MAE 223, Spring 2014
Midterm 1
05/06/2014

Guidelines:
Solve the questions. Give all formulae used in the solutions and explain your steps in each problem. It is OK to keep this sheet after the test.

QUESTION 1. The discrete Fourier series of a periodic function given on N points over a spatial grid is given by

$$u(x_j) = \sum_{n=-N/2}^{N/2-1} \widehat{u}_n \exp(ik_n x_j),$$

with $x_j = jL/N$, $k_n = 2\pi n/L$ and $u(x_j + L) = u(x_j)$.

1. Show that the Fourier coefficients of u, given by

$$\widehat{u}_n = \frac{1}{N} \sum_{j=-N/2}^{N/2-1} u(x_j) \exp(-ik_n x_j),$$

   are also periodic with period N. Use this periodicity to provide a graphical interpretation for the aliasing error. (**5 points**)

2. Show that if $u(x)$ is a smooth function ($u(x)$ is bound and all its derivatives are continuous) the Fourier series converges faster than N^{-p} for any $p > 0$. (**5 points**)

3. Describe the Gibbs phenomenon that would appear if $u(x)$ were discontinuous. (**5 points**)

QUESTION 2. Consider the 1D problem

$$\partial_t u + u\partial_x u = 0.$$

In order to use a Fourier pseudospectral method to solve integrate this problem, we will need to dealias the non-linear terms. Imagine that, instead of applying zero-padding or phase shift, each time step we use a different grid that changes by a random amount $0 \leq \Delta \leq \Delta x$.

1. How does the cost of this procedure compare to phase-shifting and zero-padding in terms of computational cost. (**5 points**)

2. Does this procedure exactly remove the aliasing error? Why? (**5 points**)

QUESTION 3. Consider the linearized Kuramoto-Sivashinski equation

$$\partial_t u = \nu\partial_{xx} u + k\partial_{xxxx} u$$

1. Discretize the equation using a Galerkin Fourier method. (**5 points**)

2. Estimate the minimum number of time steps required to integrate the the ODE obtained in the previous step with an amplification factor of $1/2$, from $t = 0$ to $t = T$ and for $\Delta x = L/N$, using

   (a) An explicit Euler scheme (**5 points**)

   (b) An implicit Euler scheme (**5 points**)

QUESTION 4.

1. How many terms (e.g. single, double and triple) does the aliasing error of the Navier-Stokes equations contain in three dimensions? Demonstrate the origin of these terms. (**5 points**)

2. Explain the evaluations of the non-linear terms on different grids that would be required each time step to cancel all the aliasing error terms using phase shifting in 3-D. (**5 points**)

EXTRA CREDIT QUESTION (HARD. It is advisable to finish all other questions before attempting this one) Explain how to solve the problem

$$\partial_t u + \cos(x) u = \nu \partial_{xx} u$$

for $0 \leq x \leq 2\pi$ with periodic boundary conditions, using a Fourier method with a Galerkin approach, and without resorting to the pseudospectral method. Can this be extrapolated efficiently to any arbitrary function instead of $\cos(x)$? (**20 points**)