which is focused on rare events as characterized by the tails of probability distributions.

2. The CDF method allows one to derive an exact computable deterministic equation for the CDF of nonlinear advection equations such as the Buckley–Leverett equation. To our knowledge, this is the first method that results in computable equations for the CDF (or PDF) for the nonlinear hyperbolic conservation laws with shocks.

3. Our approach converts the original stochastic nonlinear equation into a deterministic equation for the system state's CDF without any approximations. It allows one to preserve the underlying physics, in particular shocks, in the final solution.

4. Uncertainty in the boundary flux has a significant impact on uncertainty in predictions of the system state (water saturation).

5. The present analysis relies on the existing rarefaction solution of the deterministic Buckley–Leverett equation. However, our general CDF framework is also applicable to problems that do not admit deterministic analytical solutions.

6. Our formulation of the CDF equations may be generalized to the case of multiple shocks by considering multiple jump conditions and rarefaction zones according to the theory of hyperbolic conservation laws.

7. Derivation of the CDF equations for higher-dimensional problems with shocks might benefit from the deterministic theory of "kinetic defects" [18, 19].

Appendix A. Derivation of stochastic equation for raw CDF. Our derivation of the CDF equations is closely related to the PDF equations in turbulence [20] and their applications for uncertainty quantification [26]. We express the spatial and temporal derivatives of $\Pi(\Theta; \mathbf{x}, t)$ as

$$\nabla \Pi = \frac{\partial \Pi}{\partial s} \nabla s = -\frac{\partial \Pi}{\partial \Theta} \nabla s, \qquad \frac{\partial \Pi}{\partial t} = \frac{\partial \Pi}{\partial s}\frac{\partial s}{\partial t} = -\frac{\partial \Pi}{\partial \Theta}\frac{\partial s}{\partial t}. \tag{A.1}$$

For smooth solutions, multiplying (2.3) by $\partial \Pi / \partial \Theta$ and using the first expression in (A.1) yields

$$\frac{\partial \Pi}{\partial t} + \mathbf{v}(s) \frac{\partial \Pi}{\partial \Theta} \cdot \nabla s \ = \ 0. \tag{A.2}$$

Since $\partial \Pi / \partial \Theta = \delta(\Theta - s)$, this yields

$$\frac{\partial \Pi}{\partial t} + \mathbf{v}(s)\, \delta(\Theta - s) \ \cdot \nabla s = 0. \tag{A.3}$$

Finally, recalling that $f(s)\delta(\Theta - s) = f(\Theta)\delta(\Theta - s)$, we rewrite (A.2) as

$$\frac{\partial \Pi}{\partial t} + \mathbf{v}(\Theta) \frac{\partial \Pi}{\partial \Theta} \cdot \nabla s \ = \ 0. \tag{A.4}$$

Combining (A.4) and the second expression in (A.1) leads to (3.3).

This derivation is appropriate for smooth solutions, but in general hyperbolic equations of discontinuous solutions may form. In one spatial dimension, solutions with a single shock may be obtained by directly analyzing shock propagation. To account for shocks and entropy conditions in higher dimensions it might be possible to build upon the deterministic analysis of "kinetic defects" [18, 19].

Appendix B. Analytical solution to one-dimensional CDF equation. We use the method of characteristics to solve the linear hyperbolic equation (4.10) subject

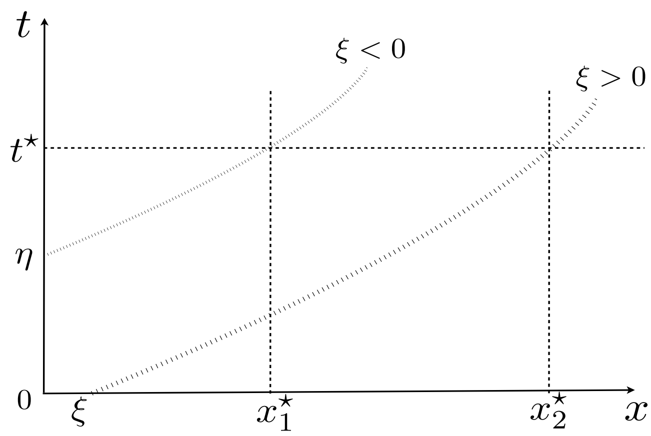

FIG. B.1. *Characteristic curves in the* (x, t) *plane for* $\Pi(Q; x, t)$.

to the following boundary and initial conditions:

$$\Pi(\Theta; x, t = 0) = \mathcal{H}(\Theta - s_{wi}), \qquad \Pi(\Theta; x = 0, t) = \mathcal{H}(\Theta - 1 + s_{oi}). \tag{B.1}$$

We derive a solution for the case in which the random velocity v is such that no shocks form in the underlying stochastic equation. This solution corresponds to a rarefaction zone when we consider a solution that does contain a shock. A family of characteristics, $x = x(t; \xi)$, is defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v(\Theta, x, t), \qquad x(t = 0) = \xi, \tag{B.2}$$

where the "label" ξ defines the origin of each characteristic line (see Figure B.1). Its solution is

$$x = \int_0^t v(\Theta, x(t'), t')\mathrm{d}t' + \xi. \tag{B.3}$$

Along these characteristics, (4.10) becomes

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = 0, \tag{B.4}$$

which is to say that Π is a function of t and ξ only, i.e., $\Pi = g(t, \xi)$:

1. For $\xi \geq 0$, the characteristics originate from the x-axis ($t = 0$) and the solution is determined by the boundary condition on t, i.e., by the initial condition in (B.1).

2. For $\xi < 0$, the characteristics originate from the t-axis ($t = \eta$) and the solution is determined by the boundary condition on x. The variable η is a solution of $\int_0^\eta v \, \mathrm{d}t' = -\xi$.

Substituting (B.3) into (B.4), and eliminating ξ in favor of x and t in the solution, yields

$$\Pi = \mathcal{H}\left(\Theta - s_{wi}\right)\mathcal{H}(x - C) + \mathcal{H}\left(\Theta - 1 + s_{oi}\right)\mathcal{H}(C - x), \tag{B.5}$$

where $C = \int_0^t v(\Theta, x(t'), t')\,\mathrm{d}t'$.

Appendix C. Approximation of random integral I_q. For $t > \tau_q$, the statistics of the integral $I_q(t)$ in (5.5) can be computed with the methods presented in [7]. One starts by subdividing the integration interval $[0, t]$ into N subintervals of length $\Delta = t/N$. Then (5.5) is rewritten as

$$I_q(t) = \sum_{i=1}^{N} \chi_i = \sum_{i=1}^{N} \int_{(i-1)\Delta}^{i\Delta} q(t')\mathrm{d}t'. \tag{C.1}$$

Since $q(t)$ is a stationary process with a continuous sample function, the integrals χ_i $(i = 1, \ldots, N)$ share the same mean

$$\mu_\chi = \mu_q \Delta \tag{C.2}$$

and variance

$$\sigma_\chi^2 = \sigma_q^2 \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{i\Delta} \rho_q(t' - t'')\mathrm{d}t'\mathrm{d}t'' = 2\sigma_q^2 \int_0^{\Delta} (\Delta - t')\rho_q(t')\mathrm{d}t'. \tag{C.3}$$

The two-point covariance between the intervals is given by

$$\mathrm{Cov}(\chi_1, \chi_i) = \sigma_q^2 \int_0^{\Delta} \int_{(i-1)\Delta}^{i\Delta} \rho_q(t' - t'')\mathrm{d}t'\mathrm{d}t'', \quad i \geq 2. \tag{C.4}$$

According to the CLT for correlated random variables [8, 34], $I_q(t) = \sum_{i=1}^{N} \chi_i$ is asymptotically (as $N \to \infty$) Gaussian with mean $N\mu_\chi$ and variance NV, where

$$V = \sigma_\chi^2 + 2\sum_{i=2}^{N} \mathrm{Cov}(\chi_1, \chi_i) < \infty. \tag{C.5}$$

For large times, $t \gg \tau_q$, this approximation is equivalent to the WN scheme in (5.8). This scheme fails when one considers small times, $t \ll \tau_q$, since the condition above (C.5) cannot be satisfied.

It is interesting to note that, as $N \to \infty$, the length of each interval tends to zero. In other words, $\Delta \ll \tau_q$, and hence the random field can be approximated as a random variable for each interval. Now (C.3) and (C.4) can be rewritten as

$$\sigma_\chi^2 = \sigma_q^2 \Delta^2, \tag{C.6}$$

$$\mathrm{Cov}(\chi_1, \chi_i) = \sigma_q^2 \Delta^2 \rho_q\left[(i - 1)\Delta\right], \quad i \geq 2. \tag{C.7}$$

On the other hand, for $t \to \infty$, this approximation scheme is equivalent to that of white noise.

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