium with the concentrated force Q at the coordinate origin. The free-body diagram of the lower half of this infinite medium is shown in Fig. 7b. The plane $z = 0$ is under the concentrated force $Q/2$ and the shear stress distribution

$$\sigma_{zr}^{Q}(r, z = 0) = -\frac{(1-2\nu)Q}{8\pi(1-\nu)} \frac{1}{r^2}\,. \tag{7.18}$$

This follows from the Kelvin problem analysis in Sect. 5 and expression (5.6).

Figure 7c shows an infinite space under uniformly distributed centers of dilatation of intensity q_* along the negative z axis. The free-body diagram of the lower half of this space ($z \geq 0$) is shown in Fig. 7d. The boundary $z = 0$ is under the concentrated force Q_* and the shear stress distribution $\sigma_{zr}^{q_*}$, given by

$$Q_* = 2\pi q_*\,, \quad \sigma_{zr}^{q_*}(r, z = 0) = -q_* \frac{1}{r^2}\,, \tag{7.19}$$

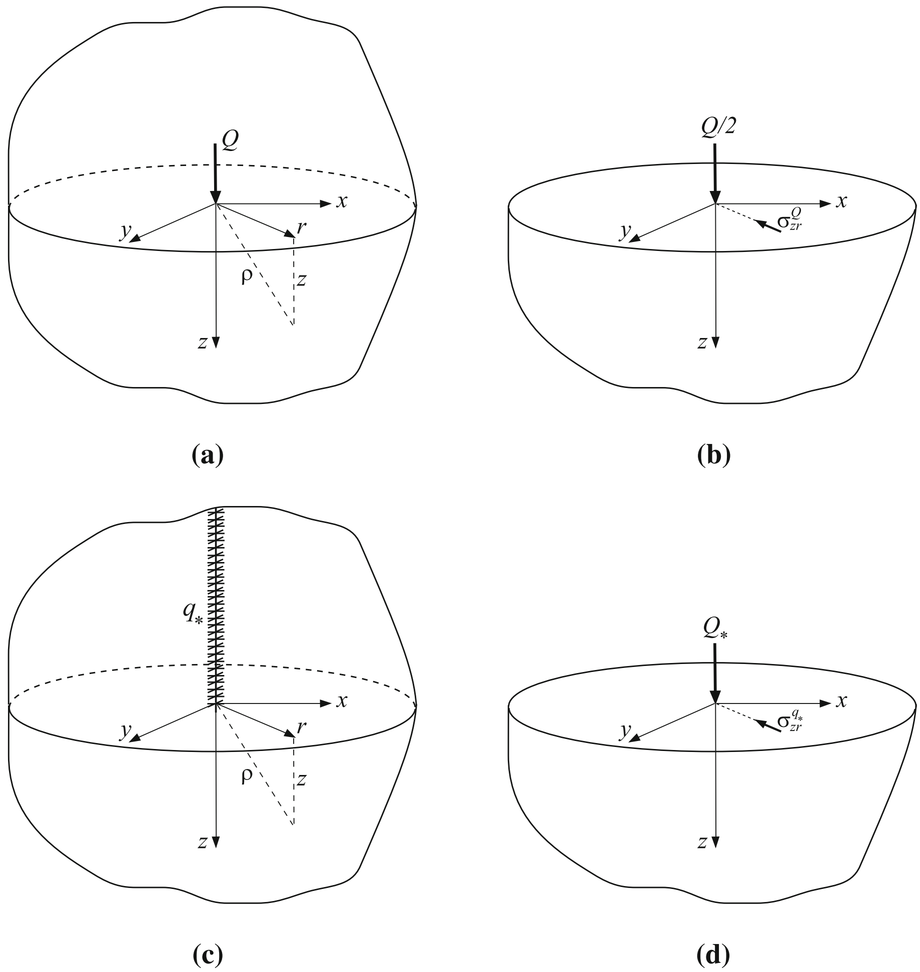


Fig. 7 **a** The concentrated force Q in the interior of an infinitely extended solid. **b** The lower portion of the solid from part (**a**). The boundary $z = 0$ is under the concentrated force $Q/2$ and the shear stress distribution $\sigma_{zr}^{Q} \sim 1/r^2$. **c** The line of centers of dilatation of intensity q_* (per unit length), uniformly distributed in an infinitely extended solid along the z axis from $z = -\infty$ to $z = 0$. **d** A lower portion of the infinite solid from part (**c**). The boundary $z = 0$ is under the concentrated force $Q_* = 2\pi q_*$ and the shear stress distribution $\sigma_{zr}^{q_*}$ specified by (7.19). The sum of problems in parts (**b**, **d**) provides the solution to the Boussinesq problem with the concentrated force P, provided that $Q_* + Q/2 = P$ and $\sigma_{zr}^{Q}(r, 0) + \sigma_{zr}^{q_*}(r, 0) = 0$

as shown in Sect. 6.4; see expressions (6.26) and (6.27).

To solve the Boussinesq problem, we determine Z and Q in such a way that the total shear stress over the plane $z = 0$ vanishes, i.e.,

$$\sigma_{zr}(r, 0) = \sigma_{zr}^{Q}(r, 0) + \sigma_{zr}^{q_*}(r, 0) = 0 \quad \Rightarrow \quad \frac{(1-2\nu)Q}{8\pi(1-\nu)} + q_* = 0\,, \tag{7.20}$$

and that the total concentrated force over the boundary ($z = 0$) of the half-space is equal to P, i.e.,

$$\frac{Q}{2} + Q_* = P\,, \quad Q_* = 2\pi q_*\,. \tag{7.21}$$

By solving the algebraic equations (7.20) and (7.21) for Q and q_*, we obtain

$$Q = 4(1-\nu)P\,, \quad q_* = -\frac{(1-2\nu)P}{2\pi}\,. \tag{7.22}$$

The value of q_* is negative, which means that the dilatation centers are actually the centers of compression. It also follows that $Q_* = -(1-2\nu)P$.

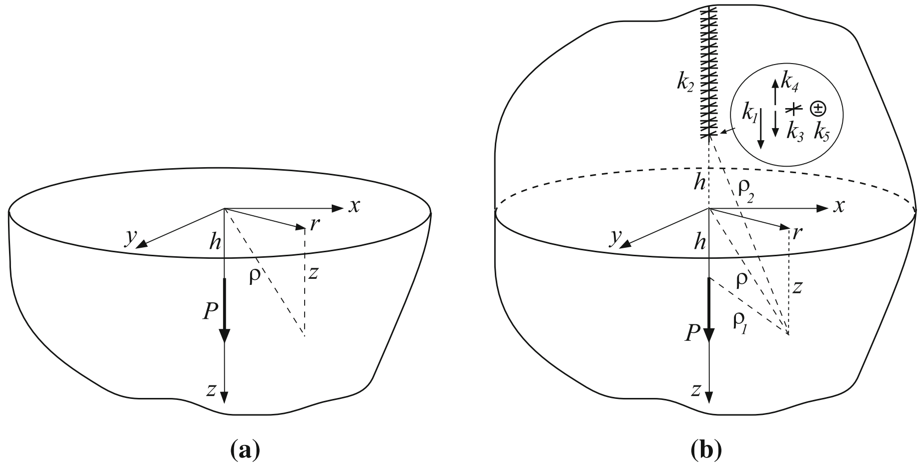


Fig. 8 **a** The vertical concentrated force P applied in the interior of a semi-infinite solid ($z \geq 0$), at a distance h beneath the free surface $z = 0$. **b** The potential function for the problem in part (**a**) can be obtained as the sum of the potential function for the force P at a point $(0, h)$ in an infinite solid, the potential function corresponding to the line of centers of compression of an appropriate intensity extending along the negative z axis from the image point $(0, -h)$ to negative infinity, and the potential functions corresponding to four additional sources of strain located at the image point $(0, -h)$: the concentrated force, the tensile doublet of forces, the center of compression, and the compression/dilatation doublet. The strengths of these nuclei of strain are specified in the text

8 Mindlin problem: force at a point in the interior of a half-space

Figure 8a shows a half-space loaded in is interior by the concentrated force P orthogonal to the traction-free surface $z = 0$, at a distance h below it. The stress and displacement fields for this problem were originally determined by Mindlin [36]. They are of importance because they represent Green's functions for the stress and displacement fields in semi-infinite media under other types of loadings [21]. To satisfy the traction-free boundary condition at $z = 0$ and the condition of vanishing stresses at $z \to \infty$, Mindlin recognized by inspection that the potential Ω for the problem in Fig. 8a can be obtained as the sum of Love's potential ($c\rho_1$) for the force P at point $(0, h)$ in an infinite medium (Kelvin problem), Love's potential corresponding to the line of centers of compression of an appropriate strength (k_2) extending along the negative z axis from the image point $(0, -h)$ to negative infinity, and the potential functions corresponding to four additional sources (nuclei) of strain in the infinite medium, located at the same image point (Fig. 8b): a concentrated force (whose potential is $k_1\rho_2$), a center of compression of strength k_3h, a tensile doublet of forces of strength k_4h, and a compression/dilatation doublet of strength k_5h^2 (see Sect. 6), where the constants k_1 to k_5 are appropriately determined, as described below. Thus, it is assumed that

$$\begin{aligned}\Omega &= k_0\rho_1 + k_1\rho_2 - k_2[\rho_2 - (z+h)\ln(\rho_2 + z + h)] - k_3h\ln(\rho_2 + z + h)\\ &\quad - k_4h\,\frac{z+h}{\rho_2} + k_5h^2\,\frac{1}{\rho_2}\,, \quad k_0 = c = \frac{P}{8\pi(1-\nu)}\end{aligned} \tag{8.1}$$

where

$$\rho_1^2 = r^2 + (z-h)^2\,, \quad \rho_2^2 = r^2 + (z+h)^2\,. \tag{8.2}$$

The constant $k_0 = c$ is as specified in (8.1), because in the limit $h \to \infty$ the potential function in (8.1) must reduce to the potential function of the Kelvin problem ($\Omega = c\rho_1$). The potential function $\Omega = c\rho_1$ can also be interpreted as a particular solution to the partial differential equation $\nabla^4\Omega = -P\delta(z-h)/(1-\nu)$, in which the body force is taken to be $b = P\delta(z-h)$.

To determine the constants k_1 to k_5, we choose a somewhat shorter path than that used by Mindlin [36], and specify the constants k_1 and k_2 first, by requiring that (8.1) reduces to Love's potential function of the Boussinesq problem in the limit $h \to 0$, i.e.,

$$c\rho + k_1\rho + k_2[z\ln(\rho_2 + z) - \rho] = \frac{P}{2\pi}\,[2\nu\rho + (1-2\nu)z\ln(\rho + z)]\,. \tag{8.3}$$

This gives

$$k_1 = (3-4\nu)c\,, \quad k_2 = 4(1-\nu)(1-2\nu)c\,. \tag{8.4}$$

The remaining constants k_3, k_4, and k_5 are then determined from the conditions that the boundary of the half-space is traction-free,

$$\sigma_{zr}(r, z=0) = 0\,, \quad \sigma_{zz}(r, z=0) = 0\,, \tag{8.5}$$

which yields the needed algebraic equations for k_3, k_4, and k_5, as shown below. The integral condition of equilibrium,

$$\int_0^\infty \sigma_{zz}(r, z>h) 2\pi r\, \mathrm{d}r = -P\,, \tag{8.6}$$

is identically satisfied, because we have already imposed the condition that the problem reduces to the Boussinesq problem in the limit $h \to 0$.

By using the stress expressions from Sect. 6, the traction components over the plane $z=0$, corresponding to (8.1), are obtained from

$$\begin{aligned} h^2\sigma_{zr} &= -(k_0+k_1)\bar{r}\left[(1-2\nu)\frac{1}{\bar{\rho}_0^3} + 3\frac{1}{\bar{\rho}_0^5}\right] + k_2\bar{r}\frac{1}{\bar{\rho}_0^3} + 3k_3\bar{r}\frac{1}{\bar{\rho}_0^5} \\ &\quad + 3k_4\bar{r}\left[(1+2\nu)\frac{1}{\bar{\rho}_0^5} - 5\frac{1}{\bar{\rho}_0^7}\right] - 3k_5\bar{r}\left(\frac{1}{\bar{\rho}_0^5} - 5\frac{1}{\bar{\rho}_0^7}\right), \end{aligned} \tag{8.7}$$

$$\begin{aligned} h^2\sigma_{zz} &= (k_0-k_1)\left[(1-2\nu)\frac{1}{\bar{\rho}_0^3} + 3\frac{1}{\bar{\rho}_0^5}\right] + k_2\frac{1}{\bar{\rho}_0^3} - k_3\left(\frac{1}{\bar{\rho}_0^3} - 3\frac{1}{\bar{\rho}_0^5}\right) \\ &\quad + k_4\left[(1-2\nu)\frac{1}{\bar{\rho}_0^3} + 6(1+\nu)\frac{1}{\bar{\rho}_0^5} - 15\frac{1}{\bar{\rho}_0^7}\right] - 3k_5\left(3\frac{1}{\bar{\rho}_0^5} - 5\frac{1}{\bar{\rho}_0^7}\right) \end{aligned} \tag{8.8}$$

where $\bar{r} = r/h$ and $\bar{\rho}_0^2 = 1 + \bar{r}^2$ are conveniently introduced non-dimensional quantities. By using (8.7) and (8.8) to cast the conditions (8.5), and upon making the coefficients in front of the terms of the order of $1/\bar{\rho}_0^3$, $1/\bar{\rho}_0^5$, and $1/\bar{\rho}_0^7$ equal to zero, the following equations are obtained:

$$\begin{aligned} (1-2\nu)k_1 - k_2 &= -(1-2\nu)c\,, \\ k_1 - k_3 - (1+2\nu)k_4 + k_5 &= -c\,, \\ (1-2\nu)k_1 - k_2 + k_3 - (1-2\nu)k_4 &= (1-2\nu)c\,, \\ k_1 - k_3 - 2(1+\nu)k_4 + 3k_5 &= c\,, \\ k_4 - k_5 &= 0\,. \end{aligned} \tag{8.9}$$

These are five linear algebraic equations for the five unknowns k_1 to k_5. When expressions (8.4) for k_1 and k_2 are substituted into (8.9), these equations reduce to $k_5 = k_4$, and

$$\begin{aligned} k_3 + 2\nu k_4 &= 4(1-\nu)c\,, \\ k_3 - (1-2\nu)k_4 &= 2(1-2\nu)c\,. \end{aligned} \tag{8.10}$$

Solving the system of algebraic equations (8.10), we find the remaining unknown constants to be

$$k_3 = 4(1-2\nu)c\,, \quad k_4 = 2c\,, \quad k_5 = 2c\,. \tag{8.11}$$

Consequently, Love's potential (8.1) becomes

$$\Omega = c\left\{\rho_1 + [8\nu(1-\nu)-1]\rho_2 + 4(1-2\nu)[(1-\nu)z - \nu h]\ln(\rho_2+z+h) - 2h\frac{z}{\rho_2}\right\}, \tag{8.12}$$

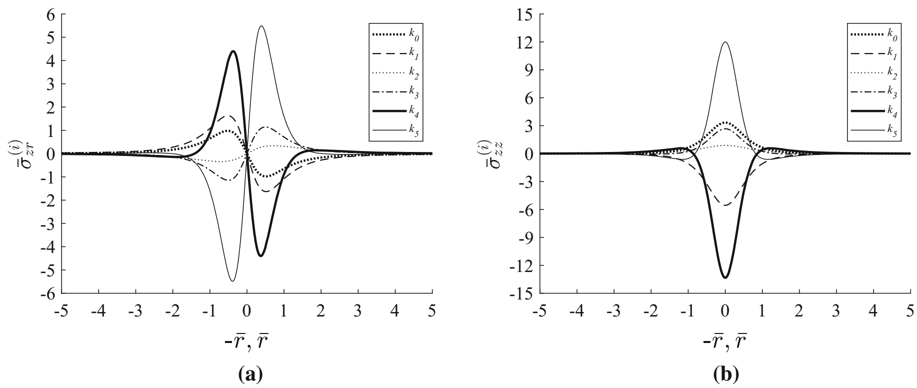


Fig. 9 The variation of the **a** normalized shear stress $\bar{\sigma}_{zr}^{(i)}(r, 0)$ and **b** normalized normal stress $\bar{\sigma}_{zz}^{(i)}(r, 0)$ with $\bar{r} = r/h$ along the plane $z = 0$ for each of the nuclei of strain ($i = 0, 1, 2, \ldots, 5$) used in (8.1). The stresses are normalized by c/h^2. The utilized value of Poisson's ratio is $\nu = 1/3$. The sums of the six shown shear stresses and the six shown normal stresses identically vanish (traction-free boundary $z = 0$)

which is Mindlin's [36] expression (8). The complete stress and displacement expressions are also listed in Mindlin's paper. Their representations in the coordinate system with the origin at the image point, at a distance h above the free surface, are listed in [27].

In retrospect, it may be noted that Mindlin's construction of the superposition of the nuclei of strain in (8.1) was guided by the physical intuition and by the observation that the traction components σ_{zr} and σ_{zz} over the plane $z = 0$ for each utilized nucleus of strain in (8.1) are linear combinations of some or all of the terms proportional to $1/\rho_0^3$, $1/\rho_0^5$ and $1/\rho_0^7$, where $\rho_0^2 = r^2 + h^2$ (see expressions (8.7) and (8.8), and Fig. 9). Thus, making the coefficients in front of these terms equal to zero in the expressions for $\sigma_{zr}(r, 0)$ and $\sigma_{zz}(r, 0)$ provides enough equations to determine the needed constants appearing in (8.1).

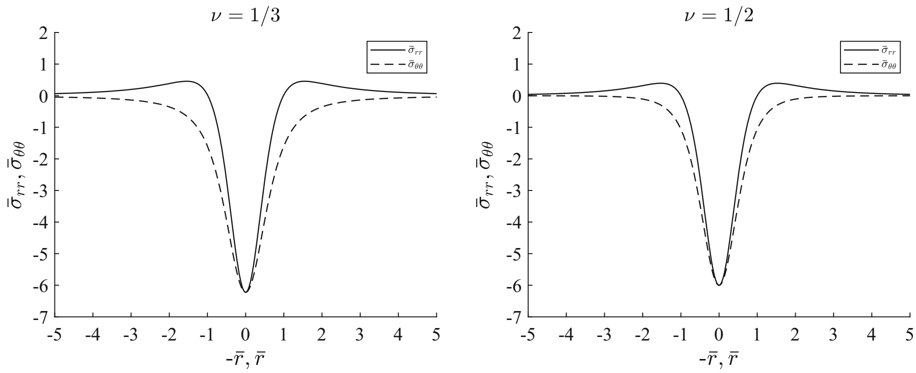
The two-dimensional version of Mindln's problem was solved by Melan [37]. As pointed out by Mindlin, the plane strain version of Melan's solution can be deduced by integration using Mindlin's solution as Green's function, while the plane stress solution follows from the plane strain solution by the usual change of elastic constants. Mindlin [36] also determined the displacement and stress fields in the case of the concentrated force parallel to the traction-free boundary of the half-space, thereby extending the Cerruti [32] solution for the concentrated force tangential to the boundary of a half-space.

Figure 9 shows the variation of the normalized shear and normal stress components $\bar{\sigma}_{zr}^{(i)}(r, 0)$ and $\bar{\sigma}_{zz}^{(i)}(r, 0)$ with $\bar{r} = r/h$ along the plane $z = 0$ for each of the nuclei of strain ($i = 0, 1, 2, \ldots, 5$) embedded in (8.1); their sums are equal to zero (traction-free boundary $z = 0$). The stresses are normalized by c/h^2, and the value of Poisson's ratio was taken to be $\nu = 1/3$. These plots are obtained by using expressions (8.7) and (8.8). Figure 10 shows the variation of the corresponding normalized normal stresses $\bar{\sigma}_{rr}(r, 0)$ and $\bar{\sigma}_{\theta\theta}(r, 0)$ along the traction-free boundary in the case $\nu = 1/3$ and $\nu = 1/2$. The latter plots are obtained from the following expressions:

$$\begin{aligned} h^2\sigma_{rr} = {} & (k_1 - k_0)\left[(1-2\nu)\,\frac{1}{\bar{\rho}_0^3} - 3\,\frac{\bar{r}^2}{\bar{\rho}_0^5}\right] - k_2\left(\frac{1}{\bar{\rho}_0^3} + \frac{1}{\bar{r}^2\bar{\rho}_0} - \frac{1}{\bar{r}^2}\right) - k_3\left(\frac{1}{\bar{\rho}_0^3} - 3\,\frac{\bar{r}^2}{\bar{\rho}_0^5}\right) \\ & + k_4\left[(1+\nu)\,\frac{2}{\bar{\rho}_0^3} - 6\nu\,\frac{1}{\bar{\rho}_0^5} - 15\,\frac{\bar{r}^2}{\bar{\rho}_0^7}\right] - 3k_5\left(\frac{1}{\bar{\rho}_0^5} - 5\,\frac{\bar{r}^2}{\bar{\rho}_0^7}\right), \end{aligned} \tag{8.13}$$

$$\begin{aligned} h^2\sigma_{\theta\theta} = {} & (k_1 - k_0)(1-2\nu)\,\frac{1}{\bar{\rho}_0^3} - k_2\left(\frac{1}{\bar{r}^2} - \frac{1}{\bar{r}^2\bar{\rho}_0}\right) - k_3\,\frac{1}{\bar{\rho}_0^3} \\ & + k_4(1-2\nu)\left(\frac{2}{\bar{\rho}_0^3} - 3\,\frac{\bar{r}^2}{\bar{\rho}_0^5}\right) - 3k_5\,\frac{1}{\bar{\rho}_0^5}\,. \end{aligned} \tag{8.14}$$

For incompressible materials ($\nu = 1/2$), the constants are $k_1 = c$, $k_2 = k_3 = 0$, $k_4 = k_5 = 2c$. Thus, in this case the solution is obtained by the superposition of the infinite-medium solutions for the force P at the points $(0, \pm h)$, the doublet of tensile forces of strength $2ch$ at the image point $(0, -h)$, and the doublet of



(a) **(b)**

Fig. 10 The variations of the normalized radial stress $\bar{\sigma}_{rr}(r, 0)$ and the normalized hoop stress $\bar{\sigma}_{\theta\theta}(r, 0)$ with $\bar{r} = r/h$ along the traction-free plane $z = 0$, according to (8.13) and (8.14). The normalizing stress factor is c/h^2. The value of Poisson's ratio used in part (**a**) is $\nu = 1/3$, and in part (**b**) $\nu = 1/2$

compression/dilatation centers of strength $2ch^2$ at the image point $(0, -h)$. The potential function in (8.12) accordingly simplifies to

$$\Omega = c\left(\rho_1 + \rho_2 - \frac{2hz}{\rho_2}\right). \tag{8.15}$$

The corresponding normal stress σ_{zz} along the z axis ($r = 0$) is

$$\sigma_{zz}(0, z) = -3c\left[\frac{\text{sign}(z-h)}{(z-h)^2} + \frac{1}{(z+h)^2} + \frac{4hz}{(z+h)^4}\right]. \tag{8.16}$$

8.1 Solution of the Mindlin problem by using Boussinesq's potentials

If the concentrated force is treated as the body force, the governing equations for the Boussinesq's potentials are

$$\nabla^2 B = -2P\delta(z-h)\,, \quad \nabla^2\beta = \frac{P}{2(1-\nu)}\, z\delta(z-h)\,. \tag{8.17}$$

The solutions to these equations can be sought as the sum of their particular and complementary parts. The particular solutions to Poisson-type partial differential equations in (8.17) are the well-known fundamental solutions

$$B_{\text{p}} = 2P\,\frac{1}{4\pi\rho_1}\,, \quad \beta_{\text{p}} = -\frac{P}{2(1-\nu)}\,\frac{h}{4\pi\rho_1}\,. \tag{8.18}$$

The expressions in (8.18) can be physically interpreted as the Boussinesq's potentials for the Kelvin problem of a force in an infinite solid, expressed in the coordinate system with the origin at a distance h above the point of the application of the force; see (5.13).

The complementary solutions are constructed by using two basic harmonic functions (f and g), which give rise to stresses and displacements that vanish at infinity, i.e.,

$$\begin{aligned} &\nabla^2 B_{\text{h}} = 0: \quad B_{\text{h}} = c_1 g + c_2 h\,\frac{\partial g}{\partial z}\,, \quad g = \frac{1}{\rho_2} \equiv \frac{\partial f}{\partial z}\,, \\ &\nabla^2 \beta_{\text{h}} = 0: \quad \beta_{\text{h}} = c_3 f + c_4 h\,\frac{\partial f}{\partial z}\,, \quad f = \ln(\rho_2 + z + h)\,. \end{aligned} \tag{8.19}$$

Thus, the complete solutions of equations (8.17) are sought in the form

$$
\begin{aligned}
B &= \frac{P}{2\pi}\,\frac{1}{\rho_1} + c_1\,\frac{1}{\rho_2} - c_2 h\,\frac{z+h}{\rho_2^3}\,,\\
\beta &= -\frac{P}{8\pi(1-\nu)}\,\frac{h}{\rho_1} + c_3\ln(\rho_2+z+h) + c_4 h\,\frac{1}{\rho_2}\,,
\end{aligned}
\tag{8.20}
$$

provided that the constants c_1, c_2, c_3, and c_4 can be determined to satisfy the traction-free boundary conditions at $z = 0$.

By using expressions (4.6.1) for the stress components, we can write

$$
\begin{aligned}
\sigma_{zr} &= \frac{1-2\nu}{4(1-\nu)}\,\frac{\partial B}{\partial r} - \frac{\partial^2\beta}{\partial r\partial z} - \frac{z}{4(1-\nu)}\,\frac{\partial^2 B}{\partial r\partial z}\,,\\
\sigma_{zz} &= -\frac{\partial^2\beta}{\partial z^2} + \frac{1}{2}\,\frac{\partial B}{\partial z} - \frac{z}{4(1-\nu)}\,\frac{\partial^2 B}{\partial z^2}\,.
\end{aligned}
\tag{8.21}
$$

Upon substituting (8.20) into (8.21), the condition $\sigma_{zr}(r, z=0) = 0$ gives

$$
c_3 - \frac{1-2\nu}{4(1-\nu)}\,c_1 = \frac{(1-2\nu)P}{8\pi(1-\nu)}\,, \qquad c_4 - \frac{1-2\nu}{4(1-\nu)}\,c_2 = -\frac{P}{8\pi(1-\nu)}\,. \tag{8.22}
$$

Similarly, the boundary condition $\sigma_{zz}(r, z=0) = 0$ gives

$$
c_4 - \frac{1}{2}\,c_2 = \frac{P}{8\pi(1-\nu)}\,, \qquad c_3 - \frac{1}{2}\,c_1 = -\frac{P}{4\pi}\,. \tag{8.23}
$$

After solving the system of four algebraic equations (8.22) and (8.23), the required constants are found to be

$$
c_1 = \frac{(3-4\nu)P}{2\pi}\,, \quad c_2 = -\frac{P}{\pi}\,, \quad c_3 = \frac{2(1-2\nu)P}{4\pi}\,, \quad c_4 = -\frac{(3-4\nu)P}{8\pi(1-\nu)}\,. \tag{8.24}
$$

Thus, the Boussinesq's potentials (8.20) take the form

$$
\begin{aligned}
B &= \frac{P}{2\pi}\left[\frac{1}{\rho_1} + (3-4\nu)\,\frac{1}{\rho_2} + 2h\,\frac{z+h}{\rho_2^3}\right],\\
\beta &= -\frac{P}{8\pi(1-\nu)}\left[\frac{h}{\rho_1} + (3-4\nu)\,\frac{h}{\rho_2} - 4(1-\nu)(1-2\nu)\ln(\rho_2+z+h)\right].
\end{aligned}
\tag{8.25}
$$

These expressions are equivalent to Mindlin's [13] expressions (39) and (40), who derived them more rigorously by using the Papkovich–Neuber potentials and by exploring the boundary conditions in conjunction with Green's integral formula. In the limit $h \to 0$, expressions (8.16) reduce to (7.11) and (7.13) for the Boussinesq problem of the concentrated force at the boundary of a half-space. We also note that the constants c_1 to c_4 are related to constants k_1 to k_4, appearing in (8.1), by

$$
c_1 = 4(1-\nu)k_1\,, \quad c_2 = -4(1-\nu)k_4\,, \quad c_3 = k_2\,, \quad c_4 = k_1 + k_2 - k_3\,. \tag{8.26}
$$

Alternative derivations of (8.25) were constructed in [38,39]. The solution by using Fourier transforms was presented in [40].

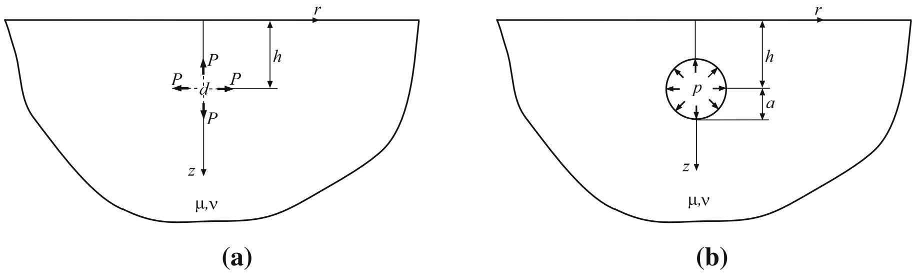


Fig. 11 **a** The center of dilatation of strength at a point $(0, h)$ beneath the free surface of a half-space, whose elastic constants are (μ, ν). **b** A pressurized spherical hole beneath the free surface of a half-space. The radius of the hole is a and its center is at the distance h from the free surface $z = 0$. The applied pressure is p

8.2 Center of dilatation at a point in the interior of half-space

The structures of Love's potential for a doublet of forces, a dilatation center, and other nuclei of strain beneath the free surface of a half-space were listed in [24]. For example, Love's potential for the center of dilatation of strength k_* at a point $(0, h)$ (Fig. 11a) is

$$\Omega = k_* \left[\ln(\rho_1 + z - h) + (1 - 4\nu)\ln(\rho_2 + z + h) + \frac{2z}{\rho_2} \right], \quad k_* = \frac{(1-2\nu)Pd}{4\pi(1-\nu)}. \tag{8.27}$$

The corresponding Boussinesq's potentials are

$$B = -8(1-\nu)k_* \frac{z+h}{\rho_2^3}, \quad \beta = k_* \left[\frac{1}{\rho_1} + (3-4\nu)\frac{1}{\rho_2} \right]. \tag{8.28}$$

The displacements and stresses associated with (8.27) or (8.28) follow from (4.4) and (4.6). For example, the displacement components are

$$\begin{aligned} u_r &= \frac{k_*}{2\mu} \left[\frac{r}{\rho_1^3} + (3-4\nu)\frac{r}{\rho_2^3} - 6\frac{rz(z+h)}{\rho_2^5} \right], \\ u_z &= \frac{k_*}{2\mu} \left[\frac{z-h}{\rho_1^3} - \frac{(1-4\nu)z + (3-4\nu)h}{\rho_2^3} - 6\frac{z(z+h)^2}{\rho_2^5} \right]. \end{aligned} \tag{8.29}$$

Expressions analogous to (8.29) were used in [41] to analyze the displacement and stress fields around a spherical inclusion beneath the free surface of a half-space; see their expression preceding Fig. 3 on page 932, and also [42]. The determination of the displacement and stress fields around a pressurized spherical hole beneath the free surface of the half-space (Fig. 11b), which is of importance for example in geomechanics (magma-chamber problem), requires the use of infinite series or numerical methods [43–45]]. The two-dimensional version of a pressurized cylindrical hole beneath the free surface was solved by Jefferey [46]. The stress and displacement fields around a cylindrical hole beneath the free surface of a half-space under the gravity load (the so-called tunnel problem) were originally derived by Mindlin [47], and have found since many important applications in rock mechanics and geomechanics [27–29].

9 Conclusions

We have reviewed in this paper the derivation of the solutions to three fundamental three-dimensional axisymmetric elasticity problems: the Kelvin problem of a concentrated force in an infinite space, the Boussinesq problem of a concentrated force orthogonal to the boundary of a half-space, and the Mindlin problem of a vertical concentrated force in the interior of a half-space. Two approaches were used in the derivations, one based on the Galerkin method and Love's potential function, and the other based on the Papkovich–Neuber displacement representation and Boussinesq's potential functions. Three different procedures were used to construct the solution to the Boussinesq problem, which are related to each other, yet are each conceptually appealing on their own. In the construction of the solution of the Mindlin problem, the solutions of the governing differential equations for the utilized potential functions are expressed as the sums of their particular and complementary parts. The particular parts follow from the solution of the Kelvin problem. The derivation of the complementary parts is simplified by imposing at the early stage of the analysis that the solution of the Mindlin problem must reduce to the solution of the Boussinesq problem in an appropriate limit. The introduced simplifications in the derivations may be appealing for the coverage of this important topic in an introductory graduate course of solid mechanics.

Acknowledgements Helpful comments and suggestions by two anonymous reviewers are gratefully acknowledged.

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