Estimation and Control of Systems with Multiplicative Noise via Linear Matrix Inequalities

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***Abstract*— This paper deals with the estimation and control problem constrained to multiplicative noise for linear discrete-time systems. First, the design of the state feedback controller, such that the closed loop system is mean square stable, is presented. Second, the sufficient conditions for the existence of the state estimator are also provided; these conditions are expressed in terms of linear matrix inequalities (LMIs), and the parametrization of all admissible solutions is addressed. Finally an estimator design approach is formulated using LMIs, and the performance of the estimator is examined by means of the numerical examples.**

I. Introduction

The filtering and control problem for the systems with multiplicative noise has recently received a great deal of attentions, and has found applications in many fields of sciences and engineering. Different from the traditional additive noise, multiplicative noise is more practical, since it allows the statistical description of the multiplicative noise be not known *a prior* but depend on the control and state solution. Such models are found in many physical systems, such as signal processing systems [2], biological movement systems [4], [13], [14], [15], [16] and aerospace engineering systems.

One important benefit of the multiplicative noise in a linear control problem is that the controllers for multiplicative systems appear robust. This is in contrast to LQG theory, where the minimum variance occurs at infinite control gain, which renders the solution of problem unstable. Therefore, the multiplicative noise system has a significant effect on the robustness of the overall control system [3], [6], [7], [12].

So far, many researchers have been working on various kinds of analysis of filtering and control in systems with multiplicative noise, and there have been several approaches for dealing with this problem, including the linear matrix inequality approach [2], [3], the Riccati difference equation method [9], [14], [16], and the game theoretic approach [8]. Since this model reflects more realistic properties in engineering, a complete theory which includes control and estimation should be developed.

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This paper focuses on the study of estimation problem for multiplicative noise system using LMIs. Reference [3] devised an LMI approach to the robust control of stochastic systems with multiplicative noise. In [5] we study the state estimation with signal-dependent noise model for the continuous time systems. The contribution of this paper is to propose an LMI method to cope with the estimation problem for the discrete time systems with multiplicative noise. We shall show that a mild additional constraint for scaling will make the problem convex.

The paper is organized as follows. In section II the filtering and control problem for discrete time system subject to multiplicative noise is formulated. The design of the state feedback controller is developed in section III. In section IV, the sufficient conditions for the existence of the state estimator are given, and an algorithm for the filtering design is derived which guarantees the performance requirement. Section V presents two numerical examples, and some concluding remarks are drawn in section VI.

Throughout the paper, the notation used is fairly standard. The transpose of a real matrix A is denoted by A^T; for symmetric matrix, the standard notation > 0 (≥ 0) is used to denote positive definite matrix (positive semi-definite matrix), and the notation < 0 (≤ 0) is used to denote negative definite matrix (negative semi-definite matrix); ε_∞ is used to denote $lim_{t\to\infty}\varepsilon(\cdot)$, where $\varepsilon(\cdot)$ denotes the expectation. We also denote by $B^\perp$ any matric such that $B^\perp B = 0$.

II. System Model and Problem Formulation

Consider the following discrete time system with state space representation

$$\begin{aligned} x_{k+1} &= (A_k + A_{s,k}\eta_k)x_k + (B_k + B_{s,k}\varepsilon_k)u_k + D_k\omega_k, \\ z_k &= (H_k + H_{s,k}\zeta_k)x_k + v_k, \\ y_k &= C_k x_k, \end{aligned} \tag{1}$$

where $x_k \in R^{n_x}$ is the state variable, $z_k \in R^{n_z}$ is the measurement output; $y_k \in R^{n_y}$ is the output of interest for performance evaluation; $\omega_k \in R^{n_w}$ and $v_k \in R^{n_y}$ are the process and measurement noises; η_k, ε_k and ζ_k are the multiplicative noises; $A_k, A_{s,k}, B_k, B_{s,k}, D_k, H_k, H_{s,k}$ and C_k are constant matrices which have proper dimensions. The independent random variable ω_k, v_k, η_k, ε_k and ζ_k have zero-mean gaussian distributions with covariances W, V, and $\Omega^\eta = \Omega^\varepsilon = \Omega^\zeta = I$ respectively.

Here we consider the following problems. First, we look for the state feedback control law $u_k = Kx_k$ such that the closed loop system (1) is mean square stable. And we will determine if there exists a control gain K such that $\varepsilon_\infty\{y_k^T y_k\} < \mu$ is satisfied for the given μ.

Second, we consider (1) where B_k and $B_{s,k}$ are zero. The objective is to design a linear filter with the state space representation

$$\hat{x}_{k+1} = A_k\hat{x}_k + F(z_k - H_k\hat{x}_k), \tag{2}$$
$$\hat{y}_k = C_k\hat{x}_k, \tag{3}$$

where $\hat{x}_k$ is the unbiased estimate of the state x_k, F is the filter gain to be determined such that $(A_k - FH_k)$ is asymptotically stable, and the estimation error has covariance less than a specified matrix. The estimation error is $\tilde{x}_k = x_k - \hat{x}_k$, and the estimation error system is given by

$$\tilde{x}_{k+1} = (A_k - FH_k)\tilde{x}_k + \bar{\omega}_k, \tag{4}$$
$$\tilde{y}_k = C_k\tilde{x}_k, \tag{5}$$

where

$$\bar{\omega}_k = D_k\omega_k + A_{s,k}\eta_k x_k - FH_{s,k}\zeta_k x_k - Fv_k, \tag{6}$$

and $\tilde{y}_k$ denotes the estimation error of particular interests. In this paper we will explore the existence condition of the state estimator. We will be able to provide the sufficient conditions for the existence of the state estimator based on Linear Matrix Inequalities (LMIs). The key idea of this filtering problem is to find the estimate $\hat{x}_k$ of x_k such that the performance criterion $\varepsilon_\infty\{\tilde{y}_k\tilde{y}_k^T\} < \Omega$ is satisfied for the given Ω.

III. State-feedback Controller Design

Consider the system (1) where z_k is excluded, the state-feedback control law is the form $u_k = Kx_k$, now the closed loop system is the following

$$\begin{aligned} x_{k+1} &= (A_k + B_kK)x_k + (A_{s,k}\eta_k + B_{s,k}K\varepsilon_k)x_k + D_k\omega_k, \\ y_k &= C_kx_k. \end{aligned} \tag{7}$$

We first find the state feedback control gain K such that the closed loop system is mean square stable. Assuming that the upper bound of state covariance matrix associated with system (7) exists, which is described as $X \geq \varepsilon_\infty\{x_kx_k^T\}$, it should satisfy the following Lyapunov inequality

$$\begin{aligned} &(A_k + B_kK)X(A_k + B_kK)^T - X + A_{s,k}XA_{s,k}^T \\ &+ B_{s,k}KXK^TB_{s,k}^T < 0. \end{aligned} \tag{8}$$

Theorem 1: There exists a state feedback controller gain K such that the closed loop system (7) is mean square stable if, and only if, there exist a symmetric positive definite matrix X and a matrix L such that

$$\begin{pmatrix} X & A_kX + B_kL & A_{s,k}X & B_{s,k}L \\ XA_k^T + L^TB_k^T & X & 0 & 0 \\ XA_{s,k}^T & 0 & X & 0 \\ L^TB_{s,k}^T & 0 & 0 & X \end{pmatrix} > 0. \tag{9}$$

And the corresponding feedback control gain is given by $K = LX^{-1}$.

Proof: With the change of variable $L := KX$, (8) can be written as

$$\begin{aligned} &(A_kX + B_kL)X^{-1}(A_k + B_kL)^T - X + A_{s,k}XA_{s,k}^T \\ &+ B_{s,k}LX^{-1}L^TB_{s,k}^T < 0. \end{aligned}$$

Utilizing the Schur complement formula, the above inequality can be immediately written as (9). ■

Theorem 2: For a given scalar $\mu > 0$, if there exist a symmetric positive definite matrix X and a matrix L such that (9) is satisfied and

$$trace[C_kXC_k^T] < \mu, \tag{10}$$

then there exists a state feedback controller gain K such that the closed loop system (7) is mean square stable and $\varepsilon_\infty\{y_k^Ty_k\} < \mu$. And the corresponding feedback control gain is given by $K = LX^{-1}$.

Proof: The result follows from Theorem 1 and

$$\begin{aligned} \varepsilon_\infty\{y_k^Ty_k\} &= trace[C\varepsilon_\infty\{x_kx_k^T\}C^T] \\ &< trace[C_kXC_k^T] < \mu, \end{aligned}$$

which concludes the proof. ■

IV. Filter Design

A. Existence Condition

To solve the filter design, we will consider the augmented adjoint system. Combining the filter (2) and the estimation error dynamics (4), it yields

$$\mathbf{x}_{k+1} = \mathcal{A}\mathbf{x}_k + \mathcal{N}\eta_k\mathbf{x}_k + \mathcal{H}\zeta_k\mathbf{x}_k + \mathcal{D}\mathbf{w}_k, \tag{11}$$

where

$$\mathbf{x_k} = \begin{pmatrix} \tilde{x}_k \\ \hat{x}_k \end{pmatrix}, \qquad \mathbf{w}_k = \begin{pmatrix} \omega_k \\ v_k \end{pmatrix}, \tag{12}$$
$$\mathcal{A} = \begin{pmatrix} A_k - FH_k & 0 \\ FH_k & A_k \end{pmatrix} = \mathcal{A}_0 + \mathcal{B}_0F\mathcal{C}_0, \tag{13}$$
$$\mathcal{N} = \begin{pmatrix} A_{s,k} & A_{s,k} \\ 0 & 0 \end{pmatrix}, \tag{14}$$
$$\mathcal{H} = \begin{pmatrix} -FH_{s,k} & -FH_{s,k} \\ FH_{s,k} & FH_{s,k} \end{pmatrix} = \mathcal{B}_0FG_0, \tag{15}$$
$$\mathcal{D} = \begin{pmatrix} D_k & -F \\ 0 & F \end{pmatrix} = \mathcal{D}_0 + \mathcal{B}_0FE_0, \tag{16}$$
$$\mathcal{A}_0 = \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix}, \qquad \mathcal{B}_0 = \begin{pmatrix} -I \\ I \end{pmatrix}, \tag{17}$$
$$\mathcal{C}_0 = \begin{pmatrix} H_k & 0 \end{pmatrix}, \qquad \mathcal{D}_0 = \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix}, \tag{18}$$
$$G_0 = \begin{pmatrix} H_{s,k} & H_{s,k} \end{pmatrix}, \quad E_0 = \begin{pmatrix} 0 & I \end{pmatrix}. \tag{19}$$

We start by defining the upper bound of the state covariance matrix of system (11) as

$$\mathcal{X} \geq \varepsilon_\infty\{\mathbf{x}_k\mathbf{x}_k^T\}, \tag{20}$$

if it exists, it should satisfy the following Lyapunov inequality:

$$0 > \mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{H}\mathcal{X}\mathcal{H}^T + \mathcal{D}\mathcal{W}\mathcal{D}^T, \tag{21}$$

where $\mathcal{W} = \begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix}$ is symmetric and positive definite. Substitution of (13), (15)-(19) into the above inequality, yields

$$\begin{aligned} 0 \quad > \quad & (\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)\mathcal{X}(\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)^T - \mathcal{X} \\ & + \mathcal{N}\mathcal{X}\mathcal{N}^T + (\mathcal{B}_0 F G_0)\mathcal{X}(\mathcal{B}_0 F G_0)^T \\ & + (\mathcal{D}_0 + \mathcal{B}_0 F E_0)\mathcal{W}(\mathcal{D}_0 + \mathcal{B}_0 F E_0)^T. \end{aligned} \quad (22)$$

Lemma 1: The inequality (22) can be rewritten in a form

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0, \quad (23)$$

where

$$\Theta = \begin{pmatrix} -\mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T & 0 & \mathcal{A}_0\mathcal{X} & \mathcal{D}_0\mathcal{W} \\ 0 & -\mathcal{X} & 0 & 0 \\ \mathcal{X}\mathcal{A}_0^T & 0 & -\mathcal{X} & 0 \\ \mathcal{W}\mathcal{D}_0^T & 0 & 0 & -\mathcal{W} \end{pmatrix}, \quad (24)$$

$$\Gamma = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (25)$$

$$\Lambda = \begin{pmatrix} 0 & G_0\mathcal{X} & \mathcal{C}_0\mathcal{X} & E_0\mathcal{W} \end{pmatrix}. \quad (26)$$

Proof: We start by using the Schur complement, the inequality (22) can be written as

$$\begin{pmatrix} -\mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T & (\mathcal{B}_0 F G_0)\mathcal{X} & \mathcal{A}\mathcal{X} & \mathcal{D}\mathcal{W} \\ \mathcal{X}(\mathcal{B}_0 F G_0)^T & -\mathcal{X} & 0 & 0 \\ \mathcal{X}\mathcal{A}^T & 0 & -\mathcal{X} & 0 \\ \mathcal{W}\mathcal{D}^T & 0 & 0 & -\mathcal{W} \end{pmatrix} < 0,$$

where $\mathcal{A}$ is defined in (13), $\mathcal{D}$ is defined in (16). Utilizing the structure of the above matrix and substituting (24) into the above inequality, it obtains

$$\begin{aligned} \Theta + & \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} F \begin{pmatrix} 0 & G_0\mathcal{X} & \mathcal{C}_0\mathcal{X} & E_0\mathcal{W} \end{pmatrix} \\ + & \left(\begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} F \begin{pmatrix} 0 & G_0\mathcal{X} & \mathcal{C}_0\mathcal{X} & E_0\mathcal{W} \end{pmatrix} \right)^T < 0, \end{aligned}$$

With the use of Γ and Λ given in (25)-(26), the above condition can be equivalently written as (23). ■

It is important to notice that the filtering design problem has been converted into looking for the solution of F in the inequality (23). In order to find the existence conditions of the state estimator and the parametrization of all the solutions, the following lemma will be introduced from [11].

Lemma 2 (Finsler's Lemma): Let $x \in \mathbf{R}^n$, $\Theta = \Theta^T \in \mathbf{R}^{n\times n}$, $\Gamma \in \mathbf{R}^{n\times m}$ and $\Lambda \in \mathbf{R}^{k\times n}$. Let $\Gamma^\perp$ be any matrix such that $\Gamma^\perp\Gamma = 0$. Let $\Lambda^{T^\perp}$ be any matrix such that $\Lambda^{T^\perp}\Lambda^T = 0$. The following statements are equivalent:

$$(i) \quad x^T\Theta x < 0, \ \forall \Gamma^T x = 0, \Lambda x = 0 \ x \neq 0. \quad (27)$$

$$(ii) \quad \Gamma^\perp \Theta \Gamma^{\perp^T} < 0, \quad (28)$$

$$\Lambda^{T^\perp} \Theta \Lambda^{T^{\perp^T}} < 0. \quad (29)$$

$$(iii) \quad \exists \mu_1, \mu_2 \in \mathbf{R} : \Theta - \mu_1 \Gamma\Gamma^T < 0, \quad (30)$$

$$\Theta - \mu_2 \Lambda^T \Lambda < 0. \quad (31)$$

$$(iv) \quad \exists F \in \mathbf{R}^{m\times k} : \Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0. \quad (32)$$

Note that Finsler's Lemma can be applied to obtain LMI formulations in control and estimation theory. By applying the Finsler's lemma, we obtain the following theorem.

Theorem 3: The condition (28) and (29) are equivalent to the following statement: there exist symmetric positive definite matrices $\mathcal{X}, P \in \mathbf{R}^{2n_x\times 2n_x}$ that satisfy

$$\mathcal{X}P = I, \quad (33)$$

$$\mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X}\mathcal{A}_0^T - \mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{W}\mathcal{D}_0^T)\mathcal{B}_0^{\perp^T} < 0, \quad (34)$$

$$\begin{pmatrix} 0 \\ G_0^T \\ \mathcal{C}_0^T \\ E_0^T \end{pmatrix}^\perp \begin{pmatrix} -P + P\mathcal{N}\mathcal{X}\mathcal{N}^T P & 0 & P\mathcal{A}_0 & P\mathcal{D}_0 \\ 0 & -P & 0 & 0 \\ \mathcal{A}_0^T P & 0 & -P & 0 \\ \mathcal{D}_0^T P & 0 & 0 & -\mathcal{W}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ G_0^T \\ \mathcal{C}_0^T \\ E_0^T \end{pmatrix}^{\perp^T} < 0. \quad (35)$$

Proof: According to Lemma 1 and Finsler's Lemma, we substitute the following matrix

$$\Gamma^\perp = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^\perp = \begin{pmatrix} \mathcal{B}_0^\perp & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and (24) into (28), it yields

$$\begin{pmatrix} \mathcal{B}_0^\perp(-\mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T)\mathcal{B}_0^{\perp^T} & 0 & \mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X}) & \mathcal{B}_0^\perp\mathcal{D}_0\mathcal{W} \\ 0 & -\mathcal{X} & 0 & 0 \\ \mathcal{X}\mathcal{A}_0^T\mathcal{B}_0^{\perp^T} & 0 & -\mathcal{X} & 0 \\ \mathcal{W}\mathcal{D}_0^T\mathcal{B}_0^{\perp^T} & 0 & 0 & -\mathcal{W} \end{pmatrix} < 0.$$

A Schur complement of this matrix is

$$\begin{pmatrix} \mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X}\mathcal{A}_0^T - \mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{W}\mathcal{D}_0^T)\mathcal{B}_0^{\perp^T} & 0 \\ 0 & -\mathcal{X} \end{pmatrix} < 0,$$

which is equivalent to (34) and $\mathcal{X} > 0$. Furthermore, noting that

$$\Lambda^{T^\perp} = \begin{pmatrix} 0 \\ G_0^T \\ \mathcal{C}_0^T \\ E_0^T \end{pmatrix}^\perp \begin{pmatrix} \mathcal{X}^{-1} & 0 & 0 & 0 \\ 0 & \mathcal{X}^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{X}^{-1} & 0 \\ 0 & 0 & 0 & \mathcal{W}^{-1} \end{pmatrix},$$

defining $\mathcal{X}^{-1} = P$, and substituting (24) and the above matrix into (29), (35) can be verified which completes the proof. ■

The previous theorem provides the existence condition for the state estimator, and the characterization given in Theorem 3 is necessary and sufficient. However, we introduce a nonconvex constraint $\mathcal{X}P = I$, which makes the problem more difficult to solve. The next theorem shows how to rewrite these conditions into convex constraints by using Finsler's Lemma again.

Theorem 4: There exists a state estimator gain F to solve (21) if there exist a symmetric matrix $P \in \mathbf{R}^{2n_x \times 2n_x}$ and $\mu_1 < 0, \mu_2 < 0 \in \mathbf{R}$ that satisfy

$$P > 0, \tag{36}$$

$$\begin{pmatrix} -P & P\mathcal{A}_0 & P\mathcal{N} & P\mathcal{D}_0 & P\mathcal{B}_0 \\ \mathcal{A}_0^T P & -P & 0 & 0 & 0 \\ \mathcal{N}^T P & 0 & -P & 0 & 0 \\ \mathcal{D}_0^T P & 0 & 0 & -\mathcal{W}^{-1} & 0 \\ \mathcal{B}_0^T P & 0 & 0 & 0 & \mu_1 I \end{pmatrix} < 0, \tag{37}$$

$$\begin{pmatrix} -P & 0 & P\mathcal{A}_0 & P\mathcal{D}_0 & 0 & P\mathcal{N} \\ 0 & -P & 0 & 0 & G_0^T & 0 \\ \mathcal{A}_0^T P & 0 & -P & 0 & \mathcal{C}_0^T & 0 \\ \mathcal{D}_0^T P & 0 & 0 & -\mathcal{W}^{-1} & E_0^T & 0 \\ 0 & G_0 & \mathcal{C}_0 & E_0 & \mu_2 I & 0 \\ \mathcal{N}^T P & 0 & 0 & 0 & 0 & -P \end{pmatrix} < 0. \tag{38}$$

Proof: The result follows from Theorem 3 and the Finsler's Lemma. We note that the inequality (34) holds if, and only if,

$$\mathcal{A}_0\mathcal{X}\mathcal{A}_0^T - \mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{W}\mathcal{D}_0^T - \mu_1\mathcal{B}_0\mathcal{B}_0^T < 0$$

holds for some $\mu_1 < 0$ by Finsler's Lemma. Applying the congruence transformation

$$\mathcal{X}^{-1}(\mathcal{A}_0\mathcal{X}\mathcal{A}_0^T - \mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{W}\mathcal{D}_0^T - \mu_1\mathcal{B}_0\mathcal{B}_0^T)\mathcal{X}^{-1} < 0,$$

with $P := \mathcal{X}^{-1} > 0$ and the Schur complement, (37) can be verified.

If the inequality (35) holds, it is equivalent to the existence of a $\mu_2 < 0$ such that

$$\begin{pmatrix} -P + P\mathcal{N}\mathcal{X}\mathcal{N}^T P & 0 & P\mathcal{A}_0 & P\mathcal{D}_0 \\ 0 & -P & 0 & 0 \\ \mathcal{A}_0^T P & 0 & -P & 0 \\ \mathcal{D}_0^T P & 0 & 0 & -\mathcal{W}^{-1} \end{pmatrix}$$

$$-\mu_2 \begin{pmatrix} 0 \\ G_0^T \\ \mathcal{C}_0^T \\ E_0^T \end{pmatrix} \begin{pmatrix} 0 & G_0 & \mathcal{C}_0 & E_0 \end{pmatrix} < 0.$$

Applying Schur complements twice, it obtains (38) which is the desired conclusion. ■

Since the inequality (37) and (38) are LMIs, the existence of a feasible solution for the state estimator is a convex problem which can be solved with the use of many available algorithms.

B. Filter Design

In the previous section, a sufficient LMI condition for checking the existence of state estimator has been given. Here it is dedicated to provide the conditions that guarantee the additional closed loop system performance. We will determine a state estimator F such that the performance criterion, $\varepsilon_\infty\{\tilde{y}_k\tilde{y}_k^T\} < \Omega$, is satisfied. The fundamental algorithm that enables us to solve the filtering problem is derived from the following theorem.

Theorem 5: There exists a state estimator gain F such that $\varepsilon_\infty\{\tilde{y}_k\tilde{y}_k^T\} < \Omega$ if there exist a positive definite symmetric matrix $P \in \mathbf{R}^{2n_x \times 2n_x}$ and $\mu_1 < 0, \mu_2 < 0 \in \mathbf{R}$ that satisfy (37), (38) and

$$\begin{pmatrix} \Omega & \bar{C}_k \\ \bar{C}_k^T & P \end{pmatrix} > 0, \tag{39}$$

where

$$\bar{C}_k = C_k \begin{bmatrix} I & 0 \end{bmatrix}. \tag{40}$$

All the solutions F are given by

$$F = -R^{-1}\Gamma^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + S^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2}, \tag{41}$$

where

$$S = R^{-1} - R^{-1}\Gamma^T[\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi]\Gamma R^{-1}. \tag{42}$$

L is an arbitrary matrix such that $\|L\| < 1$ and R is an arbitrary positive definite matrix such that

$$\Phi = (\Gamma R^{-1}\Gamma^T - \Theta)^{-1} > 0, \tag{43}$$

and

$$\Theta = \begin{pmatrix} -P^{-1} + \mathcal{N}P^{-1}\mathcal{N}^T & 0 & \mathcal{A}_0 P^{-1} & \mathcal{D}_0\mathcal{W} \\ 0 & -P^{-1} & 0 & 0 \\ P^{-1}\mathcal{A}_0^T & 0 & -P^{-1} & 0 \\ \mathcal{W}\mathcal{D}_0^T & 0 & 0 & -\mathcal{W} \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 0 & G_0 P^{-1} & \mathcal{C}_0 P^{-1} & E_0\mathcal{W} \end{pmatrix}.$$

Proof: We know that

$$\varepsilon_\infty\{\tilde{y}_k\tilde{y}_k^T\} = C_k\,\varepsilon_\infty\{\tilde{x}_k\tilde{x}_k^T\}\,C_k^T = \bar{C}_k\mathcal{X}\bar{C}_k^T < \Omega,$$

where the state covariance matrix $\mathcal{X}$ is defined in (20) and $\bar{C}_k$ is given in (40). With the definition $P := \mathcal{X}^{-1} > 0$, the inequality (39) can be manipulated by using the Schur complement. And the proof for solving F follows a similar approach in [11]. ■

We observe that the optimization approach proposed in this theorem is a convex programming problem stated as LMIs, which can be solved by efficient methods.

V. Numerical Example

In order to determine the applicability of the method, two examples to solve for the system design are presented next.

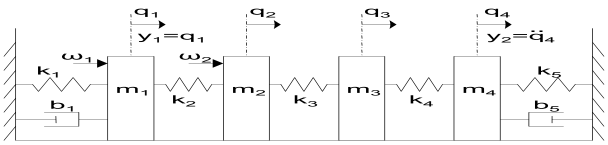

Fig. 1. Four mass mechanical system with springs and dampers

A. Four Mass Mechanical System

Consider the four mass mechanical system with springs and dampers depicted in Fig. 1. The discrete-time system dynamics is described in the following state space form

$$x_{k+1} = \begin{bmatrix} I & \Delta I \\ -\Delta M^{-1}K & I-\Delta M^{-1}G \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \Delta M^{-1}D \end{bmatrix} \omega_k, \tag{44}$$

with the measurement

$$z_k = (H + H_s\zeta_k)x_k + v_k, \tag{45}$$

and the desired output

$$y_k = Cx_k, \tag{46}$$

where

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, G = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_5 \end{bmatrix},$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, \qquad C = H,$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_4}{m_4} & -\frac{k_4+k_5}{m_4} & 0 & 0 & 0 & -\frac{b_5}{m_4} \end{bmatrix},$$

$$H_s = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$m_1 = m_2 = m_4 = 1, \quad m_3 = 2, \quad b_1 = 5, \quad b_5 = 2,$$
$$k_1 = k_3 = k_4 = 1, \quad k_2 = 2, \quad k_5 = 4.$$

Note that ω_k, v_k and ζ_k are uncorrelated zero mean Gaussian white noise sequences with unity covariance. And Δ is the time step (0.01sec). The performance criterion for the filter design is $[\varepsilon_\infty\{\tilde{y_k}\tilde{y_k}^T\}]_{i,i} < \Omega$ where $\Omega = 0.01$.

Fig. 2 and 3 demonstrate the performance of the filter introduced in this paper, where the error of each state variable is plotted. The simulation result shows that the output covariance of the estimation error are $[\varepsilon_\infty\{\tilde{y_k}\tilde{y_k}^T\}]_{1,1} = 0.001$, $[\varepsilon_\infty\{\tilde{y_k}\tilde{y_k}^T\}]_{2,2} = 0.0005$, which satisfy the design requirement, since both of 0.001 and $0.0005 < \mathbf{0.01}$.

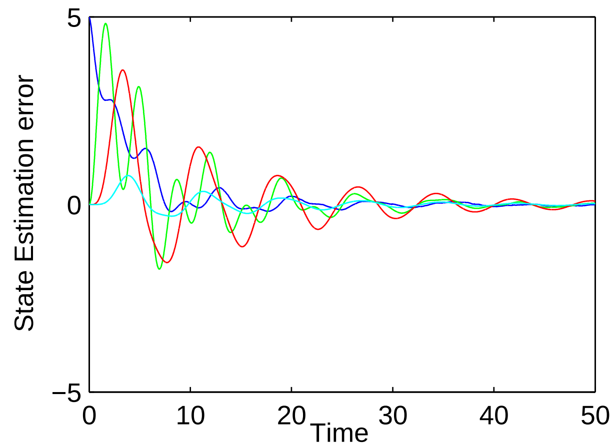

Fig. 2. Estimation error (from state 1 to 4)

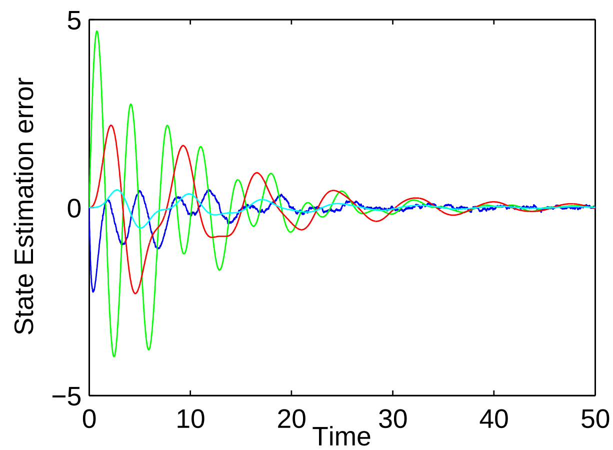

Fig. 3. Estimation error (from state 5 to 8)

B. Biomechanical Hand Movement System

Consider the hand modelled as a point mass (m = 1kg) whose one-dimensional position at time t is p(t), and the velocity at time t is v(t). The combined action of all muscles is represented with the force f(t) acting on the hand. The control signal u(t) is transformed into force by adding control-dependent noise and applying a second order muscle-like low-pass filter

$$\tau_1\tau_2 \ddot{f(t)} + (\tau_1+\tau_2)\dot{f(t)} + f(t) = u(t),$$

where $\tau_1 = \tau_2 = 0.04sec$. We know that the above filter can be written as a pair of coupled first-order filters

$$\tau_1\dot{g} + g = u, \quad \tau_2\dot{f} + f = g.$$

The sensory feedback carries the information about position, velocity and force. The discrete-time system dynamics is described as following

$$\begin{aligned} x_{k+1} &= Ax_k + B(1+\sigma_c\varepsilon_k)u_k + \omega_k, \\ z_k &= (H + H_s\zeta_k)x_k + v_k, \\ y_k &= Hx_k. \end{aligned} \tag{47}$$

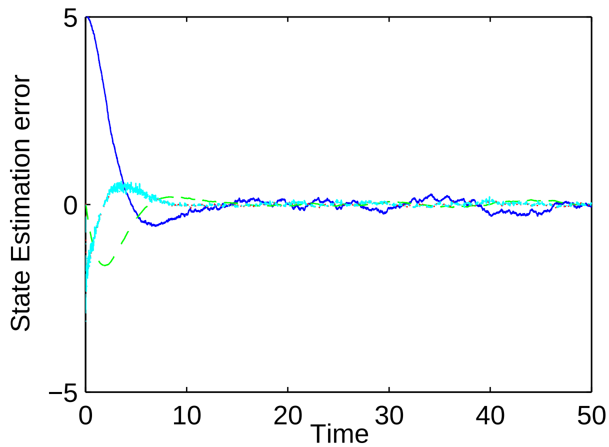

Fig. 4. Estimation error for hand movement system

where

$$x_k = \begin{pmatrix} p_k & v_k & f_k & g_k \end{pmatrix}^T, \qquad y_k = \begin{pmatrix} p_k & v_k & f_k \end{pmatrix}^T,$$

$$A = \begin{bmatrix} 1 & \Delta & 0 & 0 \\ 0 & 1 & \Delta/m & 0 \\ 0 & 0 & 1-\Delta/\tau_2 & \Delta/\tau_2 \\ 0 & 0 & 0 & 1-\Delta/\tau_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\Delta}{\tau_1} \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.125 & 0 \end{bmatrix},$$

and $\omega_k, v_k, \varepsilon_k, \zeta_k$ are independent zero-mean Gaussian white noise sequences with covariance

$$\Omega^\omega = (diag[0.01, 0.001, 0.01, 0.01])^2, \qquad \Omega^\varepsilon = I,$$
$$\Omega^v = (diag[0.01, 0.1, 0.5])^2, \qquad \Omega^\zeta = I.$$

Note that $\sigma_c = 0.5$ is a unitless quantity that defines the noise magnitude relative to the control signal magnitude. And the time step $\Delta = 0.01$sec.

Given a controller

$$u_k = \begin{bmatrix} -1.6032 & -3.0297 & -0.3361 & -2.7793 \end{bmatrix} x_k$$

such that the system (47) is mean square stable, the objective is to find a state estimator that bounds the estimation error below a specified error covariance: $[\varepsilon_\infty\{\tilde{y_k}\tilde{y_k}^T\}]_{i,i} < \Omega$ where $\Omega = 0.1$.

Fig. 4 illustrates the performance of the filter introduced in this paper, where the error of each state variable is plotted. The simulation result shows that the output covariance of the estimation error are $[\varepsilon_\infty\{\tilde{y_k}\tilde{y_k}^T\}]_{1,1} = 0.0198$, $[\varepsilon_\infty\{\tilde{y_k}\tilde{y_k}^T\}]_{2,2} = 0.0037$, $[\varepsilon_\infty\{\tilde{y_k}\tilde{y_k}^T\}]_{3,3} = 0.0018 < \mathbf{0.1}$, which satisfy the design requirement.

VI. Conclusions

Multiplicative noise models are more practical than normal additive noise models, since they allow the statistical description of the multiplicative noise be not known *a prior* but depend on the control and state solution. Such models are found in many physical systems, such as signal processing systems, biological movement systems, and aerospace engineering systems.

An LMI based approach is examined in this paper for the design of the state estimator with multiplicative noise systems. The proposed approach provides the sufficient conditions for the existence of state estimators and a parametrization of all admissible solutions. By adding a mild constrain, the original filtering problem is solved as a convex problem. The simulation results demonstrate the convergence of the system design.

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