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Stress magnification due to stretching and bending of thin ligaments between voids

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Abstract Stress magnification in thin ligaments between small and large cylindrical voids is obtained by matching the inner field approximation by beam theory to the outer rigid-body field in the bulk of the material. A void between two larger voids is modeled as a large hole within a strip of straight edges (boundaries of the holes with infinite radii of curvature). Both stretching and bending types of loading are applied to the strip. Comparison of different orders of stress magnification for different geometries and loading conditions is made. It is shown that the order of stress magnification in thin ligaments is $(R/\delta)^n$, where $n = 1/2$ in the ligament between one small and one large void, $n = 1$ in the ligament between one small void and two large voids, or between two small and two large voids, and $n = 2$ in the ligament between a large void and a small void coalescing with another large void. The relevance of these results for the study of material failure by void growth and coalescence is discussed.

Keywords Beam theory · Matched expansion · Ligament · Notched strip · Strain energy · Stress magnification · Void coalescence

1 Introduction

Large amount of research has been devoted during past several decades to the analysis of material failure by ductile void growth and coalescence. Continuum elastoplastic theories, dislocation based models, and atomistic simulations were all used to address the problem. Representative recent references include [1–8]. In this paper we present an analytical determination of the maximum stress in thin ligaments or narrow regions between close cylindrical voids. We consider ligaments between a small void and two nearby large voids, and ligaments associated with two small voids positioned in-between two large voids (Fig. 1). In addition, we consider ligaments between a small void and a nearby large void, and the ligaments between a large void and a small void coalescing with another large void (Fig. 2). The analysis is based on modeling small voids as large holes in a rectangular strip whose straight edges are the boundaries of large outer voids of infinite radii of curvature. Both stretching and bending types of loading are applied to the bulk. Stress magnification is obtained by matching the inner field approximation, which is obtained by beam theory approximation within the ligament, to the outer rigid-body field in the bulk of the material, away from the holes. The utilized approach is in an extension of the early work by Koiter [9], who constructed an elementary solution for the stress magnification in the neighborhood of a large hole symmetrically positioned within a rectangular strip under remote tension. The

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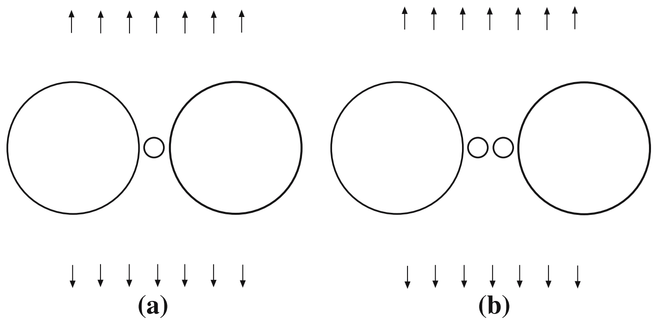

Fig. 1 **a** A small void in-between two large voids. **b** Two small voids in-between two large voids

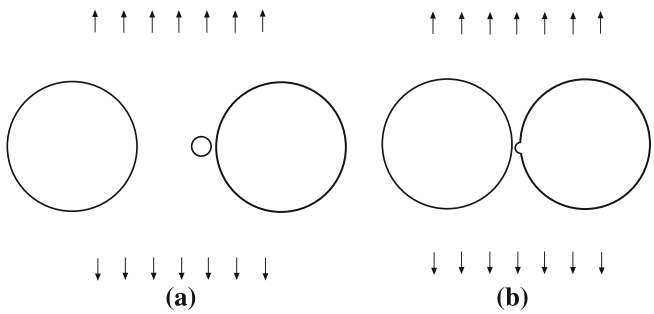

Fig. 2 **a** A small void nearby a large void. **b** A small void coalescing with one of two nearby large voids

thin ligament around the hole is modeled as a tapered beam and the integration of the corresponding fourth-order differential equation was used for the deflection of the beam. Koiter found that the maximum stress, at the root of the ligament, is $\sigma_{\max} = 2\sigma(R/\delta + 1)$, where σ is the applied stress, $\delta \ll R$ is the thickness of the ligament in the vertical plane of symmetry, and R is the radius of the hole. Since the maximum stress around the small hole in the middle of a wide strip is $\sigma_{\max} = 3\sigma$, the stress magnification factor for the large hole is of the order of R/δ. Keller [10] considered this as a problem of matched inner and outer expansions, the inner expansion being obtained by the beam theory. An integral equation method was previously used by Duan et al. [11]. Markenscoff and Dundurs [12] analyzed several cases of loading and geometries of thin ligaments, and showed that indeed the stresses obtained by the beam theory match those from the elasticity solutions to the leading order terms. Markenscoff [13] furthermore extended Koiter's analysis to an eccentric large hole under remote tension by considering, in addition to equilibrium conditions, the compatibility of the deformation due to stretching and rotation of the ligament. She found that the stress magnification factors are of the order of R/δ_k $(k = 1, 2)$, same as for the central hole.

Both Koiter's and Markenscoff's analyses were based on the integration of the differential equation of the beam. In this paper we present an alternative, but simpler approach, based on an energy analysis and the application of Castigliano's theorem. We consider not only the stretching, but also the bending of the strips that are weakened by large semicircular notches and centric or eccentric circular holes. We match the deformation of the beam to the deformation of the bulk, which we take to be a rigid body translation or rotation. The analysis delivers the stress magnification factors, as well as the geometric parameters describing the overall deformation (stretching and relative rotation of the ends of the strip). In particular, we show that the stress magnification in notched strips is of the order $(R/\delta)^2$, while in the strips weakened by large holes it is of the order R/δ. This is compared to the stress magnification of the order $(R/\delta)^{1/2}$ in the ligament near the free edge of the stretched half-plane. Furthermore, we derive a simple relationship between the maximum stresses in the outer and inner ligaments for the stretched strip weakened by two symmetrically positioned large holes. The results are also given for the bent strip weakened by two large circular holes. The derivation delivers a simple relationship between the bending moment and the axial force in the thinnest section of the ligament, $M = N\delta/6$. This relationship coincides with the one obtained by integrating the stresses in the exact elasticity solution for a hole near the edge of the half plane under tension [14], in the limit of the thin ligament between the hole and the free surface [12]. Since voids can be of very different shapes, and since the corresponding stress magnification factors depend not only on the ratio of the radius of the void to the thickness of the ligament

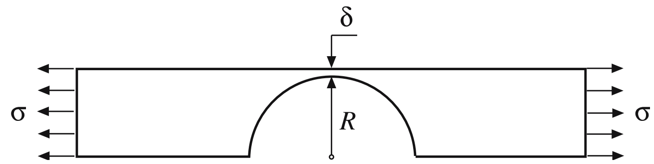

Fig. 3 A notched strip under uniform tension σ. The radius of a semicircular notch is R, and the minimum thickness of the ligament is δ

but also on the void geometry, our analysis in those cases can be applied as a first-order asymptotic analysis in which any smooth contour of the void is approximated by its osculating circle in the plane of ligament's symmetry.

The analysis and the results presented in the present paper are of interest, because an alternative numerical treatment of the considered problems, such as the one based on the FEM, requires a sensitive mesh refinement in thin regions of ligaments to capture the large stresses and their large gradients there, and does not deliver explicitly the simple expressions for the stress magnification factors in terms of the geometric parameters of the problem, nor the simple analytical relationships between the maximum stresses in different ligaments for the strips weakened by multiple holes. The obtained results allow the comparison of the severity of different loading/geometry configurations with respect to each other, which enable better understanding of the process leading to material failure. Also, large stresses in thin ligaments facilitate dislocation emission from the surface of the voids and their subsequent growth and coalescence [3].

The obtained results provide a quantitative framework for the understanding of the severity of stress in thin ligaments that will lead to material failure by void growth and coalescence.

2 Stresses and deformation in a notched strip

A long plate (strip) weakened by a large semicircular notch under remote uniform tension is shown in Fig. 3. The thickness of thin ligament in the plane of symmetry (the minimum ligament's thickness) is $\delta \ll R$, where R is the radius of the notch. If the notch is not semi-circular, but of smooth profile, the radius R can be interpreted as the local radius of the curvature in the plane of symmetry. The present analysis is thus a first-order asymptotic approach in which any smooth contour of the notch is approximated by its osculating circle in the plane of ligament's symmetry. For a sufficiently small ratio δ/R, the deformation of the strip is much greater in the thin region around the notch (ligament) than outside of it, and we shall accordingly consider this deformation only, the remote bulk of the strip moving approximately only as a rigid body. As in [9], we model the ligament as a tapered beam of variable thickness, approximated by the first two terms of the polynomial expansion as

$$t(x) = \delta + \frac{x^2}{2R}\,. \tag{1}$$

The axial force and bending moments (per unit thickness of the strip) at the two ends of an extracted segment of the ligament are shown in Fig. 4. By the moment equilibrium, they are related by

$$M(x) = M - N\frac{x^2}{4R}\,, \quad M = M(0). \tag{2}$$

The problem of a notched strip is statically determined with respect to M and N, since by the overall equilibrium of each half of the notched strip, we must have

$$N = \sigma(R+\delta)\,, \quad M = \frac{1}{2}NR\,. \tag{3}$$

The maximum tensile and compressive stresses occur at the inner and outer points of the thinnest section of the ligament, and are respectively given by

$$\begin{aligned}\sigma_{\mathrm{i}} &= \frac{N}{\delta} + \frac{6M}{\delta^2} = \sigma\left(3\frac{R^2}{\delta^2} + 4\frac{R}{\delta} + 1\right),\\ \sigma_{\mathrm{o}} &= \frac{N}{\delta} - \frac{6M}{\delta^2} = -\sigma\left(3\frac{R^2}{\delta^2} + 2\frac{R}{\delta} - 1\right).\end{aligned} \tag{4}$$

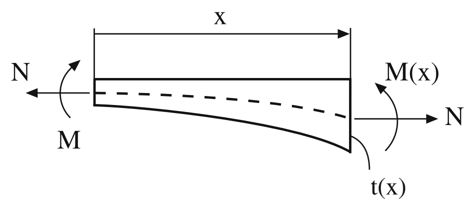

Fig. 4 A ligament segment under axial forces and bending moments. A variable thickness of the ligament is $t(x)$, with the corresponding bending moment $M(x)$

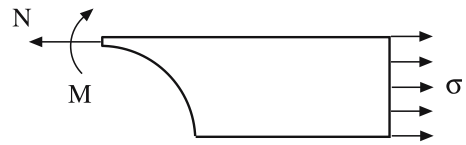

Fig. 5 One half of the notched strip under remote tension σ, which is balanced by the axial force N and the bending moment M acting in the thinnest section of the ligament

Since the maximum stress in a stretched strip weakened by a small semicircular notch [15,16] is $\sigma_{\max} \approx 3.1\sigma$, we conclude that the magnification factor for a large notch is of the order $(R/\delta)^2$.

We note that the stress contribution in the thinnest section of the ligament due to the bending moment dominates over that due to the axial force. Indeed, since $M = NR/2$, (4) can be recast as

$$\sigma_{\mathrm{i}} = \frac{N}{\delta}\left(1 + 3\frac{R}{\delta}\right), \quad \sigma_{\mathrm{o}} = \frac{N}{\delta}\left(1 - 3\frac{R}{\delta}\right), \tag{5}$$

so that the bending contribution to stress is $3R/\delta$ times greater than that due to the axial force. The much greater stress due to bending was, of course, anticipated, because the bending mode dominates the overall deformation of the heavily notched strip in Fig. 3. In fact, the relative rotation (2φ) of the two ends of the strip can be easily calculated from an expression for the strain energy in the ligament and Castigliano's theorem. Since the deformation rapidly decreases with the distance x from the mid-section of the ligament, the strain energy in each half of the ligament is

$$U = \int_0^\infty \frac{M^2(x)\mathrm{d}x}{2EI(x)} + \int_0^\infty \frac{N^2\mathrm{d}x}{2Et(x)}, \tag{6}$$

where $I(x) = t^3(x)/12$. The application of Castigliano's theorem (see Fig. 5) then gives

$$\varphi = \frac{\partial U}{\partial M} = \int_0^\infty \frac{M(x)\dfrac{\partial M(x)}{\partial M}\mathrm{d}x}{EI(x)} = \frac{3\pi\sqrt{2}NR}{8E}\left(3 - \frac{\delta}{R}\right)\left(\frac{R}{\delta}\right)^{5/2}, \tag{7}$$

which is the generalized displacement (rotation) conjugate to M. In the integration procedure, the following general result is useful (e.g., [17])

$$\int_0^\infty \frac{\mathrm{d}x}{(a^2 + x^2)^n} = \frac{\pi(2n-3)!!}{2a^{2n-1}(2n-2)!!}, \quad n = 2, 3, \ldots, \quad a > 0. \tag{8}$$

The longitudinal displacement Δ, conjugate to N, can be calculated from

$$\Delta = \frac{\partial U}{\partial N} = \int_0^\infty \frac{M(x)\dfrac{\partial M(x)}{\partial N}\mathrm{d}x}{EI(x)} + \int_0^\infty \frac{N\mathrm{d}x}{Et(x)}, \tag{9}$$

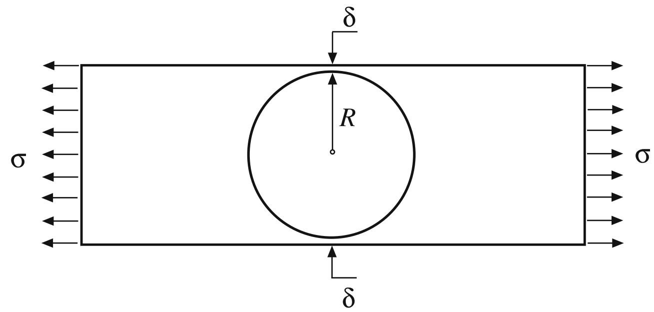

Fig. 6 A rectangular strip weakened by a large circular hole of radius R under uniform remote tension σ

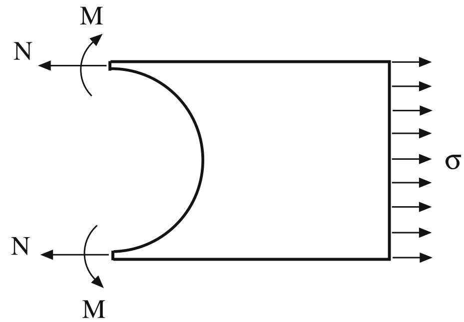

Fig. 7 One half of the strip with a large circular hole. The remote tension σ is balanced by the axial forces N and bending moments M in the mid-planes of the ligaments

which gives

$$\Delta = -\frac{\pi\sqrt{2}N}{16E}\left(3 - 17\frac{\delta}{R}\right)\left(\frac{R}{\delta}\right)^{3/2}. \tag{10}$$

As commonly done in structural analysis, M and N are treated as independent when applying the Castigliano theorem to determine the conjugate displacements $\varphi = \partial U/\partial M$ and $\Delta = \partial U/\partial N$. The actual relationship between M and N, given by (3), is substituted upon the differentiation. This lead to Eqs. (7) and (10).

If the notched strip is subjected to the bending moment $\mathcal{M}$ at its remote ends, we have $N = 0$ and $M = \mathcal{M}$. By a straightforward analysis, it follows that

$$\sigma_{\mathrm{i}} = -\sigma_{\mathrm{o}} = \frac{6\mathcal{M}}{\delta^2}, \tag{11}$$

and

$$\varphi = \frac{9\pi\sqrt{2}\mathcal{M}}{4E\delta^2}\left(\frac{R}{\delta}\right)^{1/2}, \quad \Delta = -\frac{3\pi\sqrt{2}\mathcal{M}}{8E\delta}\left(\frac{R}{\delta}\right)^{1/2}. \tag{12}$$

Recalling that $\sigma_{\max} \approx 3.1\mathcal{M}/R^2$ for a very small notch in a strip of width R, we again conclude that the stress magnification factor is of the order $(R/\delta)^2$. The dummy load $N = 0$ was applied to determine Δ in (12).

3 Stresses and deformation in a strip with a large central hole due to stretching

A rectangular strip with a large central hole of radius R, loaded at the remote ends of the strip by uniform tension σ (or a nonuniform tension statically equivalent to an axial force $2N$ passing through the centroidal axis of the strip), is shown in Fig. 6. In contrast to the notched strip, this problem is statically undetermined with respect to the bending moment M in the vertical plane of symmetry (Fig. 7), while the axial force N is statically determined and equal to

$$N = \sigma(R + \delta). \tag{13}$$

The bending moment M can be obtained by requiring that the slope in the vertical plane of symmetry vanishes, i.e.,

$$\frac{\partial U}{\partial M} = 0\,. \tag{14}$$

Since U is given by (6), and since

$$M(x) = M - N\frac{x^2}{4R}\,, \tag{15}$$

the condition (14) becomes

$$\int_0^\infty \frac{M(x)\dfrac{\partial M(x)}{\partial M}\mathrm{d}x}{EI(x)} = \int_0^\infty \frac{(M - Nx^2/4R)\mathrm{d}x}{(E/12)(\delta + x^2/2R)^3} = 0\,. \tag{16}$$

Upon integration, we obtain a simple relationship between the bending moment and axial force in the midsection of the ligament,

$$M = \frac{1}{6}N\delta\,. \tag{17}$$

The corresponding stresses at the inner and outer points of the ligament are

$$\sigma_\mathrm{i} = \frac{N}{\delta} + \frac{6M}{\delta^2} = 2\frac{N}{\delta}\,, \quad \sigma_\mathrm{o} = \frac{N}{\delta} - \frac{6M}{\delta^2} = 0\,. \tag{18}$$

In contrast to notched strips from Sec. 2, where the stress contribution from the bending moment M dominates over the part from the axial force, in the strip weakened by a central hole the contributions to maximum stress from the bending moment and axial force are equal to each other. This is because of large difference in the magnitude of the bending moment in two cases: for the notched strip $M = NR/2$, while for the strip with a central hole, $M = N\delta/6$ ($3R/\delta$ times smaller). From the deformation point of view, a strip with a central hole is much stiffer in bending than a notched strip and thus the difference in the magnitude of the stress contributions from the axial force and the bending moment in two cases. When expressed in terms of the applied stress σ, (18) can be rewritten as

$$\sigma_\mathrm{i} = 2\sigma\left(\frac{R}{\delta} + 1\right), \quad \sigma_\mathrm{o} = 0\,. \tag{19}$$

Since the maximum stress in a stretched strip with a small hole is $\sigma_{\max} = 3\sigma$ (the stress concentration factor thus being equal to 3), we conclude that the magnification factor for a large hole is of the order R/δ (relative to 3σ). Recall that the stress magnification factor in the notched strip is of the order $(R/\delta)^2$. It should also be recalled that in the thin ligament between the hole and the free surface of a stretched semi-infinite plate the stress magnification factor is of the order $(R/\delta)^{1/2}$ [18]. Indeed, Mindlin [19] derived a remarkably simple expression for the total force across the ligament between the hole and the free edge of a semi-infinite plate (Fig. 8), for any ratio δ/R, which is

$$N = \sigma d\,, \quad d^2 = (R + \delta)^2 - R^2\,, \tag{20}$$

where d is geometrically the length of the tangent line to the circle from the top of the ligament. For $\delta \ll R$, we obtain from (20), $d \approx (2R\delta)^{1/2}$ (which is also the range of validity in x of the beam theory approximation of the thin ligament), and since for thin ligaments $M = N\delta/6$, to first order, the maximum stress in the ligament is $\sigma_{\max} = 2N/\delta = 2\sigma(2R/\delta)^{1/2}$. It is also recalled that the stress magnification factor in a thin ligament between a void and a free surface due to a dislocation on the other side of the void is also of the order $(R/\delta)^{1/2}$ (relative to the stress measure Eb/R, where b is the magnitude of the Burgers vector, [20]). When the loading is due to body forces, as in the Mindlin's [14] tunnel problem, the stress magnification factor in a thin ligament above the tunnel is of the order $(R/\delta)^{1/2}$ (relative to the stress scale $R\rho g$, where ρ is the mass density and g is the acceleration of gravity, [21]).

Returning to the rectangular strip in Fig. 6, the stretching of the strip is equal to 2Δ, where Δ can be calculated from

$$\Delta = \frac{\partial U}{\partial N} = \int_0^\infty \frac{M(x)\dfrac{\partial M(x)}{\partial N}\mathrm{d}x}{EI(x)} + \int_0^\infty \frac{N\mathrm{d}x}{Et(x)}\,. \tag{21}$$

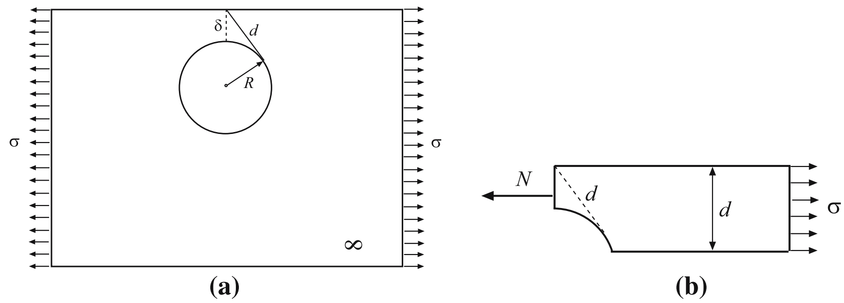

Fig. 8 **a** A circular hole under the straight edge of a semi-infinite plate. The total horizontal force across the thickness δ of the ligament above the hole is $N = \sigma d$, where σ is the applied remote stress and d is the length of the indicated *tangent line*. **b** The net horizontal force on the lower side of the extracted portion of the half-plane, at the depth d from the free edge, is equal to zero, so that the axial force in the ligament is $N = \sigma d$

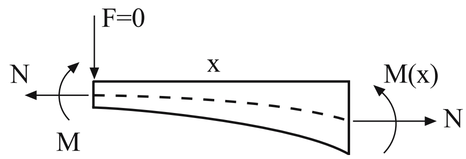

Fig. 9 A virtual force $F = 0$ is applied to the ligament in the direction of the desired lateral displacement

This gives

$$\Delta = \frac{\pi\sqrt{2}N}{E}\left(\frac{R}{\delta}\right)^{1/2}, \tag{22}$$

or

$$\Delta = \frac{\pi\sqrt{2}\sigma R}{E}\left(1+\frac{\delta}{R}\right)\left(\frac{R}{\delta}\right)^{1/2}. \tag{23}$$

The lateral displacement of the points in the midsection of the ligament can be determined by applying the virtual (zero) force in the direction of the lateral displacement (Fig. 9), and by using Castigliano's theorem in the form

$$v = \left(\frac{\partial U}{\partial F}\right)_{F=0}. \tag{24}$$

Since

$$M(x) = \left(M - N\frac{x^2}{4R} - Fx\right)_{F=0}, \quad \frac{\partial M(x)}{\partial F} = -x, \tag{25}$$

there follows

$$v = \int_0^\infty \frac{M(x)\dfrac{\partial M(x)}{\partial F}\mathrm{d}x}{EI(x)} = \int_0^\infty \frac{(M - Nx^2/4R)(-x)\mathrm{d}x}{(E/12)(\delta + x^2/2R)^3}. \tag{26}$$

The integration gives

$$v = \frac{2N}{E}\frac{R}{\delta}, \tag{27}$$

or

$$v = \frac{2\sigma R}{E}\left(\frac{R}{\delta}+1\right), \tag{28}$$

in agreement with Koiter's result obtained by the more involved integration of the differential equation of the tapered beam.

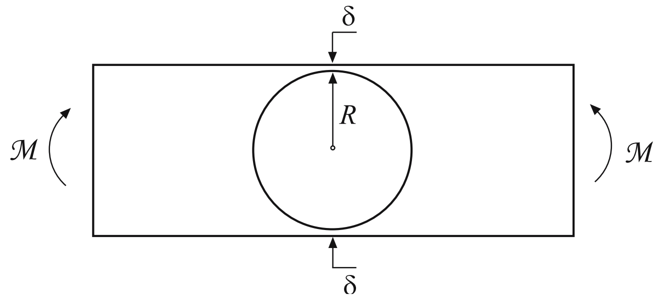

Fig. 10 A rectangular strip weakened by a large circular hole of radius R under uniform remote tension σ

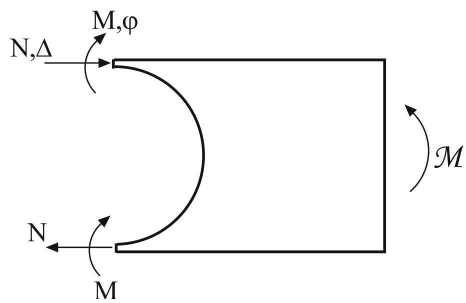

Fig. 11 One half of the strip with a large circular hole. The remote tension σ is balanced by the axial forces N and bending moments M carried by the thinnest section of the ligament

4 Stresses and deformation in a strip with a large central hole due to bending

A rectangular strip weakened by a large central hole and loaded at its remote ends by the bending moments $\mathcal{M}$ is shown in Fig. 10. A free body diagram of one half of the strip is sketched in Fig. 11. The moment equilibrium requires that

$$\mathcal{M} - 2M - N(2R + \delta) = 0\,. \tag{29}$$

The additional equation comes from the compatibility condition. Suppose that the right end of the strip in Fig. 11 is fixed. Then the axial displacement Δ and the rotation φ of the midsection of the ligament must be related by

$$\varphi = \frac{\Delta}{R + \delta/2}\,. \tag{30}$$

The simplest way to calculate Δ and φ is again by employing Castigliano's theorem, i.e.,

$$\varphi = \frac{\partial U}{\partial M}\,, \quad \Delta = \frac{\partial U}{\partial N}\,. \tag{31}$$

Since the strain energy U is given by (6), and since

$$M(x) = M + N\frac{x^2}{4R}\,, \tag{32}$$

we obtain

$$\varphi = \frac{3\pi\sqrt{2}}{8E\delta}\left(\frac{R}{\delta}\right)^{1/2}\left(\frac{6M}{\delta} + N\right), \tag{33}$$

and

$$\Delta = \frac{\pi\sqrt{2}}{16E}\left(\frac{R}{\delta}\right)^{1/2}\left(\frac{6M}{\delta} + 17N\right). \tag{34}$$

When these are substituted into (30), and the result is combined with (29), we obtain a system of two algebraic equations for M and N. Its solution, to first order, is

$$N = \frac{\mathcal{M}}{6R}\left(3 - \frac{\delta}{R}\right), \quad M = -\frac{1}{6}N\delta\,. \tag{35}$$

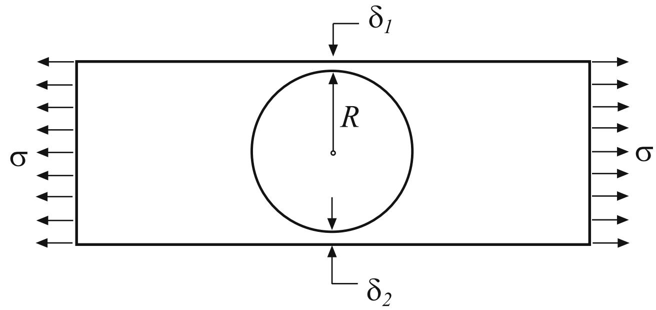

Fig. 12 A rectangular strip weakened by a large eccentric circular hole of radius R under uniform remote tension σ. The ligament thickness in the vertical plane of symmetry is δ_1 for the upper ligament, and δ_2 for the lower ligament

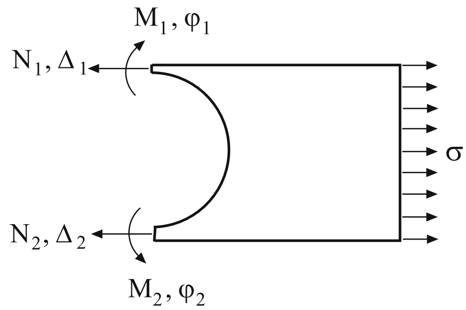

Fig. 13 One half of the strip with a large eccentric circular hole. The remote tension σ is balanced by the axial forces N_1 and N_2, and the bending moments M_1 and M_2 in the mid-planes of the two ligaments. The conjugate stretching and rotations are denoted by Δ_1, Δ_2 and φ_1, φ_2

The corresponding stresses are

$$\sigma_\mathrm{i} = -\frac{\mathcal{M}}{R\delta}\left(1 - \frac{1}{3}\frac{\delta}{R}\right), \quad \sigma_\mathrm{o} = 0\,. \tag{36}$$

The stress at the outer point of the ligament vanishes, because of the first order approximation imbedded in the relationship $M = -N\delta/6$. In the second-order approximation one has

$$N = \frac{\mathcal{M}}{6R}\left(3 - \frac{\delta}{R}\right), \quad M = -\frac{\mathcal{M}\delta}{12R}\left(1 - 3\frac{\delta}{R}\right) = -\frac{N\delta}{6}\left(1 - \frac{8}{3}\frac{\delta}{R}\right), \tag{37}$$

and the corresponding stresses would be

$$\sigma_\mathrm{i} = -\frac{\mathcal{M}}{R\delta}\left(1 - \frac{5}{3}\frac{\delta}{R}\right), \quad \sigma_\mathrm{o} = -\frac{4\mathcal{M}}{3R^2}\,. \tag{38}$$

The rotation of the two ends of the strip follows from (33) to (35). The result is

$$2\varphi = \frac{\pi\sqrt{2}\mathcal{M}}{ER^2}\left(\frac{R}{\delta}\right)^{1/2} \tag{39}$$

5 Strip with a large eccentric hole under remote tension

A rectangular strip with a large eccentric hole of radius R, loaded at the remote ends by uniform tension σ, is shown in Fig. 12. The ligament thicknesses in the vertical plane of symmetry are δ_1 and δ_2. The equilibrium conditions for one half of the strip (Fig. 12) are

$$N_1 + N_2 = \sigma(2R + \delta_1 + \delta_2)\,, \tag{40}$$

$$N_2\left(2R+\frac{\delta_1+\delta_2}{2}\right)+M_1-M_2-\sigma(2R+\delta_1+\delta_2)\left(R+\frac{\delta_2}{2}\right)=0\,. \tag{41}$$

The problem is two times statically undetermined with respect to axial forces and bending moments in the vertical plane of symmetry. The two additional equations are obtained from the compatibility consideration. Suppose that the right-hand side of one half of the strip in Fig. 13 is fixed. Then, we require, as in [13], that the axial displacements and the rotations of two ligaments are related to each other according to

$$\varphi_1=-\varphi_2\,, \tag{42}$$

$$\Delta_2-\Delta_1=\left(2R+\frac{\delta_1+\delta_2}{2}\right)\varphi_1\,. \tag{43}$$

The displacements and rotations are most readily calculated from

$$\varphi_k=\frac{\partial U_k}{\partial M_k}\,,\quad \Delta_k=\frac{\partial U_k}{\partial N_k}\,,\quad (k=1,2). \tag{44}$$

The compatibility conditions (42) and (43) can be again interpreted as matching the inner beam theory field in the thin ligament with the outer rigid-body field in the bulk of the strip. The strain energy in each ligament is given by

$$U_k=\int_0^\infty\frac{M_k^2(x)\mathrm{d}x}{2EI_k(x)}+\int_0^\infty\frac{N_k^2\mathrm{d}x}{2Et_k(x)}\,,\quad (k=1,2), \tag{45}$$

with

$$t_k(x)=\delta_k+\frac{x^2}{2R}\,,\quad M_k(x)=M_k-N_k\frac{x^2}{4R}\,,\quad (k=1,2). \tag{46}$$

It readily follows that

$$\varphi_k=\frac{3\pi\sqrt{2}}{8E\delta_k}\left(\frac{R}{\delta_k}\right)^{1/2}\left(\frac{6M_k}{\delta_k}-N_k\right), \tag{47}$$

$$\Delta_k=\frac{\pi\sqrt{2}}{16E}\left(\frac{R}{\delta_k}\right)^{1/2}\left(-\frac{6M_k}{\delta_k}+17N_k\right),\quad (k=1,2). \tag{48}$$

To first order in δ_k/R, the solution of Eqs. (40)–(43) is

$$M_1=\frac{1}{6}\,N_1\delta_1\,,\quad M_2=\frac{1}{6}\,N_2\delta_2\,, \tag{49}$$

$$N_1=\frac{1}{6}\,\sigma R\left(6+5\frac{\delta_1}{R}+\frac{\delta_2}{R}\right),\quad N_2=\frac{1}{6}\,\sigma R\left(6+\frac{\delta_1}{R}+6\frac{\delta_2}{R}\right). \tag{50}$$

The corresponding stresses at the inner and outer points of the upper ligament are

$$\sigma_\mathrm{i}=\frac{\sigma R}{3\delta_1}\left(6+5\frac{\delta_1}{R}+\frac{\delta_2}{R}\right),\quad \sigma_\mathrm{o}=0\,, \tag{51}$$

while at the inner and outer points of the lower ligament,

$$\sigma_\mathrm{i}=\frac{\sigma R}{3\delta_2}\left(6+\frac{\delta_1}{R}+5\frac{\delta_2}{R}\right),\quad \sigma_\mathrm{o}=0\,. \tag{52}$$

If $\delta_1=\delta_2=\delta$, we recover the results (19) from Sec. 3.

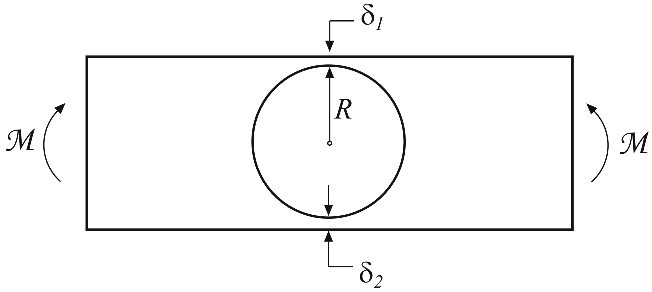

Fig. 14 A rectangular strip weakened by a large eccentric circular hole of radius R under remote bending moments $\mathcal{M}$. The ligament thickness in the vertical plane of symmetry is δ_1 for the upper ligament, and δ_2 for the lower ligament

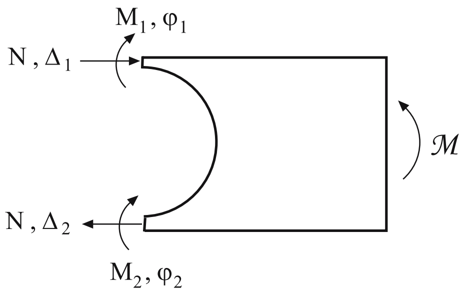

Fig. 15 One half of the strip with a large eccentric circular hole. The remote bending moment $\mathcal{M}$ is balanced by the axial forces N and the bending moments M_1 and M_2 in the mid-planes of the two ligaments. The conjugate stretching and rotations are denoted by Δ_1, Δ_2 and φ_1, φ_2

6 Strip with a large eccentric hole under remote bending

A rectangular strip with a large eccentric hole under remote bending moments $\mathcal{M}$ is shown in Fig. 14. The ligament thicknesses in the vertical plane of symmetry are again δ_1 and δ_2. The equilibrium condition for one half of the strip shown in Fig. 15 are

$$N\left(2R + \frac{\delta_1 + \delta_2}{2}\right) + M_1 + M_2 = \mathcal{M}\,. \tag{53}$$

The problem is two times statically determined with respect to axial force N and the bending moments M_1 and M_2. The two additional equations are obtained from the compatibility consideration. Assuming that the right-hand side of one half of the strip in Fig. 15 is fixed, we require that the axial displacements and the rotations of two ligaments are related to each other according to

$$\varphi_1 = \varphi_2\,, \tag{54}$$

$$\Delta_1 + \Delta_2 = \left(2R + \frac{\delta_1 + \delta_2}{2}\right)\varphi_1\,, \tag{55}$$

where φ_k and Δ_k are defined by (44). The strain energy in each ligament is given by (45), with $t_k(x) = \delta_k + x^2/2R$, $k = 1, 2$, and

$$M_1(x) = M_1 + N\frac{x^2}{4R}\,, \quad M_2(x) = -M_2 - N\frac{x^2}{4R}\,. \tag{56}$$

It readily follows that

$$\varphi_k = \frac{3\pi\sqrt{2}}{8E\delta_k}\left(\frac{R}{\delta_k}\right)^{1/2}\left(\frac{6M_k}{\delta_k} + N\right), \tag{57}$$

$$\Delta_k = \frac{\pi\sqrt{2}}{16E}\left(\frac{R}{\delta_k}\right)^{1/2}\left(\frac{6M_k}{\delta_k} + 17N\right), \quad (k = 1, 2). \tag{58}$$

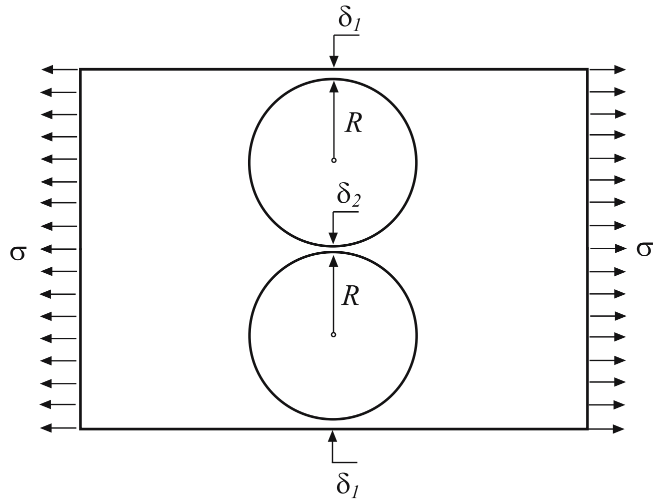

Fig. 16 A rectangular strip weakened by two large circular holes of radius R under uniform remote tension σ. The inner minimum ligament thickness is δ_2, and the outer is δ_1

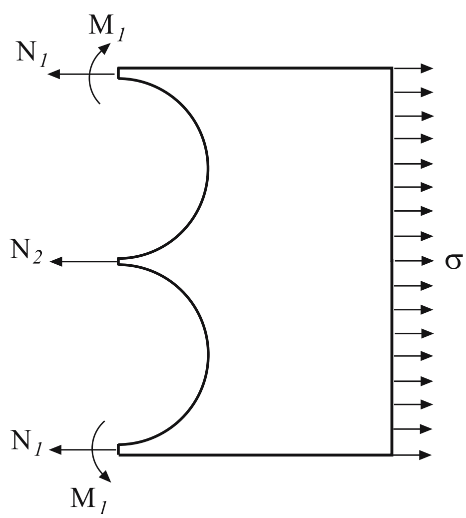

Fig. 17 One half of the strip with two large circular holes. The remote tension σ is balanced by the axial forces N_1 and N_2, and the bending moments M_1 in the mid-plane of the outer ligaments

To first order in δ_k/R, the solution of Eqs. (53)–(55) is

$$M_1 = -\frac{1}{6}N\delta_1\,, \quad M_2 = -\frac{1}{6}N\delta_2\,, \tag{59}$$

$$N = \frac{\mathcal{M}}{6R}\left(3 - \frac{\delta_1+\delta_2}{2R}\right). \tag{60}$$

The corresponding stresses at the outer points of the ligaments are equal to zero, while at the inner points of the upper and lower ligament they are, respectively,

$$\sigma_{\mathrm{i}} = -\frac{\mathcal{M}}{R\delta_1}\left(1 - \frac{\delta_1+\delta_2}{6R}\right), \quad \sigma_{\mathrm{i}} = \frac{\mathcal{M}}{R\delta_2}\left(1 - \frac{\delta_1+\delta_2}{6R}\right). \tag{61}$$

If $\delta_1 = \delta_2 = \delta$, we recover the results (36) from Sec. 4 for the bent strip with a central hole.

7 Strip weakened by two large holes under tension

The foregoing analysis can be extended to strips weakened by more than one large holes. We illustrate this by considering a strip weakened by two large holes, each of radius R, symmetrically positioned within the strip, as shown in Fig. 16. The outer ligament thickness is δ_1 and the inner ligament thickness is δ_2. The strip is loaded at its remote ends by uniform tension σ. The free-body diagram of one half of the strip is shown in Fig. 17. Due to symmetry with respect to the horizontal mid-plane of the strip, there is no bending moment in the inner ligament, while the axial forces and the bending moments in two outer ligaments are equal to each other. The equilibrium condition is then

$$2N_1 + N_2 = \sigma(4R + 2\delta_1 + \delta_2)\,. \tag{62}$$

The bending moment M_1 can be determined by requiring that the slope in the vertical plane of symmetry vanishes for each ligament. Proceeding as in Sec. 3, we obtain that

$$M_1 = \frac{1}{6}N_1\delta_1\,. \tag{63}$$

The remaining equation is the compatibility condition, which matches the deformation of ligaments with the rigid-body displacement of the outer bulk of the strip. This is

$$\Delta_1 = \Delta_2\,, \tag{64}$$

where

$$\Delta_1 = \int_0^\infty \frac{M_1(x)\dfrac{\partial M_1(x)}{\partial N_1}\mathrm{d}x}{EI_1(x)} + \int_0^\infty \frac{N_1\mathrm{d}x}{Et_1(x)}\,, \tag{65}$$

and

$$\Delta_2 = \int_0^\infty \frac{N_2\mathrm{d}x}{Et_2(x)}\,. \tag{66}$$

The bending moments and the ligament thicknesses at an arbitrary x within the ligaments are

$$M_1(x) = M_1 - N_1\frac{x^2}{4R}\,, \quad t_1(x) = \delta_1 + \frac{x^2}{2R}\,, \quad t_2(x) = \delta_2 + \frac{x^2}{R}\,. \tag{67}$$

Upon integration and substitution into (64), it follows that

$$N_2 = 2\left(\frac{2\delta_2}{\delta_1}\right)^{1/2}\,. \tag{68}$$

When this is combined with (62), we obtain, to the leading-order terms,

$$N_1 = \frac{2R}{1+(2\delta_2/\delta_1)^{1/2}}\,\sigma\,, \tag{69}$$

$$N_2 = \frac{4R(2\delta_2/\delta_1)^{1/2}}{1+(2\delta_2/\delta_1)^{1/2}}\,\sigma\,. \tag{70}$$

The stress in the inner ligament is

$$\sigma_2 = \frac{N_2}{\delta_2} = \frac{4(2\delta_2/\delta_1)^{1/2}}{1+(2\delta_2/\delta_1)^{1/2}}\,\frac{R}{\delta_2}\,\sigma\,, \tag{71}$$

while the maximum stress in the outer ligaments is

$$\sigma_1^{\max} = \frac{N_1}{\delta_1} + \frac{6M_1}{\delta_1^2} = 2\frac{N_1}{\delta_1} = \frac{4}{1+(2\delta_2/\delta_1)^{1/2}}\,\frac{R}{\delta_1}\,\sigma\,. \tag{72}$$

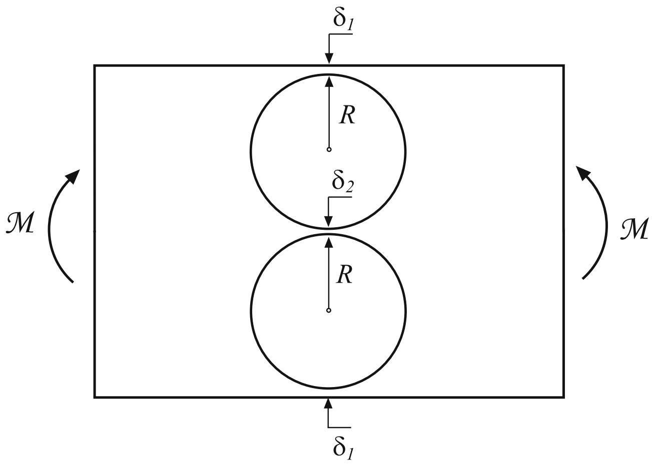

Fig. 18 A rectangular strip weakened by two large holes of radius R under remote bending moments $\mathcal{M}$

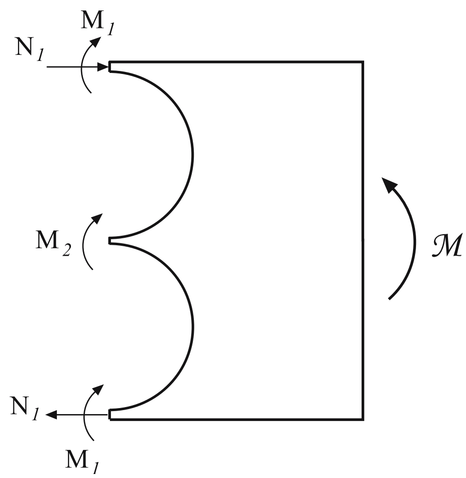

Fig. 19 One half of the strip weakened by two large holes. The remote bending moment $\mathcal{M}$ is balanced by axial forces N_1 and bending moments M_1 and M_2

The stress magnification factors are of the order R/δ_2 and R/δ_1, respectively. Evidently,

$$\frac{\sigma_2}{\sigma_1^{\max}} = \left(\frac{2\delta_2}{\delta_1}\right)^{1/2}, \tag{73}$$

which reveals a remarkably simple condition that $\sigma_2 > \sigma_1^{\max}$ if $\delta_2 < 2\delta_1$. It should be recalled in this context that the stress magnification in a thin ligament between two holes in a stretched infinite medium is only of the order $(R/\delta)^{1/2}$, which was obtained in [22] by using an asymptotic analysis of the two-dimensional elasticity solution.

8 Strip weakened by two large holes under bending

A rectangular strip weakened by two large holes and loaded at its remote ends by the bending moments $\mathcal{M}$ is shown in Fig. 18, with a free body diagram of one half of the strip shown in Fig. 19. The moment equilibrium requires that

$$\mathcal{M} - 2M_1 - M_2 - N_1(4R + \delta_1 + \delta_2) = 0\,. \tag{74}$$

The additional two equations needed to calculate the three unknown quantities (M_1, M_2, N_1) come from the compatibility conditions

$$\varphi_1 = \varphi_2\,, \tag{75}$$

$$\varphi_1 = \frac{\Delta_1}{2R + \delta_1/2 + \delta_2/2}\,. \tag{76}$$

It readily follows that

$$\varphi_1 = \frac{3\pi\sqrt{2}}{8E\delta_1}\left(\frac{R}{\delta_1}\right)^{1/2}\left(\frac{6M_1}{\delta_1} + N_1\right), \tag{77}$$

$$\Delta_1 = \frac{\pi\sqrt{2}}{16E}\left(\frac{R}{\delta_1}\right)^{1/2}\left(\frac{6M_1}{\delta_1} + 17N_1\right). \tag{78}$$

$$\varphi_2 = \frac{3\pi}{4E}\left(\frac{R}{\delta_2}\right)^{1/2}\frac{3M_2}{\delta_2^2}\,. \tag{79}$$

When these are substituted into (74)–(76), we obtain a system of three algebraic equations for M_1, M_2, and N_1, whose solution is

$$N_1 = \frac{\mathcal{M}}{4R}\left(1 - \frac{1}{6}\frac{\delta_1}{R} - \frac{1}{4}\frac{\delta_2}{R}\right), \tag{80}$$

$$M_1 = -\frac{\mathcal{M}\delta_1}{24R}\left(1 - \frac{3}{2}\frac{\delta_1}{R} - \frac{1}{4}\frac{\delta_2}{R}\right), \tag{81}$$

$$M_2 = \frac{\sqrt{2}\mathcal{M}}{18}\left(\frac{\delta_2}{\delta_1}\right)^{1/2}\left(\frac{\delta_2}{R}\right)^2\left(1 - \frac{1}{12}\frac{\delta_1}{R} - \frac{1}{4}\frac{\delta_2}{R}\right). \tag{82}$$

To the leading-order terms this is equivalent to

$$N_1 = \frac{\mathcal{M}}{4R}\,,\quad M_1 = -\frac{\mathcal{M}\delta_1}{24R}\,,\quad M_2 = \frac{\sqrt{2}\mathcal{M}}{18}\left(\frac{\delta_2}{\delta_1}\right)^{1/2}\left(\frac{\delta_2}{R}\right)^2, \tag{83}$$

with $M_1 = -N_1\delta_1/6$. The corresponding maximum stresses are

$$\sigma_1^{\max} = \frac{N_1}{\delta_1} - \frac{6M_1}{\delta_1^2} = 2\frac{N_1}{\delta_1} = \frac{\mathcal{M}}{2R\delta_1}\,,\quad \sigma_2^{\max} = \frac{6M_2}{\delta_2^2} = \frac{\sqrt{2}\mathcal{M}}{3R^2}\left(\frac{\delta_2}{\delta_1}\right)^{1/2}. \tag{84}$$

As expected on physical grounds, the remote ligaments are dominantly carrying the bending moment $\mathcal{M}$, so that $\sigma_1^{\max}$ is of the order R/δ_1 greater than $\sigma_2^{\max}$. Indeed, the bending moment M_2 is only of the order δ_2/R times M_1, since from (84)

$$\frac{M_2}{M_1} = -\frac{4\sqrt{2}}{3}\left(\frac{\delta_2}{\delta_1}\right)^{3/2}\frac{\delta_2}{R}\,. \tag{85}$$

9 Conclusions

We have presented a simple method of calculating the stress magnification factors in thin ligaments between small and large cylindrical voids. This is accomplished by considering large semicircular notches or circular holes in rectangular strips under stretching or bending types of loading. The utilized method is based on the consideration of the strain energy in thin ligaments modeled as tapered beams, and the matching of the inner beam theory approximation in thin ligaments to the outer rigid-body field in the bulk of the material. It is shown that for the strip weakened by a large circular hole (modeling a small void in close vicinity of two large voids), the stress magnification factor is of the order R/δ. The same magnification factor is found in the outer ligaments between two small and two large voids. For this case we also derived a simple relationship between the maximum stresses in the outer and inner ligaments. For the bent strip weakened by two large holes we showed that the maximum stress in the outer ligament is of the order R/δ greater than the maximum stress in the inner ligament. For the strip weakened by a large semicircular notch (modeling a coalescing void), the stress magnification factor is of the order $(R/\delta)^2$. Finally, the stress magnification factor is of the order $(R/\delta)^{1/2}$ in the ligament between one small and one large void. The analysis presented in this paper is of interest, because an alternative numerical approach to the considered problems requires a sensitive mesh refinement in thin regions of ligaments and does not deliver explicitly the simple analytical expressions for the stress magnification factors nor the relationships among maximum stresses in different ligaments. Moreover, in some numerical schemes incorporating the order of the singularity may improve the convergence and the accuracy. The obtained results provide a quantitative framework for the understanding of the severity of stress in thin ligaments that will lead to material failure by void growth and coalescence.

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References

1. Bandstra, J.P., Koss, D.A., Geltmacher, A., Matic, P., Everett, R.K.: Modeling void coalescence during ductile fracture of a steel. Mater Sci Eng A **366**, 269–281 (2004)
2. Gologanu, M., Leblond, J.B., Perrin, G., Devaux, J.: Theoretical models for void coalescence in porous ductile solids: coalescence in layers. Int J Solids Struct **38**, 5581–5594 (2001)
3. Lubarda, V.A., Schneider, M.S., Kalantar, D.H., Remington, B.A., Meyers, M.A.: Void growth by dislocation emission. Acta Mater **52**, 1397–1408 (2004)
4. Marian, J., Knap, J., Ortiz, M.: Nanovoid deformation in aluminum under simple shear. Acta Mater **53**, 2893–2900 (2005)
5. Nakayama, Y., Tvergaard, V.: Interaction of two closely spaced voids during growth to coalescence. Key Eng Materials **274–276**, 81–86 (2004)
6. Ragab, A.R.: A model for ductile fracture based on internal necking of spheroidal voids. Acta Mater **52**, 3997–4009 (2004)
7. Seppala, E.T., Belak, J., Rudd, R.E.: Three-dimensional molecular dynamics simulations of void coalescence during dynamic fracture of ductile metals. Phys Review B 71 (2005) Art. No. 064112
8. Asaro, R.J., Lubarda, V.A.: Mechanics of solids and materials. Cambridge: Cambridge University Press, 2006
9. Koiter, W.T.: An elementary solution of two stress concentration problems in the neighborhood of a hole. Q Appl Math **15**, 303–308 (1957)
10. Keller, J.B.: Stresses in narrow regions. J Appl Mech **60**, 1054–1056 (1993)
11. Duan, Z.P., Kienzler, R., Herrmann, G.: An integral equation method and its application to defect mechanics. J Mech Phys Solids **34**, 539–561 (1986)
12. Markenscoff, X., Dundurs, J.: Amplification of stresses in thin ligaments. Int J Solids Struct **29**, 1883–1888 (1992)
13. Markenscoff, X.: Stress amplification in the neighborhood of an eccentric large hole in a strip in tension. Z Angew Math Phys **51**, 550–554 (2000)
14. Mindlin, R.D.: Stress distribution around a tunnel. Trans ASCE **105**, 1117–1153 (1940)
15. Noda, N.A., Nisitani, H.: Stress concentration of a strip with a single edge notch. Eng Fract Mech **28**, 223–238 (1987)
16. Teh, L.S., Brennan, F.P.: Stress intensity factors for cracks emanating from two-dimensional semicircular notches using the composition of SIF weight functions. Fatigue Fract Eng Mater Struct **28**, 423–435 (2005)
17. Råde, L., Westergren, B.: Mathematics handbook for science and engineering. Berlin Heidelberg New York: Springer, 1999
18. Callias, C.J., Markenscoff, X.: Singular asymptotics analysis for the singularity at a hole near a boundary. Q Appl Math **47**, 233–245 (1989)
19. Mindlin, R.D.: Stress distribution around a hole near the edge of a plate under tension. Proc SESA **5**, 56–67 (1948)
20. Lubarda, V.A., Markenscoff, X.: The stress field for a screw dislocation near cavities and straight boundaries. Mater Sci Eng A **349**, 327–334 (2003)
21. Wu, L., Markenscoff, X.: Asymptotics for thin elastic ligaments with applications to body force and thermal loading. J Mech Phys Solids **45**, 2033–2054 (1997)
22. Wu, L., Markenscoff, X.: Singular stress amplification between two holes in tension. J Elasticity **44**, 131–144 (1996)