A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number*

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The hypotheses concerning the local structure of turbulence at high Reynolds number, developed in the years 1939–41 by myself and Oboukhov (Kolmogorov 1941 *a, b, c*; Oboukhov 1941 *a, b*) were based physically on Richardson's idea of the existence in the turbulent flow of vortices on all possible scales $l < r < L$ between the 'external scale' L and the 'internal scale' l and of a certain uniform mechanism of energy transfer from the coarser-scaled vortices to the finer.

These hypotheses were arrived at independently by a number of other authors and have achieved very wide acceptance. But quite soon after they originated, Landau noticed that they did not take into account a circumstance which arises directly from the assumption of the essentially accidental and random character of the mechanism of transfer of energy from the coarser vortices to the finer: with increase of the ratio $L:l$, the variation of the dissipation of energy

$$\epsilon = \tfrac{1}{2}\nu \sum_{\alpha} \sum_{\beta} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)^2$$

should increase without limit. More accurately, it is natural to suppose that when $L/l \gg 1$ the dispersion of the logarithm of ϵ has the asymptotic behaviour

$$\sigma^2_{\log \epsilon} \sim A + k' \log L/l, \tag{1}$$

where k' is some universal constant.

However, Oboukhov has now discovered how to refine our previous results in a way which takes Landau's comments into consideration. The method consists in examining the dissipation

$$\epsilon_r(\mathbf{x}, t) = \frac{3}{4\pi r^3} \int_{|\mathbf{h}| \leqslant r} \epsilon(\mathbf{x} + \mathbf{h}, t)\, d\mathbf{h}$$

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averaged for a sphere of radius r, and in assuming that for large L/r the logarithm of $\epsilon_r(\mathbf{x}, t)$ has a normal distribution. It is natural to suppose that the variance of $\log \epsilon_r(\mathbf{x}, t)$ is given by

$$\sigma_r^2(\mathbf{x}, t) = A(\mathbf{x}, t) + 9k \log L/r, \tag{2}$$

where k is a universal constant, the factor 9 is inserted for later convenience, and $A(\mathbf{x}, t)$ depends on the macrostructure of the flow.

I have formulated appropriate modifications to the two similarity hypotheses that I put forward in 1941, and I therefore wish to give an interpretation of Oboukhov's new idea in the light of this earlier work. The assumed normality of the distribution of $\log \epsilon_r$, together with formula (2) for its variance, constitutes a third hypothesis.

I shall show that the equation

$$B_{dd}(r) = Cr^{\frac{2}{3}}\bar{\epsilon}^{\frac{2}{3}}$$

from Kolmogorov (1941*a*) now takes the modified form

$$B_{dd}(r) = C(\mathbf{x}, t)\, r^{\frac{2}{3}}\bar{\epsilon}^{\frac{2}{3}}(L/r)^{-k}, \tag{3}$$

where k is the constant in equation (2) and the factor $C(\mathbf{x}, t)$ depends on the macrostructure of the flow; and that the theorem of constancy of skewness

$$S(r) = B_{ddd}(r)/B_{dd}^{\frac{3}{2}}(r)$$

when $l \ll r \ll L$, derived in Kolmogorov (1941*b*), is now replaced by the equation

$$S(r) = S_0(L/r)^{\frac{3}{2}k} \tag{4}$$

where the coefficient S_0 also depends on the macrostructure of the flow.

Let us associate with the length scale r and the point $(\mathbf{x}, t)$ scales of time and velocity

$$T_r = r^{\frac{2}{3}}\epsilon_r^{-\frac{1}{3}}, \quad U_r = r^{\frac{1}{3}}\epsilon_r^{\frac{1}{3}},$$

and an internal length scale,

$$l_r = \nu^{\frac{3}{4}}\epsilon_r^{-\frac{1}{4}}.$$

Obviously the Reynolds number formed from U_r and r is determined by the ratio $r : l_r$ through the equation

$$Re_r = \frac{U_r r}{\nu} = \left(\frac{r}{l_r}\right)^{\frac{4}{3}}. \tag{5}$$

Let x'_α, t' be the co-ordinates of any point $(\mathbf{x}', t)$ in the neighbourhood of of $(\mathbf{x}, t)$. Then we define dimensionless co-ordinates ξ_α and time τ by equations

$$x'_\alpha = x_\alpha + \xi_\alpha r, \quad t' = t + \tau T_r,$$

and dimensionless relative velocities $v_\alpha(\boldsymbol{\xi}, \tau)$ by equations

$$v_\alpha(\boldsymbol{\xi}, \tau) = \frac{u_\alpha(\mathbf{x} + \boldsymbol{\xi} r, t + \tau T_r) - u_\alpha(\mathbf{x}, t)}{U_r}$$

The first similarity hypothesis

If $r \ll L$, then for assigned numbers $\xi_\alpha^{(k)}, \tau^{(k)}$ ($\alpha = 1, 2, 3$; $k = 1, 2, \ldots, n$) the probability distribution of the $3n$ variables $v_\alpha(\boldsymbol{\xi}^{(k)}, \tau^{(k)})$, for the fixed value $Re_r = Re$, depends only on Re and is the same in all turbulent flows.

The second similarity hypothesis

When $Re \gg 1$, the distribution given in the first hypothesis does not depend on Re.

We shall indicate an absolute mathematical expectation (mean) by an overbar. Since $\bar{\epsilon}$ is almost constant in regions which are small in comparison with the external scale L, when $r \ll L$ it may be supposed that

$$\bar{\epsilon}_r = \bar{\epsilon}. \tag{6}$$

Let us examine the difference between the linear components of velocities at two points distant r apart,

$$\Delta_{dd}(r) = u_1(x_1 + r, x_2, x_3, t) - u_1(x_1, x_2, x_3, t).$$

From the definitions of v_α and U_r,

$$\Delta_{dd}(r) = v_1(1, 0, 0, 0)\, r^{\frac{1}{3}} \epsilon_r^{\frac{1}{3}}. \tag{7}$$

If $r \gg l$, where l is the upper limit (excluding cases of negligibly small probability) of the internal scale l_r, then from equations (6) and (7) and the above two hypotheses, it follows that

$$\overline{|\Delta_{dd}(r)|^3} = Cr\bar{\epsilon} \tag{8}$$

where C is an absolute constant.

We now use the formula pointed out by Oboukhov for moments of quantities having a logarithmically normal law of distribution, viz.

$$\overline{\zeta^p} = \exp\left(pm + \tfrac{1}{2}p^2\sigma^2\right), \tag{9}$$

where m and σ^2 are the mean and variance of $\log \zeta$. From equations (2), (8) and (9), it follows that

$$\overline{|\Delta_{dd}(r)|^p} = C_p(\mathbf{x}, t)\,(r\bar{\epsilon})^{p/3}\,(L/r)^{\frac{1}{2}kp(p-3)}. \tag{10}$$

In particular, when $p = 2$,

$$\overline{\Delta^2_{dd}(r)} = C_2(\mathbf{x}, t)\,(r\bar{\epsilon})^{\frac{2}{3}}\,(L/r)^{-k},$$

and this is the same as equation (3).

Since the formula

$$B_{ddd}(r) = -\tfrac{4}{5}\bar{\epsilon}r$$

given in Kolmogorov (1941*b*) remains valid, equation (4) now follows from equation (3).

Our presentation can be freed from the special selection of the values of $\epsilon_r(\mathbf{x}, t)$ forming the basis of Oboukhov's examination. The two similarity hypotheses are then formulated as follows:

First similarity hypothesis

If $|\mathbf{x}^{(k)} - \mathbf{x}| \ll L$, $(k = 0, 1, 2, \ldots, n)$ then the probability distribution of the values of

$$\frac{u_\alpha(\mathbf{x}^{(k)}) - u_\alpha(\mathbf{x})}{u_\alpha(\mathbf{x}^{(0)}) - u_\alpha(\mathbf{x})} \quad (\alpha = 1, 2, 3;\, k = 1, 2, \ldots, n) \tag{11}$$

depends only on the Reynolds number

$$Re = \frac{|u_\alpha(\mathbf{x}^{(0)}) - u_\alpha(\mathbf{x})|\,|\mathbf{x}^{(0)} - \mathbf{x}|}{\nu}$$

Second similarity hypothesis

When $Re \gg 1$, the distribution given in the first hypothesis does not depend on Re.

The essential content of Oboukhov's additional assumptions can be formulated as the

Third hypothesis

Two subsets of values in the set (11) are stochastically independent, if in the first set $|\mathbf{x}^{(k)} - \mathbf{x}| \geqslant r_1$, in the second $|\mathbf{x}^{(k)} - \mathbf{x}| \leqslant r_2$, and $r_1 \gg r_2$.

Naturally the formulation of this hypothesis must be refined if mathematical rigour is desired and if it is to be used to derive the logarithmic normality of the distribution of velocity differences and the formula for the dispersion of the logarithms of these differences analogous to equations (1) and (3).

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