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Effective hydraulic conductivity of bounded, strongly heterogeneous porous media

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Abstract. We develop analytical expressions for the effective hydraulic conductivity K_e of a three-dimensional, heterogeneous porous medium in the presence of randomly prescribed head and flux boundaries. The log hydraulic conductivity Y forms a Gaussian, statistically homogeneous and anisotropic random field with an exponential autocovariance. By effective hydraulic conductivity of a finite volume in such a field, we mean the ensemble mean (expected value) of all random equivalent conductivities that one could associate with a similar volume under uniform mean flow. We start by deriving a first-order approximation of an exact expression developed in 1993 by Neuman and Orr. We then generalize this to strongly heterogeneous media by invoking the Landau-Lifshitz conjecture. Upon evaluating our expressions, we find that K_e decreases rapidly from the arithmetic mean K_A toward an asymptotic value as distance between the prescribed head boundaries increases from zero to about eight integral scales of Y. The more heterogeneous is the medium, the larger is K_e relative to its asymptote at any given separation distance. Our theory compares well with published results of spatially power-averaged expressions and with a first-order expression developed intuitively by Kitanidis in 1990.

Introduction

We consider the steady state flow of groundwater in a randomly heterogeneous flow domain, Ω. The Darcy flux $\mathbf{q}(\mathbf{x})$ and the hydraulic head $h(\mathbf{x})$ obey the continuity equation and Darcy's law,

$$\nabla \cdot \mathbf{q}(\mathbf{x}) = 0 \qquad \mathbf{q}(\mathbf{x}) = -K(\mathbf{x})\nabla h(\mathbf{x}), \tag{1}$$

subject to the boundary conditions

$$h(\mathbf{x}) = H(\mathbf{x}) \qquad \text{on} \quad \Gamma_D \tag{2}$$

$$-\mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = Q(\mathbf{x}) \qquad \text{on} \quad \Gamma_N. \tag{3}$$

Here $K(\mathbf{x})$ is a scalar hydraulic conductivity which varies randomly in space, $H(\mathbf{x})$ is a randomly prescribed head on Dirichlet boundary segments Γ_D, $Q(\mathbf{x})$ is a randomly prescribed flux into Ω across Neumann boundary segments Γ_N, $\mathbf{n}(\mathbf{x})$ is a unit vector outward normal to the boundary Γ, and Γ is the union of Γ_D and Γ_N. Both $H(\mathbf{x})$ and $Q(\mathbf{x})$ are prescribed in a statistically independent manner.

Due to the random nature of $K(\mathbf{x})$, $H(\mathbf{x})$, and $Q(\mathbf{x})$, (1)–(3) constitute a stochastic system of equations. Taking the ensemble mean (expectation), expressed by angle brackets, of (1)–(3) yields the deterministic system

$$\nabla \cdot \langle \mathbf{q}(\mathbf{x}) \rangle = 0 \qquad \langle \mathbf{q}(\mathbf{x}) \rangle = -\langle K(\mathbf{x}) \rangle \nabla \langle h(\mathbf{x}) \rangle + \mathbf{r}(\mathbf{x}) \tag{4}$$

subject to

$$\langle h(\mathbf{x}) \rangle = \langle H(\mathbf{x}) \rangle \qquad \text{on} \quad \Gamma_D \tag{5}$$

$$-\langle \mathbf{q}(\mathbf{x}) \rangle \cdot \mathbf{n}(\mathbf{x}) = \langle Q(\mathbf{x}) \rangle \qquad \text{on} \quad \Gamma_N. \tag{6}$$

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The term $\mathbf{r}(\mathbf{x})$ was termed residual flux by *Neuman and Orr* [1993]. According to their equation (F7), it is given exactly by

$$\mathbf{r}(\mathbf{x}) = \int_\Omega \mathbf{a}(\mathbf{x}, \boldsymbol{\chi}) \nabla_\chi \langle h(\boldsymbol{\chi}) \rangle \, d\boldsymbol{\chi} + \int_\Omega \mathbf{b}(\mathbf{x}, \boldsymbol{\chi}) \mathbf{r}(\boldsymbol{\chi}) \, d\boldsymbol{\chi} \tag{7}$$

where $\mathbf{a}(\mathbf{x}, \boldsymbol{\chi})$ and $\mathbf{b}(\mathbf{x}, \boldsymbol{\chi})$ are kernels independent of $\langle h(\mathbf{x}) \rangle$. The latter are defined as

$$\mathbf{a}(\mathbf{x}, \boldsymbol{\chi}) = \langle K'(\mathbf{x}) K'(\chi) \nabla \nabla_\chi^T \mathcal{G}(\boldsymbol{\chi}, \mathbf{x}) \rangle \tag{8}$$

$$\mathbf{b}(\mathbf{x}, \boldsymbol{\chi}) = \langle K'(\mathbf{x}) \nabla \nabla_\chi^T \mathcal{G}(\boldsymbol{\chi}, \mathbf{x}) \rangle \tag{9}$$

where primed quantities represent zero mean random fluctuations and $\mathcal{G}$ is the random Green's function associated with (1)–(3). Both kernels are positive semidefinite tensors of second rank, the first symmetric, the second nonsymmetric. Since $\mathbf{r}(\mathbf{x})$ and hence $\langle \mathbf{q}(\mathbf{x}) \rangle$ depend on mean head gradients at points other than $\mathbf{x}$, the above problem is nonlocal. Since $\langle \mathbf{q}(\mathbf{x}) \rangle$ is not proportional to $\nabla \langle h(\mathbf{x}) \rangle$, the ensemble mean flux is generally non-Darcian.

Neuman and Orr [1993] have pointed out that for $\mathbf{r}(\mathbf{x})$ (and hence $\langle \mathbf{q}(\mathbf{x}) \rangle$) to be Darcian (and hence local) it is necessary that there exists a symmetric, positive semidefinite tensor $\boldsymbol{\kappa}(\mathbf{x})$ of second rank whose principal values (eigenvalues) do not exceed $\langle K(\mathbf{x}) \rangle$ and which is additionally independent of $\langle h(\mathbf{x}) \rangle$ and its gradient, such that

$$\mathbf{r}(\mathbf{x}) = \boldsymbol{\kappa}(\mathbf{x}) \nabla \langle h(\mathbf{x}) \rangle. \tag{10}$$

Then, and only then, is it strictly proper to write

$$\langle \mathbf{q}(\mathbf{x}) \rangle = -\mathbf{K}_e(\mathbf{x}) \nabla \langle h(\mathbf{x}) \rangle \tag{11}$$

where $\mathbf{K}_e(\mathbf{x})$ is a symmetric, positive definite "effective (or equivalent) hydraulic conductivity tensor (dyadic)" given by

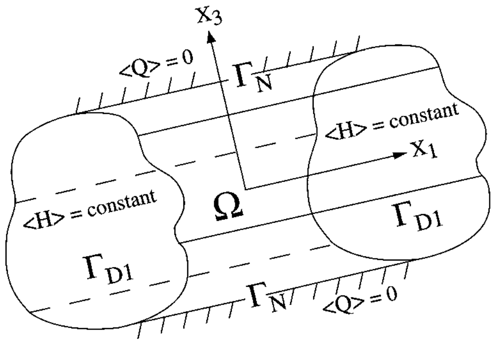

Figure 1. Bounded flow domain Ω.

$$\mathbf{K}_e(\mathbf{x}) = \langle K(\mathbf{x})\rangle \mathbf{I} - \boldsymbol{\kappa}(\mathbf{x}), \tag{12}$$

$\mathbf{I}$ being the identity tensor. Note that $\mathbf{K}_e(\mathbf{x})$ does not generally apply to a single realization of the random functions entering into (1)–(3). Instead, it relates the flux $\mathbf{q}(\mathbf{x})$ averaged over the ensemble of all realizations to a similarly averaged hydraulic gradient at $\mathbf{x}$. Only if the random fields $K(\mathbf{x})$, $\nabla h(\mathbf{x})$, and $\mathbf{q}(\mathbf{x})$ are ergodic can one interpret $\mathbf{K}_e$ in (11) as relating spatial averages of $\mathbf{q}$ to those of ∇h over a sufficiently large volume of porous medium. Neuman and Orr discussed several conditions under which (11) is strictly valid. They showed that in the limit as Ω tends to ω, where ω is the smallest scale on which $K(\mathbf{x})$, $\nabla h(\mathbf{x})$, and $\mathbf{q}(\mathbf{x})$ have operational meaning, $\mathbf{K}_e(\mathbf{x})$ tends to $\langle K(\mathbf{x})\rangle \mathbf{I}$.

The case most often discussed in the stochastic literature is that of an unbounded domain Ω_∞ in which the hydraulic conductivity field can be decomposed into a constant mean, $\langle K(\mathbf{x})\rangle \equiv \langle K\rangle = \text{const}$, and a zero-mean statistically homogeneous perturbation $K'(\mathbf{x})$. Here flow is controlled by boundary conditions at infinity, which result in a uniform mean hydraulic gradient $\nabla\langle h(\mathbf{x})\rangle \equiv \mathbf{J} = \text{const}$. Then $\langle \mathbf{q}\rangle$ and $\langle \mathbf{r}\rangle$ are also constant, and so are the kernels $\mathbf{a}(\mathbf{x}, \boldsymbol{\chi})$ and $\mathbf{b}(\mathbf{x}, \boldsymbol{\chi})$. As such, the latter can be evaluated at arbitrary $\mathbf{x}$. By doing so at the center, $\mathbf{x} = 0$, of the mean uniform flow field, $\mathbf{b}(0, \boldsymbol{\chi})$ becomes odd in $\boldsymbol{\chi}$ and the second integral in (7) vanishes. Hence we obtain

$$\langle \mathbf{q}\rangle = -\mathbf{K}_e \mathbf{J} \qquad \mathbf{K}_e = \langle K\rangle \mathbf{I} - \boldsymbol{\kappa} \tag{13}$$

where

$$\boldsymbol{\kappa} = \int_\Omega \langle K'(\mathbf{x}) K'(\boldsymbol{\chi}) \nabla\nabla_\chi^T \mathcal{G}(\boldsymbol{\chi}, \mathbf{x})\rangle|_{\mathbf{x}=0}\, d\boldsymbol{\chi} \tag{14}$$

and all terms are constant. Under the ergodic hypothesis, which in this case appears justified, one can interpret $\langle \mathbf{q}\rangle$ and $\mathbf{J}$ as spatial averages over a block of porous medium that is large compared to the spatial correlation (integral) scale of $K(\mathbf{x})$. The smallest block size that satisfies this equivalence between ensemble and spatial averages of $\mathbf{q}(\mathbf{x})$ and $\nabla h(\mathbf{x})$, to some acceptable tolerance, qualifies as a representative elementary volume (REV) for these particular flow conditions. For blocks of size REV or larger, the spatial averages of $\mathbf{q}(\mathbf{x})$ and $\nabla h(\mathbf{x})$ vary slowly enough with block size to justify treating them as deterministic; the same, of course, applies to $\mathbf{K}_e$. For blocks of scale smaller than an REV, the equivalent hydraulic conductivity is a random variable. We consider below the ensemble mean $\mathbf{K}_e$ of such random quantities and refer to it as "effective" block conductivity.

In this paper we wish to focus on two questions: what constitutes an REV, and what happens to $\mathbf{K}_e$ in (13) on sub-REV scales (where the flow domain Ω is smaller than an REV). These questions have so far not been investigated analytically except (as mentioned earlier) in the limit as Ω tends to ω. Otherwise, published analytical studies of bounded domains have focused largely on the effects that boundaries may have on second moments involving head [*Naff and Vecchia*, 1986; *Rubin and Dagan*, 1988, 1989]. More directly relevant to our concerns is the work of *Kitanidis* [1990]. He derived equations which must be satisfied by the effective hydraulic conductivity tensor of a nonrandom, periodic $K(\mathbf{x})$ field via volume averaging and a modification of the Taylor-Aris method of moments. Kitanidis then extended his approach to random, statistically homogeneous $K(\mathbf{x})$ fields exhibiting small fluctuations by invoking [*Kitanidis*, 1990, p. 1204] "intuitive arguments rather than rigorous mathematical analysis." We shall see later that, despite this lack of rigor, the solution of Kitanidis as implemented numerically by *Dykaar and Kitanidis* [1992] appears to capture correctly the effect of boundaries on the effective hydraulic conductivity of mildly heterogeneous random media.

Many numerical studies have been devoted to the issue of upscaling [e.g., *Desbarats and Dimitrakopoulos*, 1990; *Rubin and Gómez-Hernández*, 1990; *Durlofsky*, 1991; *Desbarats*, 1992; *Fenton and Griffiths*, 1993]. Upscaling is the process of assigning equivalent properties to the grid blocks of a numerical model within which these properties are known to vary. Several authors [*Journel et al.*, 1986; *Deutsch*, 1989; *Gómez-Hernández and Gorelick*, 1989; *Ababou and Wood*, 1990; *Desbarats*, 1992] have concluded, based on numerical experiments, that block hydraulic conductivity may be written as a power average, or α-norm, of hydraulic conductivity values according to $K_\alpha = \{K^\alpha\}^{1/\alpha}$, where {} represents ensemble (in the case of *Desbarats* [1992] spatial) average and α is an empirical parameter. A more rigorous theoretical approach to the upscaling of mildly fluctuating random hydraulic conductivities has been proposed by *Indelman and Dagan* [1993a, b]. However, their "upscaling is carried out in a region sufficiently far from the boundary" [*Indelman and Dagan* [1993a, p. 920], so that its effect is not felt. This may explain why the upscaled hydraulic conductivity in Figure 5 of *Indelman* [1993] tends to the geometric mean of $K(\mathbf{x})$, rather than to its arithmetic mean as required by theory [*Neuman and Orr*, 1993], when the domain size approaches zero.

In this paper, we derive an explicit expression for the effective hydraulic conductivity of a bounded, randomly heterogeneous domain based on (13) and (14). We first derive a linearized expression which is valid only for mildly heterogeneous media. We then extend this expression to strongly heterogeneous media by adopting a conjecture that has proven to work well for large domains. We conclude by evaluating our expressions numerically and comparing them with previously published results.

Linearized Analysis

We consider a flow domain Ω bounded by two parallel, planar Dirichlet boundaries of equal shape and size (Figure 1). The boundaries are connected by a surface normal to them which acts as a Neumann boundary. They are spaced a distance $2L$ apart along the x_1 coordinate of a Cartesian system that has its origin midway between the planes. The mean head $\langle H\rangle$ is uniform on each Dirichlet boundary. On the Neumann boundary, the mean normal flux is equal to zero, $\langle Q \rangle = 0$. The log hydraulic conductivity, $Y(\mathbf{x}) = \ln K(\mathbf{x})$, in Ω is a statistically homogeneous field with mean $\langle Y \rangle$ and variance σ_Y^2, both constant. The covariance of $Y(\mathbf{x})$, $C_Y(\mathbf{x} - \boldsymbol{\chi}) = \langle Y'(\mathbf{x}) Y'(\boldsymbol{\chi}) \rangle$, is exponential and exhibits elliptical anisotropy with principal directions parallel to the coordinates.

It has been shown by *Neuman and Orr* [1993] that, under these circumstances, the mean flux has only one nonzero component parallel to x_1 which can be expressed by means of a scalar form of Darcy's law,

$$\langle q_1(x_2, x_3) \rangle = -K_{e1}(x_2, x_3) J_1$$
$$K_{e1}(x_2, x_3) = \langle K \rangle - \kappa_1(x_2, x_3). \tag{15}$$

Here J_1 is the constant mean hydraulic gradient imposed throughout Ω by the Dirichlet boundaries, $K_{e1}(x_2, x_3)$ is an effective directional hydraulic conductivity, and (14) implies that $\kappa_1(x_2, x_3)$ is given by

$$\kappa_1(x_2, x_3) = \int_\Omega \langle K'(\mathbf{x}) K'(\boldsymbol{\chi}) \partial^2 \mathcal{G}(\boldsymbol{\chi}, \mathbf{x}) / \partial x_1 \partial \chi_1 \rangle \, d\boldsymbol{\chi} \,. \tag{16}$$

Since homogeneity prevails on planes parallel to the Dirichlet boundaries with $\langle q_2 \rangle = \langle q_3 \rangle = 0$, and by virtue of the continuity equation (1), we have that $d\langle q_1 \rangle / dx_1 = 0$. Hence $\langle q_1 \rangle$ (and consequently, K_{e1}) in (15) is constant in the x_1 direction. In (16) the integrand is thus evaluated at an arbitrary value of x_1, and Green's function $\mathcal{G}(\boldsymbol{\chi}, \mathbf{x})$ is the random solution of (1)–(3) due to a point source of unit strength at $\boldsymbol{\chi}$, given by the Dirac delta function $\delta(\mathbf{x} - \boldsymbol{\chi})$, subject to homogeneous boundary conditions $H(\mathbf{x}) = Q(\mathbf{x}) \equiv 0$. It is important to appreciate [*Neumann and Orr*, 1993] that K_{e1} is strictly directional and does not form the principal component of a tensor when Ω is finite.

To evaluate $K_{e1}(x_2, x_3)$, we restrict our attention, for the time being, to mildly heterogeneous media with $\sigma_Y^2 < 1$. This justifies approximating $\langle K \rangle$ and $K_{e1}(x_2, x_3)$ to first order in σ_Y^2. As a first step, we approximate $\mathcal{G}(\boldsymbol{\chi}, \mathbf{x})$ to zero order by means of the deterministic Green's function $G_0(\boldsymbol{\chi}, \mathbf{x})$. The latter is defined as the solution of (4)–(6) with $\mathbf{r}(\mathbf{x})$ set equal to zero, due to a point source of unit strength at $\boldsymbol{\chi}$, subject to homogeneous mean boundary conditions $\langle H(\mathbf{x}) \rangle = \langle Q(\mathbf{x}) \rangle \equiv 0$. Next, we take $Y(\mathbf{x})$ to be Gaussian, so that [e.g., *Neuman and Orr*, 1993] $\langle K \rangle = K_G \exp(\sigma_Y^2/2)$ and $\langle K'(\mathbf{x}) K'(\boldsymbol{\chi}) \rangle = K_G^2 \exp(\sigma_Y^2) \{\exp[C_Y(\mathbf{x} - \boldsymbol{\chi})] - 1\}$, where $K_G = \exp\langle Y \rangle$ is the geometric mean of $K(\mathbf{x})$. Upon approximating these moments to first-order in σ_Y^2 and combining (15) and (16), we obtain the following linearized expression for the effective conductivity,

$$K_{e1}(x_2, x_3) \approx K_G \left(1 + \frac{\sigma_Y^2}{2} - \int_\Omega C_Y(\mathbf{x} - \boldsymbol{\chi}) \frac{\partial^2 G(\boldsymbol{\chi}, \mathbf{x})}{\partial x_1 \partial \chi_1} d\boldsymbol{\chi} \right) \tag{17}$$

where $G(\boldsymbol{\chi}, \mathbf{x}) = \langle K \rangle G_0(\boldsymbol{\chi}, \mathbf{x})$ satisfies $\nabla^2 G(\boldsymbol{\chi}, \mathbf{x}) + \delta(\mathbf{x} - \boldsymbol{\chi}) = 0$ in Ω, subject to homogeneous boundary conditions [*Greenberg*, 1971].

For Dirichlet boundaries at infinity, (17) is identical to linearized expressions published earlier by *Gutjahr et al*. [1978], *Gelhar and Axness* [1983], *Neuman and Depner* [1988], and *Dagan* [1989] for infinite anisotropic media and by *Kitanidis* [1990, equation (81)] for finite volume.

To simplify the evaluation of (17), we consider only points (x_2, x_3) that are at distances of at least three integral scales of $Y(\mathbf{x})$ from the lateral Neumann boundary. At such distances, this mean no-flow boundary has negligible effect on the flow [*Rubin and Dagan*, 1989], and we are therefore justified placing it at infinity for mathematical convenience. This renders K_{e1} in (17) independent of $\mathbf{x}$, and we arbitrarily evaluate the corresponding integrals at $\mathbf{x} = 0$.

The evaluation of (17) at $\mathbf{x} = 0$ for the case of lateral Neumann boundaries at infinity is described in Appendix A. The result can be expressed as

$$K_{e1} = K_G [1 + \sigma_Y^2 (\tfrac{1}{2} - D)] \tag{18}$$

where D is a domain integral. We present below two alternative but equally valid expressions for D, based on two different representations of the Green's function. One alternative, which converges rapidly for large $\rho = L/\lambda_1$, is developed via the method of images in Appendix A. The corresponding expression is

$$D = 1 - \frac{\rho \varepsilon_2 \varepsilon_3}{4\pi} \sum_{j=-\infty}^{+\infty} \int_{\xi=-1}^{+1} \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} r^2 [1 + (rR)^2]^{-3/2}$$
$$\cdot [1 + \rho|\xi| \sqrt{1 + (rR)^2}] \exp(-\rho|\xi| \sqrt{1 + (rR)^2})$$
$$\cdot \exp(-\rho|\xi - 2j| r) \, dr \, d\theta \, d\xi \tag{19}$$

where $\varepsilon_2 = \lambda_2/\lambda_1$, $\varepsilon_3 = \lambda_3/\lambda_1$, $R^2 = \varepsilon_2^2 \cos^2 \theta + \varepsilon_3^2 \sin^2 \theta$, and λ_1, λ_2, λ_3 are the principal integral scales of $Y(\mathbf{x})$. For statistically isotropic media, the order of integration is reduced by one,

$$D = 1 - \frac{\rho}{2} \sum_{j=-\infty}^{+\infty} \int_{\xi=-1}^{+1} \int_{r=0}^{\infty} r^2 (1 + r^2)^{-3/2} (1 + \rho|\xi| \sqrt{1 + r^2})$$
$$\cdot \exp(-\rho|\xi| \sqrt{1 + r^2}) \exp(-\rho|\xi - 2j| r) \, dr \, d\xi \tag{20}$$

The other alternative, which converges rapidly for small ρ, is developed via an eigenfunction expansion in Appendix B. The corresponding expression is

$$D = 4\pi \sum_{n=1}^{\infty} n^2 \int_0^{+\infty} \int_0^{+\infty} \frac{\rho^2}{n^2\pi^2 + [(k_2^*/\varepsilon_2)^2 + (k_3^*/\varepsilon_3)^2] \rho^2}$$
$$\cdot \left[(-1)^{n+1} \exp(-\rho \sqrt{1 + k^{*2}}) \left(\frac{2\rho}{\rho^2 (1 + k^{*2}) + n^2 \pi^2} + \frac{1}{(1 + k^{*2})^{1/2}} \right) + \frac{2\rho}{\rho^2 (1 + k^{*2}) + n^2 \pi^2} \right]$$
$$\cdot \frac{dk_2^* dk_3^*}{\rho^2 (1 + k^{*2}) + n^2 \pi^2}. \tag{21}$$

Generalization to Strongly Heterogeneous Media

Gelhar and Axness [1983] developed a first-order expression for effective hydraulic conductivities K_{ei} $(i = 1, 2, 3)$, parallel to the principal directions of statistical anisotropy, in an infinite domain Ω_∞. Whereas their result involves the spectral density of $Y(\mathbf{x})$, *Neuman and Depner* [1988] and *Dagan* [1989]

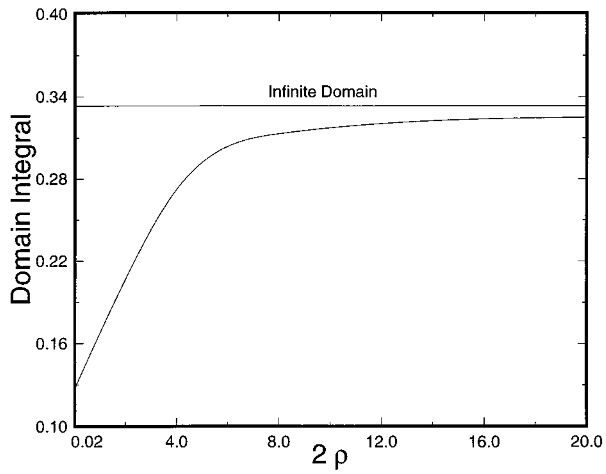

Figure 2. Domain integral D versus 2ρ.

showed that K_{ei} depend solely on σ_Y^2 and the integral scales of $Y(\mathbf{x})$. According to Neuman and Depner,

$$K_{ei} \approx K_G\left(1 + \frac{\sigma_Y^2}{2} - F_i\right) \qquad i = 1, 2, 3 \qquad (22)$$

where

$$F_i = \frac{2\sigma_Y^2}{\pi\lambda_i^2}\int_0^{\pi/2}\int_0^{\pi/2}\frac{f_i^2}{\mathbf{f}^T\boldsymbol{\lambda}^{-2}\mathbf{f}}\sin\phi\, d\phi\, d\theta, \qquad (23)$$

$\boldsymbol{\lambda}$ being a diagonal matrix of principal integral scales λ_i, and

$$\mathbf{f}^T(\phi, \theta) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi). \qquad (24)$$

To obtain an expression valid for large σ_Y^2, they adopted a conjecture used previously by *Gelhar and Axness* [1983], similar to that proposed earlier for isotropic fields by *Landau and Lifshitz* [1960], *Shvidler* [1962], and *Matheron* [1967], that the expression within parentheses in (22) constitutes the first two terms in a series expansion of exp $(\sigma_Y^2/2 - F_i)$, yielding

$$K_{ei} \approx K_G \exp\left(\frac{\sigma_Y^2}{2} - F_i\right) \qquad i = 1, 2, 3. \qquad (25)$$

We shall refer to the above as the Landau-Lifshitz conjecture. The conjecture is rigorously valid under one-dimensional flow where it yields the harmonic mean, K_H, of $K(\mathbf{x})$. It is rigorously valid under two-dimensional flow in lognormal, statistically isotropic conductivity fields, where it yields the geometric mean, K_G [*Matheron*, 1967; see also *Neuman and Orr*, 1993]. Attempts to prove (25) rigorously for three-dimensional flow in such fields have been reported by *King* [1989] and *Noetinger* [1990]. According to *Dagan* [1993], all these attempts involve some approximations. He has shown that, in this case, the conjecture is valid at least to second order in σ_Y^2. Numerical Monte Carlo simulations have shown that it holds at least up to $\sigma_Y^2 = 7$ [*Neuman et al.*, 1992]. When (25) was applied by *Neuman and Depner* [1988] to data obtained from relatively small-scale single-hole packer tests in fractured granites, according to which $\sigma_Y^2 > 7$ and $Y(\mathbf{x})$ is statistically anisotropic, (25) showed consistency with the results of much larger scale cross-hole tests conducted and interpreted independently of the single-hole tests. Although *Indelman and Abramovich* [1994] have found that statistical anisotropy renders K_{ei} dependent on the shape of the covariance function C_Y at second order in σ_Y^2, this effect appears to be [*Indelman and Abramovich*, 1994, p. 1862] "rather small."

Given these results, we feel reasonably comfortable applying the Landau-Lifshitz conjecture to our bounded case. It transforms (18) into the generalized expression

$$K_{e1} = K_G \exp\left[\sigma_Y^2\left(\tfrac{1}{2} - D\right)\right]. \qquad (26)$$

In the limit as the distance between the Dirichlet boundaries goes to infinity, $\rho \to \infty$, (26) in conjunction with (20) yield (Appendix A)

$$K_{e1} \to K_\infty = K_G e^{\sigma_Y^2/6}. \qquad (27)$$

This is the result originally conjectured by *Landau and Lifshitz* [1960], *Shvidler* [1962], *Matheron* [1967], and *Gelhar and Axness* [1983], whose exact proof was sought by *King* [1989] and *Noetinger* [1990] and achieved to second-order by *Dagan* [1993]. The same follows directly from (22)–(24).

In the limit as the Dirichlet boundaries approach each other, $\rho \to 0$, the domain integral vanishes. This follows from (21) and the Dominated Convergence Theorem, and is explained in Appendix B. Then the effective hydraulic conductivity in (18) reduces to the arithmetic mean, as predicted theoretically by *Neuman and Orr* [1993].

Evaluation and Comparison With Published Results

The linearized equation (18) and the generalized equation (26) were evaluated numerically for statistically isotropic media. We evaluated (20) for $\rho = L/\lambda_1$ ranging from 0.01 to 50 and σ_Y^2 ranging from 1 to 7. Results for $\sigma_Y^2 = 1$ are listed in Table 1. The evaluation entailed computing the domain integral D in (20) by means of Gaussian quadrature. All calculations were performed on the University of Arizona Convex C240 with four vector processors, except for the computationally intensive case of $\rho = 0.01$, which was run on the Connection machine at Los Alamos National Laboratories. Equation (19) for anisotropic media involves an additional integration and is therefore more difficult to evaluate. Figure 2 shows how the domain integral D varies with 2ρ. It is important to notice from this figure that D decreases toward the asymptotic value of zero, expected theoretically [*Neuman and Orr*, 1993]) at $\rho = 0$, where the Dirichlet boundaries are in contact. At $2\rho = 20$, D differs by less than 4% from the theoretical value of 1/3 for an infinite domain ($\rho \to \infty$). At $2\rho = 100$ this difference is less than 2.5%.

Table 1 also lists K_{e1}/K_G, as computed by means of the linearized equation (18) and the generalized equation (26), for $\sigma_Y^2 = 1$ and various values of 2ρ. At $2\rho = 0.02$, K_{e1}/K_G from (18) differs by 8.5% from 1.5, the linearized value of K_A/K_G at $\rho = 0$; K_{e1}/K_G from (26) differs by 12% from 1.6487, the exact value of K_A/K_G at $\rho = 0$. At $2\rho = 8$, K_{e1}/K_G from (18) differs by only 1.4% from $1 + 1/6$, the linearized value expected theoretically at $\rho \to \infty$; K_{e1}/K_G from (26) differs by 1.6% from K_∞/K_G = exp (1/6), the conjectured [*Landau and Lifshitz*, 1960] value corresponding to an infinite medium. If one accepts this difference as sufficiently small, one concludes that a separation distance of eight integral scales between the Dirichlet boundaries is enough to consider the medium as being unbounded for the purpose of assigning to it an effective hydraulic conductivity in this case.

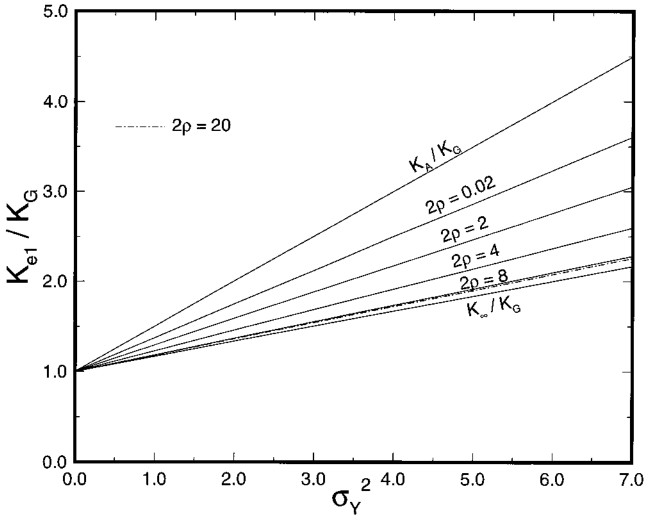

Figure 3. Linearized K_{e1}/K_G versus σ_Y^2 and 2ρ for statistically isotropic media.

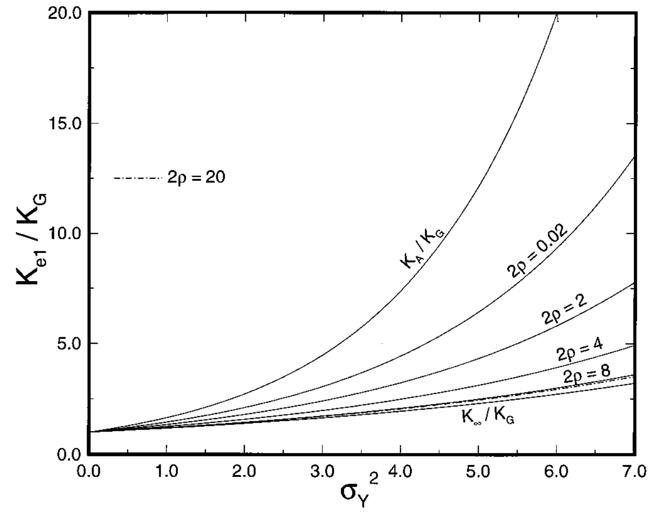

Figure 4. Generalized K_{e1}/K_G versus σ_Y^2 and 2ρ for statistically isotropic media.

Figure 3 shows how the linearized ratio K_{e1}/K_G varies with σ_Y^2, for various values of 2ρ, according to (18). Figure 4 depicts similar variations of the generalized ratio K_{e1}/K_G according to (26). Figure 5 is a cross plot of Figure 4, showing how the generalized ratio varies with 2ρ for various values of σ_Y^2. It is clear from Figures 3 and 4 that, for decreasing 2ρ, K_{e1} tends to K_A for all σ_Y^2. At $2\rho = 8$, K_{e1} is only slightly larger than the conjectured value for an infinite domain, $K_\infty = K_G \exp(\sigma_Y^2/6)$.

Figure 5 shows how fast the asymptote of K_{e1}/K_G is approached for a given σ_Y^2 as the separation distance grows. The effect of the Dirichlet boundaries is seen to diminish rapidly as the characteristic length of the domain increases toward eight integral scales of Y ($2\rho \to 8$), and much more slowly thereafter. The larger is σ_Y^2, the steeper is the descent of K_{e1} toward its asymptote.

Our results compare both qualitatively and quantitatively with those of other investigators. For example, *Desbarats and Dimitrakopoulos* [1990, Figure 7] and *Desbarats* [1992, Figure 1] found that for small σ_Y^2 their empirical spatially averaged equivalent hydraulic conductivity is close to K_A when the ratio between length of the field and integral scale of $Y(\mathbf{x})$ is 0.01 and 0.1, respectively. They also found that their equivalent effective conductivity is virtually free of boundary influences when the above ratio exceeds 10. Our results for K_{e1}/K_G in Table 1, computed through (26), show a very good agreement with Figure 1 of Desbarats for cubic fields of sides greater than or equal to 4.

Table 1. Effective Hydraulic Conductivity in Bounded Domains

2ρ	D	K_{e1}/K_G (Equation (18))	K_{e1}/K_G (Equation (26))
0.02	0.1275	1.3725	1.4513
2	0.2068	1.2932	1.3407
4	0.2722	1.2278	1.2558
8	0.3171	1.1829	1.2007
20	0.3205	1.1795	1.1966
100	0.3252	1.1748	1.1910

We mentioned in our introduction that *Kitanidis* [1990] had derived intuitively a first-order expression for the effective hydraulic conductivity of a lognormal $K(\mathbf{x})$ field in a finite flow domain. *Dykaar and Kitanidis* [1992, Figure 1] presented corresponding values of normalized effective conductivity, versus the ratio between domain length and integral scale, for a statistically isotropic $Y(\mathbf{x})$ field with $\sigma_Y^2 = 1$ and either exponential or Gaussian autocovariance. In two dimensions the normalization is done with respect to K_G; in three dimensions with respect to $K_\infty = K_G(1 + 1/6)$. We compare in Table 2 their three-dimensional results, as read by us off the graph labeled three-dimensional exponential covariance in Figure 1 of *Dykaar and Kitanidis* [1992], with our first-order results in Table 1. The difference between their results and ours is seen to be at most 1.2%. Some of this difference may be due to the fact that we place our lateral boundary at infinity. The difference is, in our view, sufficiently small to conclude that the intuitive solution of *Kitanidis* [1990] adequately captures the effect of boundaries on effective hydraulic conductivity in this case.

Conclusions

The following major conclusions can be drawn from this paper.

1. The theory of *Neuman and Orr* [1993] accounts rigorously for boundary effects on steady state flow in randomly heterogeneous porous media through an appropriate Green's function. We developed a first-order approximation of their expres-

Table 2. Comparison With *Dykaar and Kitanidis* [1992, Figure 1]

	Dykaar and Kitanidis		This Paper (Equation (18))	
2ρ	K_{e1}/K_∞	K_{e1}/K_G	K_{e1}/K_G	Percent Difference
0	1.270	1.482	1.500	1.2
2	1.100	1.283	1.2932	0.8
4	1.050	1.225	1.2278	0.2
8	1.015	1.184	1.1829	0.1

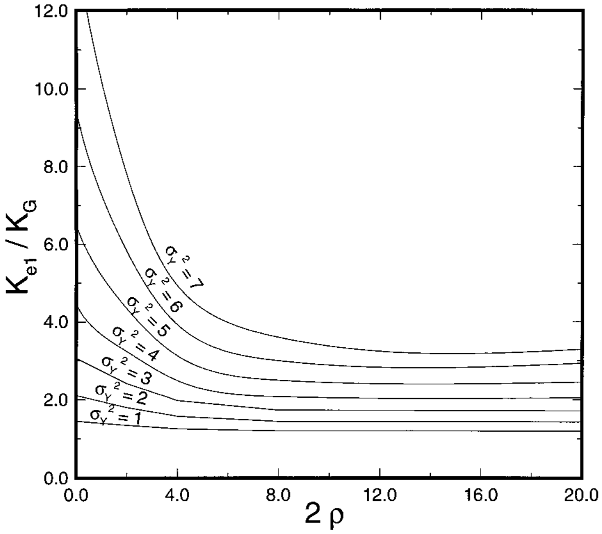

Figure 5. Generalized K_{e1}/K_G versus 2ρ and σ_Y^2 for statistically isotropic media.

sion for the effective hydraulic conductivity, K_e, of a domain bounded by two parallel boundaries on which head is constant in the mean. Our solution applies to a lognormal, statistically homogeneous and anisotropic conductivity field with an exponential covariance and principal axes normal and parallel to the mean Dirichlet boundaries. It is valid when the variance, σ_Y^2, of the log conductivity, $Y = \ln K$, is small, and lateral boundaries (across which flow is zero in the mean) are separated by at least a few integral scales of Y. The effect of large σ_Y^2 is included via the *Landau and Lifshitz* [1960] conjecture.

2. The effective hydraulic conductivity tends to the arithmetic mean, K_A, in the limit as the mean Dirichlet boundaries approach each other. This was anticipated on theoretical grounds by *Neuman and Orr* [1993]. As these boundaries separate, the effective conductivity tends to a known result, K_∞, corresponding to infinite media.

3. In statistically isotropic media, the effective conductivity diminishes rapidly from K_A toward its asymptotic value, K_∞, as the separation distance between the Dirichlet boundaries approaches eight integral scales. The larger is σ_Y^2, the steeper is this descent of K_e toward its asymptote. At larger separation distances, the rate at which K_e approaches K_∞ is relatively slow. When $\sigma_Y^2 = 1$, K_e exceeds K_∞ by 1.6% at a relative separation distance of 8. When $\sigma_Y^2 = 7$, K_e exceeds K_∞ by about 9% at a relative separation distance of 20.

4. The effective hydraulic conductivity we speak of is the ensemble average (expectation) of values one would obtain by performing measurements on porous blocks selected at random from a statistically homogeneous medium. When the block size is very small, the mean of the measurements is K_A and the variance of their natural logarithms is σ_Y^2. As the block size increases, the mean of the measurements diminishes toward the asymptote of K_e, and their variance diminishes toward zero. If one is willing to consider K_e as being sufficiently close to its asymptote, and the measurement variance as being sufficiently close to zero, when the block exceeds a certain size, then this block size takes the role of a representative elementary volume (REV) with respect to hydraulic conductivity. It is clear that the definition of an REV is subjective. Our results suggest to us that, in media such as those considered here, a separation distance of at least eight integral scales between the Dirichlet boundaries may be necessary for this distance to qualify as an REV scale for practical purposes. The more heterogeneous is the medium (the larger is σ_Y^2), the larger is its REV scale. Whereas blocks of size equal to or exceeding the REV scale may be assigned a deterministic hydraulic conductivity equal to K_∞, blocks of sub-REV scale must be assigned random conductivity values with mean K_e and be treated stochastically.

5. Our results agree with those from a spatially power-averaged expression by *Desbarats* [1992]. They also compare well with results published for $\sigma_Y^2 = 1$ by *Dykaar and Kitanidis* [1992]. These authors used a solution developed intuitively by *Kitanidis* [1990] in analogy to one he had derived for nonrandom periodic media. We conclude that his intuitive solution captures adequately the effect of boundaries on the effective hydraulic conductivity of random media.

Appendix A

The Green's function in (17) is obtained via the method of images [*Stakgold*, 1979],

$$G(\boldsymbol{\chi}, \mathbf{x}, L) = \frac{1}{4\pi} \cdot \sum_{j=-\infty}^{+\infty} \frac{(-1)^j}{\{[\chi_1 - 2jL - (-1)^j x_1]^2 + (\chi_2 - x_2)^2 + (\chi_3 - x_3)^2\}^{1/2}} \cdot \tag{A1}$$

$C_Y(\boldsymbol{\chi} - \mathbf{x})$ is taken to be the anisotropic exponential function

$$C_Y(\boldsymbol{\eta}) = \sigma_Y^2 \exp\,[-(\eta_i^2/\lambda_i^2)^{1/2}], \tag{A2}$$

where $\boldsymbol{\eta} = \boldsymbol{\chi} - \mathbf{x}$, λ_i are principal integral scales, and repeated indices imply summation.

Let

$$\mathcal{H} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_Y(\boldsymbol{\eta})\, \frac{\partial^2}{\partial x_1 \partial \chi_1}\, G(\boldsymbol{\chi}, \mathbf{x})\; d\chi_2\, d\chi_3 \tag{A3}$$

$$\Lambda(-k_2, -k_3; \chi_1, \mathbf{x}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_Y(\boldsymbol{\chi}'; \chi_1, \mathbf{x}) \cdot \exp\,(-i\mathbf{k}' \cdot \boldsymbol{\chi}')\; d\boldsymbol{\chi}' \tag{A4}$$

$$\mathcal{F}(k_2, k_3; \chi_1, \mathbf{x}, L) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cdot \frac{\partial^2}{\partial x_1 \partial \chi_1}\, G(\boldsymbol{\chi}'; \chi_1, \mathbf{x}, L)\, \exp\,(i\mathbf{k}' \cdot \boldsymbol{\chi}')\; d\boldsymbol{\chi}' \tag{A5}$$

where $\mathbf{k}' = (k_2,\ k_3)^T$ is wave number vector and $\boldsymbol{\chi}' = (\chi_2,\ \chi_3)^T$. With the aid of Parseval's identity [*Paleologos*, 1994], $\mathcal{H}$ is transformed into

$$\mathcal{H}(\chi_1, x_1, L) = \int_{\mathbf{k}'} \Lambda \mathcal{F}\; d\mathbf{k}'. \tag{A6}$$

Substituting (A2) into (A4) and setting $\mathbf{x}' = (x_2,\ x_3)^T$, $\boldsymbol{\chi}^* = (\eta_2/\lambda_2,\ \eta_3/\lambda_3)^T$, and $\gamma = |\eta_1|/\lambda_1$ yields

$$\Lambda = \exp(-i\mathbf{k}'\cdot\mathbf{x}')\,\frac{\lambda_2\lambda_3}{2\pi}\,\sigma_Y^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left[-(\gamma^2+\chi^{*2})^{1/2}\right]$$

$$\cdot\exp(-i\mathbf{k}^*\cdot\boldsymbol{\chi}^*)\,d\chi_2^*\,d\chi_3^*. \tag{A7}$$

Here $\mathbf{k}^* = \mathbf{L}'\mathbf{k}'$, where $\mathbf{L}'$ is a 2×2 diagonal matrix with elements λ_2 and λ_3, and norms of vectors are represented by unaccented versions of their boldface symbols. Transforming (A7) into polar coordinates so that $\mathbf{k}^*$ is aligned with $\boldsymbol{\chi}^* = (1, 0)^T$, and evaluating the angular integral [*Gradshteyn and Ryzhik*, 1980, p. 482, equation [3.915.5]], yields

$$\Lambda = \exp(-i\mathbf{k}'\cdot\mathbf{x}')\lambda_2\lambda_3\sigma_Y^2\int_{\chi^*=0}^{+\infty}\chi^*\exp\left[-(\gamma^2+\chi^{*2})^{1/2}\right]$$

$$\cdot J_0(k^*\chi^*)\,d\chi^*, \tag{A8}$$

where $J_0(k^*\chi^*)$ is the Bessel function of order zero. Combining the solutions for $\chi_1 \neq x_1$ (or $\gamma \neq 0$) [*Erdelyi et al.*, 1954, p. 9, equation [23]] and $\chi_1 = x_1$ (or $\gamma = 0$) [*Erdelyi et al.*, 1954, p. 9, equation [20]] we obtain (for all χ_1)

$$\Lambda(\mathbf{k}';\chi_1,\mathbf{x}) = \exp(-i\mathbf{k}'\cdot\mathbf{x}')\lambda_2\lambda_3\sigma_Y^2(1+k^{*2})^{-3/2}$$

$$\cdot\left[1+\gamma\sqrt{1+k^{*2}}\right]\exp\left[-\gamma(1+k^{*2})^{1/2}\right]. \tag{A9}$$

Next we note that the two-dimensional Fourier transform (FT) of G, $\hat{g}$, satisfies

$$\frac{\partial^2}{\partial\chi_1^2}\hat{g} - (k_2^2+k_3^2)\hat{g} = -\frac{1}{2\pi}\,\delta(\chi_1-x_1)\exp(i\mathbf{k}'\cdot\mathbf{x}') \tag{A10}$$

subject to homogeneous boundary conditions. Hence [*Greenberg*, 1971]

$$\hat{g} = \frac{c}{2k'}\sum_{j=-\infty}^{+\infty}(-1)^j\exp\left[-k'|\chi_1-2jL-(-1)^jx_1|\right] \tag{A11}$$

where $c = (1/2\pi)\exp(i\mathbf{k}'\cdot\mathbf{x}')$, and then the second mixed derivative of $\hat{g}$ is given by

$$\hat{g}_{x_1\chi_1} = \frac{\partial^2\hat{g}}{\partial x_1\partial\chi_1} = -\frac{ck'}{2}\sum_{j=-\infty}^{+\infty}\exp\left[-k'|\chi_1-2jL-(-1)^jx_1|\right]$$

$$+\,c\sum_{j=-\infty}^{+\infty}\delta(\chi_1-2jL-(-1)^jx_1)$$

$$\cdot\exp\left[-k'|\chi_1-2jL-(-1)^jx_1|\right]. \tag{A12}$$

Here we have used the relationships

$$\frac{\partial}{\partial x_1}\operatorname{sgn}\left[\chi_1-2jL-(-1)^jx_1\right]$$

$$= -2(-1)^j\delta[\chi_1-2jL-(-1)^jx_1] \tag{A13}$$

and

$$\operatorname{sgn}^2\left[\chi_1-2jL-(-1)^jx_1\right] = 1. \tag{A14}$$

This and the properties of the delta function [*Paleologos*, 1994] allow us to rewrite (A5) as

$$\mathscr{F}(\mathbf{k}';\chi_1,\mathbf{x}) = \frac{\exp(i\mathbf{k}'\cdot\mathbf{x}')}{2\pi}\Bigg(\delta(\chi_1-x_1)$$

$$-\frac{k'}{2}\sum_{j=-\infty}^{+\infty}\exp\left[-k'|\chi_1-2jL-(-1)^jx_1|\right]\Bigg). \tag{A15}$$

Substituting (A9) and (A15) into (A6) allows us to rewrite $\mathscr{H}$ in Fourier space,

$$\mathscr{H}(\chi_1,x_1,L) = \frac{1}{2\pi}\,\delta(\eta_1)\int_{\mathbf{k}'}\Lambda'(\mathbf{k}';\eta_1)\,d\mathbf{k}' - \frac{1}{4\pi}$$

$$\cdot\sum_{j=-\infty}^{+\infty}\int_{\mathbf{k}'}k'\exp\left[-k'|\chi_1-2jL\right.$$

$$\left.-(-1)^jx_1|\right]\Lambda'(\mathbf{k}';\eta_1)\,d\mathbf{k}', \tag{A16}$$

where $\Lambda'(\mathbf{k}';\eta_1) = \Lambda(\mathbf{k}';\chi_1,\mathbf{x})\exp(i\mathbf{k}'\cdot\mathbf{x}')$.

We note from (17) and (18) that the domain integral over Ω, D and the boundary integral over Γ_{D1}, B can be expressed using (A3) (or (A16)) as

$$D = (1/\sigma_Y^2)\int_{-L}^{L}\mathscr{H}\,d\chi_1 \tag{A17}$$

The definition of the inverse FT of $C_Y(\boldsymbol{\chi}-\mathbf{x})$ [*Dagan*, 1989] and expression (A2) allow rewriting the first term of (A16) as

$$\delta(\eta_1)\left(\frac{1}{2\pi}\int_{\mathbf{k}'}\Lambda'(\mathbf{k}';\eta_1)\,d\mathbf{k}'\right) = \delta(\eta_1)\sigma_Y^2\exp(-\eta_1/\lambda_1). \tag{A18}$$

Substituting (A16), (A18), and (A9) into (A17) and transforming into polar coordinates, we obtain

$$D = 1-\frac{\lambda_2\lambda_3}{4\pi}\sum_{j=-\infty}^{+\infty}\int_{\chi_1=-L}^{+L}\int_{k'=0}^{\infty}\int_{\theta=0}^{2\pi}k'^2(1+k^{*2})^{-3/2}$$

$$\cdot\left[1+\gamma(1+k^{*2})^{1/2}\right]\exp\left[-\gamma(1+k^{*2})^{1/2}\right]$$

$$\cdot\exp\left[-k'|\chi_1-2jL-(-1)^jx_1|\right]dk'\,d\theta\,d\chi_1. \tag{A19}$$

Upon expressing $\mathbf{k}'$ in polar coordinates, $\mathbf{k}' = k'\boldsymbol{\sigma} = k'(\cos\theta, \sin\theta)^T$, the norm of $\mathbf{k}^*$ becomes

$$k^{*2} = k'^2\|\mathbf{L}'\boldsymbol{\sigma}\|^2 = k'^2(\boldsymbol{\sigma}^T\mathbf{L}'^2\boldsymbol{\sigma})$$

$$= k'^2\lambda_1^2R^2, \tag{A20}$$

where the scalar R^2 is given by $R^2 = \varepsilon_2^2\cos^2\theta + \varepsilon_3^2\sin^2\theta$, and $\varepsilon_2 = \lambda_2/\lambda_1$ and $\varepsilon_3 = \lambda_3/\lambda_1$. This and the transformations $\rho = L/\lambda_1$, $\xi = \chi_1/L$, and $r = \lambda_1k'$ lead, with $x_1 = 0$, directly to (19).

As $\rho\to\infty$, the domain integral in statistically isotropic media is given by

$$D = 1-\frac{1}{2}\int_0^{\infty}\frac{r^2}{(1+r^2)^{3/2}}\int_{-\infty}^{\infty}(1+A|z|)e^{-(A+r)|z|}\,dz\,dr, \tag{A21}$$

where we have set $z = \rho\xi$, $A = (1 + r^2)^{1/2}$, and let $\rho \to \infty$ in the limits of integration. After straightforward integration, D reduces to

$$D = 1 - \int_0^\infty \frac{r^2}{(1 + r^2)^{3/2}} \frac{1}{A + r}\, dr - \int_0^\infty \frac{r^2}{(1 + r^2)^{3/2}} \frac{A}{(A + r)^2}\, dr. \quad \text{(A22)}$$

Following some tedious algebra, D reduces to 1/3 in the limit as $\rho \to \infty$ [*Paleologos*, 1994] and (26) yields (27).

Appendix B

The solution to (A10), subject to homogeneous boundary conditions, can alternatively be derived through the method of eigenfunction expansion [*Zauderer*, 1983]. Then the second mixed derivative of $\hat{g}$ is given, for $\mathbf{x} = 0$, by

$$\mathcal{F}(k', \chi_1) = \frac{\pi}{2L} \sum_{n=1}^{\infty} \frac{n^2}{n^2\pi^2 + k'^2L^2} \cos \frac{n\pi\chi_1}{L}. \quad \text{(B1)}$$

Substituting (A9) and (B1) into (A6) allows us to rewrite $\mathcal{H}$ in Fourier space,

$$\mathcal{H}(\chi_1, \mathbf{k}') = \frac{\pi}{2L} \sum_{n=1}^{\infty} n^2 \cos \frac{n\pi\chi_1}{L} \int_{\mathbf{k}'} \frac{\Lambda'(\mathbf{k}'; \chi_1)}{n^2\pi^2 + k'^2L^2}\, d\mathbf{k}', \quad \text{(B2)}$$

where $\Lambda'(\mathbf{k}'; \chi_1)$ is given in (A9). The domain integral in (A17) is given now by

$$D = \frac{\pi\lambda_2\lambda_3}{2L} \sum_{n=1}^{\infty} n^2 \int_{\mathbf{k}'} \frac{(1 + k^{*2})^{-3/2}}{n^2\pi^2 + k'^2L^2} \int_{-L}^{L} \cos \frac{n\pi\chi_1}{L} \cdot \left[1 + \frac{|\chi_1|}{\lambda_1}(1 + k^{*2})^{1/2}\right] \cdot \exp\left(-\frac{|\chi_1|}{\lambda_1}(1 + k^{*2})^{1/2}\right) d\chi_1\, d\mathbf{k}'. \quad \text{(B3)}$$

Evaluating the integral with respect to χ_1 [*Gradshteyn and Ryzhik*, 1980, p. 196, equation [2.663.3] and p. 198, equation [2.667.6]] leads directly to (21). Since the integrand of (21) is not singular at the limits of integration, we can invoke the Dominated Convergence theorem [*Ray*, 1988, p. 202] to interchange the order of limit and integration and thus obtain $\lim_{\rho\to 0} D = 0$.

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