Dynamics and control of tensegrity systems

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Abstract

This paper introduces rigid body dynamics in a new form, as a matrix differential equation, rather than the traditional vector differential equation. We characterize the dynamics of a broad class of systems of Rigid Bodies in a compact form, requiring no inversion of a mass matrix, yet preserving the second order character of the dynamics (that state space formulations do not preserve). For a system of n_b rigid bodies, the equations are characterized by a $3 \times 2n_b$ *configuration matrix*, and the dynamics are written as a second order differential equation in the *configuration matrix*. One significance of these equations is the complete characterization of the statics and dynamics of all class 1 tensegrity structures, where bar lengths are constant and the strings may be controlled. The form of the equations allow much easier integration of structure and control design since the control parameters (force density in the strings) and the topology of the structure (configuration space) appear in bilinear form, which might be a significant help to the control design tasks.

1 Introduction

The characterization of the dynamics of systems of rigid bodies is well-documented in a variety of vector forms. This paper introduces rigid body dynamics in a new form, as a matrix differential equation, rather than the traditional vector equation. This paper describes the dynamics and the static equilibria of a system of discontinuous rigid bodies, connected via a continuous set of strings to stabilize the system. In our theory, the "strings" are "springs" which can take compression or tension. However, in the special application of greatest interest, the "strings" can only take tension. All equilibria of such bar and string connections are described, and the dynamics of such systems are described in a new form, a second order differential equation of a $3 \times 2n_b$ matrix, called the *configuration matrix*. By parametrizing the configuration in terms of the components of vectors, the usual nonlinearities of angles, angular velocities and coordinate transformations are avoided. Indeed, there are no trigonometric functions in this formulation. We seek simplicity in the analytical form of the dynamics, for ease in designing control laws later. We believe these equations will turn out to be the simplest of all the available equations for a system of rigid bodies.

In the 18th and 19th centuries, the fundamental principles of dynamics of mechanical systems emerged. Many papers were written to uncover the secrets of *Hamilton's Principle*, to try to determine an answer to the question "When is Hamilton's Principle an extremal principle?". Unfortunately, many papers and books quoted Hamilton's principle incorrectly, setting off a hundred years of debate of the issue. Many books on dynamics characterize "Hamilton's Principle" in terms of a *fixed-end-point variation*. Hamilton's original work [**?**]attributes the idea of a *fixed end-point variation* to Lagrange, but Hamilton defines a variation with possibly *free end-points*. This subtle distinction about the boundary conditions will not change the dynamics of the motion, but the question of minimization involves the boundary conditions. Therefore, texts that state Hamilton's principle as the Lagrangian variety with fixed end-points will come to a different conclusion than those that state Hamilton's principle in the way that Hamilton did. The book by Lancos [**?**] gives Hamilton's principle correctly. Of course, control researchers have always been interested in minimization, so it is natural that Hamilton's work was revisited by the control community since the 1980s, under various labels as "Hamiltonian Systems" [**?**]. These papers present control and dynamics in a common setting of vector differential equations.

1.1 Vector-Second Order Form

In the 60s and 70s, a variety of Newtonian and energy approaches (Hamilton and Lagrange) were introduced and traded for numerical efficiencies. NASA had great interest in building accurate deployable spacecraft simulations [**?**], [**?**] composed of a large number of rigid bodies in a topological tree. Rules of thumb emerged from these studies, such as "energy approaches (based upon Hamilton's principle)" are more efficient than Newtonian approaches when there are many constraints, compared to degrees of freedom" [**?**]. Of course, the human energy to write the equations versus the computer resources required to solve the equations were traded, as well. Much of the human energy involves the book-keeping associated with constraints, and much of the computer energy involves inverting the "mass" matrix $M(q)$ at every time step in the simulation. The typical form of these equations was vector-second order as shown in (1) below.

$$M(q)\ddot{q} + D(q, \dot{q})\dot{q} + K(q)q = f \tag{1}$$

where $q \in \mathbb{R}^n$, $f \in \mathbb{R}^n$.

1.2 State Form

The first order form of dynamic systems was introduced by Hamilton using his concept of *generalized momenta* [**?**]. The lack of computers prevented the generalized momenta (*state form*) from gaining wide acceptance in the 19th century. 125 years later the first order form was given a name, the *state* equations. In the 1960s the space age rapidly developed computer technology, allowing state space computation to be popular. Indeed, computer availability and the state space form of system dynamics combined to promote a rapid acceleration of modern control theory in the 60s [?]. However, much structural information might be lost in a generic state form, and knowledge of this structural information is vital to the design of successful control strategies. Much effort has focused on the creation of analytical or numerical inversion of the mass matrix, $M(q)$. Rodriques [?] and others developed elegant factorization strategies, but the effort to factorize the mass matrix required significant amounts of computation also. This paper derives the dynamic equations in a form which, in effect, inverts the "mass" matrix analytically, without the need for any numerical factorization schemes or matrix inversion.

For a large class of problems it is reasonable to assume that the rigid bodies are rod-shaped and have negligible inertia about their longitudinal axes. We will make this assumption, although the fundamental method we employ does not require this assumption.

1.3 Tensegrity Systems

Class k tensegrity systems [?] are defined by the number (k) of rigid bodies that connect to each other (with ball frictionless joints) at a specific point (node). This paper entertains only class 1 tensegrity systems, so no rigid bodies are in contact, and the system is stabilized only by the presence of tensile members connecting the rigid bodies. Such a system has only axially-loaded members, since the rigid bodies do not touch each other and the strings connected to the rigid bodies cannot apply torques at the site of the attachment. These features not only simplify the equations of motion, but the resulting models will be much more accurate than models of rigid bodies that are subject to bending moments. That is, the internal stresses in the rigid bodies have a specific direction. Creating structures, and models of structures, that are accurate for forces in multiple directions requires complicated design decisions that eventually require additional mass in the structure, and less confidence in the structural model. Of course rigid bodies do not exist, except in our imagination and our engineering idealization, but one can justify the "rigid body assumptions" in our multi-body analysis if the rigid body can tolerate the worst force is the worst direction within specified "small enough" deformations. This is a formidable task, to analyze all possible loading conditions in a 1000 body system to choose the mass properties of each rigid body to justify the "rigid body assumptions", to yield a minimal mass design. Hence, if 1000 bars are used in the structure, and 3 moments and 1 axial force is applied to each bar, there are 4000 load directions to be analyzed for reasonableness of the rigid body assumptions (and the corresponding design of each rigid body). In the absence of reliable methods for this design task, engineers today make conservative assumptions, and over-design the system at the expense of mass.

This task becomes much less formidable if the load directions within each body can be reduced by a factor of 4. Hence, if each bar can be loaded in only one direction (axially), there are only 1000 possible loads to analyze in our example, instead of 4000. Furthermore, the *material* choices can be specialized to handle loads in a prespecified direction with much less mass than would be required of material choices that must take loads in a variety of directions. Sand and mortar are very good materials in compression, but one would not use them if the dynamic loads ever allowed the structural member to experience tension.

Tensegrity systems have been around for over 50 years as an artform [**?**], with some architectural appeal [**?**], but analytical tools to design engineering structures from tensegrity concepts are still inadequate. The primary motivation for this paper is to provide a convenient analytical tool to describe both the statics and dynamics of class 1 tensegrity systems. Moreover, the results are more general than tensegrity systems, having application to a large class of systems of rigid bodies.

1.4 Notation

Definition 1 *The set of vectors* $\underline{e}_i$, *i=1,2,3, form a* dextral set*, if the dot products satisfy* $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ *(where* δ_{ij} *is a Kronecker delta), and the cross products satisfy* $\underline{e}_i \times \underline{e}_j = \underline{e}_k$*, where the indices i,j,k form the cyclic permutations, i,j,k=1,2,3 or 2,3,1, or 3,1,2.*

Definition 2 *Let* $\underline{e}_i, i = 1, 2, 3$ *define a* dextral set *of unit vectors fixed in an inertial frame, and define the* vectrix $\underline{E}$ *by* $\underline{E} = \left[\begin{array}{ccc} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{array} \right]$.

The Gibbs concept of a *vector* [**?**] from dynamics, and the linear *vector* space from systems theory have distinctions that must be made clear. Let $\underline{r}$ be the label we use to represent a (Gibbs) vector in the three-dimensional (non-relativistic) space in which we live. This vector can be described in any chosen reference frame. Consider two reference frames, described by the dextral sets $\underline{E}$ and $\underline{X}$, where the coordinate transformation between these two frames is described by the 3×3 direction cosine matrix Ψ. Hence, $\underline{E} = \underline{X}\Psi$. Let the 3×1 matrices r^X and r^E describe the components of the same vector $\underline{r}$ in the two reference frames $\underline{X}$ and $\underline{E}$, respectively. Hence, if we wish to describe the relationship between the components of the same vector $\underline{r}$, described in two different reference frames, then

$$\underline{r} = \underline{X} r^X = \underline{E} r^E \tag{2}$$

$$\underline{E} = \underline{X}\Psi \tag{3}$$

$$\underline{X} r^X = \underline{E} r^E = \underline{X}\Psi r^E, \tag{4}$$

yields

$$r^X = \Psi r^E. \tag{5}$$

The item we call $\underline{r}$ is a Gibbs vector. The items we call r^X and r^E are vectors in the linear vector spaces of linear algebra, where we use the notation, $r^X \in \mathbb{R}^3$ and $r^E \in \mathbb{R}^3$ to denote that the items r^X and r^E live in a real three-dimensional space. However, the items r^X and r^E tell us nothing unless we have previously specified the frames of reference $\underline{X}$ and $\underline{E}$ for these quantities.

If we must assign a "dimension" to these quantities $\underline{X}$ and $\underline{E}$, then we must say they are 3×1 arrays, composed of the three elements $\underline{e_i}, i = 1, 2, 3$. However, these arrays contain quantities we call Gibbs vectors $\underline{e}_i$. So the 3×1 item $\underline{E}$ is not a *vector* in either the sense of Gibbs, nor in the sense of linear algebra. For these reasons [?] makes the logical choice to call the quantity $\underline{E}$ a *vectrix*.

This paper uses only one coordinate frame to describe all vectors, unlike many problems in aerospace, where the convenience of multiple coordinate frames is utilized. Since we always use the same frame of reference, the inertial frame, described by the vectrix $\underline{E}$, we shall not complicate the notation of vectors with different superscripts, as would be required above to distinguish between components of a vector represented in different frames. Hence, we use the notation for the vector $\underline{n}_i$, as follows

$$\underline{n}_i = \underline{E} n_i, \tag{6}$$

where $n_i^T = \begin{bmatrix} n_{i_1}, & n_{i_2}, & n_{i_3} \end{bmatrix}$ describes the components of the vector n_i in coordinates $\underline{E}$. Note that we have dropped the superscript E that would be used in the more complete and more general notation above (n_i^E), and we will write only n_i, hereafter, instead of n_i^E.

We generate a diagonal $n \times n$ matrix from an n-dimensional vector $v^T = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & \dots \end{bmatrix}$, by denoting the *hat* operator by

$$\hat{v} = diag \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & \dots \end{bmatrix}. \tag{7}$$

We generate a 3×3 matrix $\tilde{v}$ from the 3-dimensional vector $v^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ by the *tilde* operator as follows

$$\tilde{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_3 & 0 \end{bmatrix}. \tag{8}$$

We often use the fact that for any two n-dimensional vectors $v \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$,

$$\hat{v}x = \hat{x}v. \tag{9}$$

We refer to the *Schur compliments* of the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ as the matrices $A - BC^{-1}B^T$, and $C - B^T A^{-1} B$. It is well known that the positive definiteness of the original (big) matrix is equivalent to the positive definiteness of the *Schur compliments*, given that the diagonal blocks A and C are positive definite.

2 Description of a Network of Bars/Strings

We show below how to describe all dynamics in the $\underline{E}$ frame, after the usual definition of dot and cross products. The 3×1 matrix b_i represents the components of vector $\underline{b}_i$ with respect to the fixed frame $\underline{E}$. That is,

$$\underline{b}_i = \sum_{j=1}^{3} \underline{e}_j b_{i_j} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{bmatrix} \begin{bmatrix} b_{i_1} \\ b_{i_2} \\ b_{i_3} \end{bmatrix} = \underline{E} b_i. \tag{10}$$

Lemma 1 *Let for some chosen inertial reference frame $\underline{E}$,*

$$\underline{b}_i = \underline{E} b_i, \qquad \underline{f}_i = \underline{E} f_i, \qquad \underline{n}_i = \underline{E} n_i.$$

Then the cross product is given by

$$\underline{b}_i \times \underline{f}_{i+n_b} = (\underline{E} b_i) \times (\underline{E} f_{i+n_b}) = \underline{E} \tilde{b}_i f_{i+n_b}$$

where

$$\tilde{b}_i = \begin{bmatrix} 0 & -b_{i_3} & b_{i_2} \\ b_{i_3} & 0 & -b_{i_1} \\ -b_{i_2} & b_{i_1} & 0 \end{bmatrix}, \qquad b_i = \begin{bmatrix} b_{i1} \\ b_{i2} \\ b_{i3} \end{bmatrix},$$

and the dot product is given by,

$$\underline{b}_i \cdot \underline{f}_{i+n_b} = (\underline{E} b_i) \cdot (\underline{E} f_{i+n_b}) = b_i^T \underline{E}^T \cdot \underline{E} f_{i+n_b} = b_i^T f_{i+n_b}$$

where the dot product $\underline{E}^T \cdot \underline{E} = I$ since $\underline{e}_i, i = 1, 2, 3$ form a dextral set of unit vectors.

Let a structural system be composed of n_b bars and n_s strings. The definitions below will later allow us to describe the connections between the rigid members and the strings.

Definition 3 *A* node *(the i^{th} node $\underline{n}_i$) of a structural system is a point in space at which members of the structure are connected. The coordinates of this point in the $\underline{E}$ frame are $n_i \in \mathbb{R}^3$, as in (6).*

Definition 4 *A* string *(the i^{th} string) is characterized by these properties:*

- *A massless structural member connecting two nodes.*
- *A vector connecting these two nodes is $\underline{s}_i$. The direction of $\underline{s}_i$ is arbitrarily assigned.*
- *The string provides a force to resist lengthening it beyond its rest-length, but provides no force to resist shortening the string below its rest-length.*
- *A* string *has no bending stiffness.*

Definition 5 *A* bar *(the i^{th} bar of a n_b bar system) is characterized by these properties:*

- *A structural member connecting two nodes $\underline{n}_i$ and $\underline{n}_{i+n_b}$.*
- *The vector along the bar connecting nodes $\underline{n}_i$ and $\underline{n}_{i+n_b}$ is $\underline{b}_i = \underline{n}_{i+n_b} - \underline{n}_i, i = 1, 2, \ldots n_b$.*
- *The bar $\underline{b}_i$ has* length *$\|b_i\| = L_i = \sqrt{b_i^T b_i}$.*

Definition 6 *The vector $\underline{r}_i$ locates the mass center of bar $\underline{b}_i$, and $\underline{r}_i = \underline{E} r_i$.*

Definition 7 *The vector* $\underline{t}_i$ *represents the force exerted on a node by string* $\underline{s}_i$*, where the direction of* $\underline{t}_i$ *is defined to be parallel to string vector* $\underline{s}_i$*. That is,* $\underline{t}_i = \gamma_i \underline{s}_i$ *and hence,* $t_i = \gamma_i s_i$ *for some positive scalar* γ_i*.*

Definition 8 *The* force density γ_i *in string* s_i *is defined by* $\gamma_i = \frac{\|t_i\|}{\|s_i\|}$.

Definition 9 $\underline{f}_i$ *represents the net sum of vector forces external to bar* $\underline{b}_i$ *terminating at node* $\underline{n}_i$*. The set of all nodal forces external to the bar* $\underline{b}_i$ *is described by figure 1.*

Figure 1: Bar force definition

From these definitions, define matrices, $F \in \mathbb{R}^{3\times 2n_b}$, $N \in \mathbb{R}^{3\times 2n_b}$, $T \in \mathbb{R}^{3\times n_s}$, $S \in \mathbb{R}^{3\times n_s}$, $B \in \mathbb{R}^{3\times n_b}$, $\Gamma \in \mathbb{R}^{n_s\times n_s}$, as follows,

$$F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \dots & f_{n_b} & | & f_{n_b+1} & \dots & f_{2n_b} \end{bmatrix} \tag{11}$$
$$N = \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & \dots & n_{n_b} & | & n_{n_b+1} & \dots & n_{2n_b} \end{bmatrix} \tag{12}$$
$$T = \begin{bmatrix} t_1 & t_2 & \dots & t_{n_s} \end{bmatrix} \tag{13}$$
$$S = \begin{bmatrix} s_1 & s_2 & \dots & s_{n_s} \end{bmatrix} \tag{14}$$
$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_{n_b} \end{bmatrix} \tag{15}$$
$$R = \begin{bmatrix} r_1 & r_2 & \dots & r_{n_b} \end{bmatrix} \tag{16}$$
$$\hat{\gamma} = \Gamma = \text{diag} \begin{bmatrix} \gamma_1 & \dots & \gamma_{n_s} \end{bmatrix}. \tag{17}$$

It follows from (12), (15), and Definition 5 that

$$B = N_2 - N_1 = N \begin{bmatrix} -I \\ I \end{bmatrix}, \tag{18}$$

and the locations of the mass centers of all bars are described by

$$R = N_1 + \frac{1}{2}B. \tag{19}$$

It follows from Definition 8 and (13), (14) that

$$T = S\Gamma. \tag{20}$$

Lemma 2 *Assume that the mass of the bar is uniformly distributed only along its length, and that its length is much longer than its diameter. Then the angular momentum of the bar* b_i *about the center of mass of bar* b_i*, expressed in the* $\underline{E}$ *frame, is*

$$h_i = \frac{m_i}{12}\tilde{b}_i\dot{b}_i \tag{21}$$

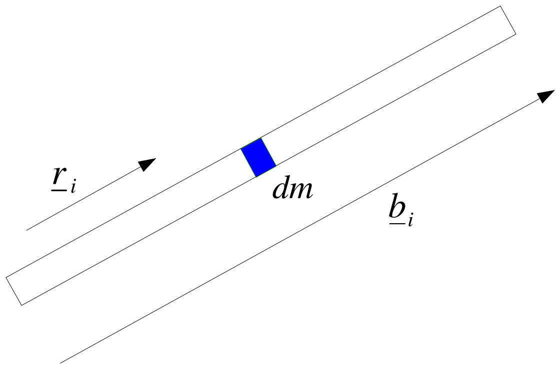

Figure 2: Rectilinear mass distribution of bar

Proof: The angular momentum vector of a bar about its mass center is described by (see Fig 2)

$$\underline{h}_i = \int_m \underline{\beta}_i \times \dot{\underline{\beta}}_i dm$$

where $dm_i = m_i d\mu$, $\underline{\beta}_i = \underline{b}_i \mu$, and $-\frac{1}{2} \leq \mu \leq \frac{1}{2}$. Hence,

$$\underline{h}_i = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\mu \underline{b}_i) \times (\mu \dot{\underline{b}}) m_i d\mu = \frac{m_i}{12} \underline{b}_i \times \dot{\underline{b}}_i$$

From lemma 1

$$\underline{b}_i \times \dot{\underline{b}}_i = \underline{E} \tilde{b}_i \dot{b}_i$$

Hence $\underline{h}_i = \underline{E} h_i$, where h_i is given by (21). $\Box$

3 Dynamics of a rigid bar

Refer to figure 1 for a single bar, with bar vector $\underline{b}$, nodes $\underline{n}_1$ and $\underline{n}_2$, at which are applied forces $\underline{f}_1$ and $\underline{f}_2$.

Lemma 3 *The translation of the mass center of bar* $\underline{b}$ *, located at position* $\underline{r}$ *obeys*

$$m \ddot{\underline{r}} = \underline{f}_1 + \underline{f}_2 \tag{22}$$

or, in the $\underline{E}$ *frame of reference,*

$$m \ddot{r} = f_1 + f_2 \tag{23}$$

Lemma 4 *The rotation of bar* b_i *about it mass center obeys*

$$\frac{m}{12} \tilde{b} \ddot{b} = \frac{1}{2} \tilde{b} (f_2 - f_1). \tag{24}$$

Proof: From angular momentum and Newton's second law

$$\dot{\underline{h}} = \frac{1}{2}\underline{b} \times (\underline{f}_2 - \underline{f}_1), \qquad \underline{h} = \frac{m}{12}\underline{b} \times \dot{\underline{b}}. \tag{25}$$

Note that

$$\dot{\underline{h}} = \frac{m}{12}(\dot{\underline{b}} \times \dot{\underline{b}} + \underline{b} \times \ddot{\underline{b}}) = \frac{m}{12}(\underline{b} \times \ddot{\underline{b}}). \tag{26}$$

Hence, in the $\underline{E}$ frame

$$\dot{h} = \frac{1}{2}\tilde{b}(f_2 - f_1), \qquad h = \frac{m}{12}\tilde{b}\dot{b}. \tag{27}$$

Thus,

$$\dot{h} = \frac{m}{12}\tilde{b}\ddot{b}. \tag{28}$$

□

3.1 Constrained Dynamics

We now wish to develop the dynamics constrained for constant bar lengths. We add a non-working constraint force f to get

$$\frac{m}{12}\tilde{b}\ddot{b} = \frac{1}{2}\tilde{b}(f_2 - f_1) + f \tag{29}$$

$$\Phi = b^T b - L^2 = 0, \tag{30}$$

where the added constraint is $\Phi = 0$, and f is the non-working force associated with this constraint. The force f does no work in the presence of any feasible perturbation of the generalized coordinate b. Hence, $f^T \delta b = 0$. The constraint must also hold in the presence of a feasible perturbation. Hence, $d\Phi = (\frac{\partial \Phi}{\partial b})^T \delta b = 0$. Thus,

$$\begin{bmatrix} f^T \\ (\frac{\partial \Phi}{\partial b})^T \end{bmatrix} \delta b = 0, \tag{31}$$

requiring that the matrix coefficient of δb must have deficient rank. Thus, $f = (\frac{\partial \Phi}{\partial b})\zeta$, for some ζ (called a Lagrange multiplier). Furthermore, $\frac{\partial \Phi}{\partial b} = 2b$. Hence, the constrained dynamic system obeys,

$$\frac{m}{12}\tilde{b}\ddot{b} = \frac{1}{2}\tilde{b}(f_2 - f_1) + b\zeta \tag{32}$$

$$\Phi = b^T b - L^2 = 0. \tag{33}$$

where we have absorbed some constants into the scalar ζ. Note that the constraint holds over time, hence $\Phi = \dot{\Phi} = \ddot{\Phi} = 0$. Differentiating the constraint yields,

$$\dot{b}^T b + b^T \dot{b} = 0 = 2b^T \dot{b}. \tag{34}$$

Differentiating (34) yields

$$\dot{b}^T \dot{b} + b^T \ddot{b} = 0,$$

or ,

$$b^T\ddot{b} = -\dot{b}^T\dot{b}. \tag{35}$$

The conclusion thus far is that constant length rigid bar rotations obey, for some scalar ζ,

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix} \ddot{b} = \begin{bmatrix} \tilde{b}(f_2 - f_1)\frac{6}{m} \\ -\dot{b}^T\dot{b} \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} \zeta. \tag{36}$$

The following identity will be useful.

Lemma 5

$$\tilde{b}^2 = bb^T - b^T bI$$

Lemma 6 *The unique Moore-Penrose inverse of* $\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}$ *is given by*

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}^+ = \begin{bmatrix} -\tilde{b} & b \end{bmatrix} L^{-2}$$

Proof: First prove that

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}^T \begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix} = L^2 I$$

Then

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}^+ = \left(\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}^T \begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix} \right)^{-1} \begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}^T = \begin{bmatrix} \tilde{b}^T & b \end{bmatrix} L^{-2}$$

where $\tilde{b}^T = -\tilde{b}$. $\square$

Lemma 7 *The solution of (36) for* $\ddot{b}$ *has the unique solution*

$$\ddot{b} = \frac{6}{m}(f_2 - f_1) - b\left(\frac{\dot{b}^T\dot{b}}{L^2} + \frac{6}{mL^2} b^T(f_2 - f_1) \right) \tag{37}$$

Proof: Solve (36) for $\ddot{b}$ using lemma 6 to get

$$\begin{aligned} \ddot{b} &= L^{-2} \begin{bmatrix} -\tilde{b} & b \end{bmatrix} \left(\begin{bmatrix} \tilde{b}(f_2 - f_1)\frac{6}{m} \\ -\dot{b}^T\dot{b} \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} \zeta \right) && (38) \\ &= L^{-2} \left(-\tilde{b}^2(f_2 - f_1)\tfrac{6}{m} - b\dot{b}^T\dot{b} \right), && (39) \end{aligned}$$

since, $\tilde{b}b = 0$. Lemma 5 and (39) yield

$$\ddot{b} = \left(-(bb^T - b^T bI)(f_2 - f_1)\frac{6}{m} - b\dot{b}^T\dot{b} \right) L^{-2}$$

where $b^T b = L^2$. Hence (37) follows. $\square$

3.2 Example: Constrained Dynamics of a two-Bar System

Lemma 8 *In a system of two bars, the dynamics for the bar b_i with fixed length L_i (i=1,2), are described by*

$$\ddot{b}_i + b_i\left(\frac{6H_i}{m_i} + V_i\right) = (f_{n_b+i} - f_i)\frac{6}{m_i} \tag{40}$$

$$\ddot{r}_i = \frac{1}{m_i}(f_i + f_{n_b+i}), \tag{41}$$

where

$$V_i = \frac{\|\dot{b}_i\|^2}{L_i^2}, \qquad H_i = \frac{b_i^T(f_{n_b+i} - f_i)}{L_i^2}$$

Proof: (37) yields (40) and (22) yields (41). □

It follows from (40, 41) that a *two-bar* system satisfies

$$\begin{bmatrix} \ddot{b}_1 & \ddot{b}_2 & \ddot{r}_1 & \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & r_1 & r_2 \end{bmatrix} \begin{bmatrix} (\frac{6H_1}{m_1} + V_1) & 0 & 0 & 0 \\ 0 & (\frac{6H_2}{m_2} + V_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} \frac{-6}{m_1} & 0 & \frac{1}{m_1} & 0 \\ 0 & \frac{-6}{m_2} & 0 & \frac{1}{m_2} \\ \frac{6}{m_1} & 0 & \frac{1}{m_1} & 0 \\ 0 & \frac{6}{m_2} & 0 & \frac{1}{m_2} \end{bmatrix}$$

or

$$\ddot{\mathcal{Q}} + \mathcal{Q}K_0 = FJ^{-1},$$

$$J^{-1} = \begin{bmatrix} -6M^{-1} & M^{-1} \\ 6M^{-1} & M^{-1} \end{bmatrix}, \quad M = \mathrm{diag}\begin{bmatrix} \dots & m_i & \dots \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} B & R \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} = N_2 - N_1, \quad N = \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix},$$

$$H = \mathrm{diag}\begin{bmatrix} \dots & H_i & \dots \end{bmatrix}, \qquad H_i = b_i^T(f_{n_b+i} - f_i)/L_i^2,$$

$$V = \mathrm{diag}\begin{bmatrix} \dots & V_i & \dots \end{bmatrix}, \qquad V_i = \|\dot{b}_i\|^2/L_i^2,$$

$$K_0 = \begin{bmatrix} I \\ 0 \end{bmatrix}(6HM^{-1} + V)\begin{bmatrix} I & 0 \end{bmatrix}.$$

3.3 An n_b-Bar System

Theorem 1 *Consider an n_b-bar system with constant length bar vectors $b_i, i = 1, 2, \ldots, n_b$, and matrices defined by,*

$$R = N_1 + \frac{1}{2}B \tag{42}$$

$$B = \left[\begin{array}{ccc} b_1 & b_2 \ldots & b_{n_b} \end{array}\right] = N_2 - N_1, \quad N = \left[\begin{array}{cc} N_1 & N_2 \end{array}\right] \tag{43}$$

$$N_1 = \left[\begin{array}{ccc} n_1 & n_2 \ldots & n_{n_b} \end{array}\right], \quad N_2 = \left[\begin{array}{ccc} n_{n_b+1} & \ldots & n_{2n_b} \end{array}\right] \tag{44}$$

$$F = \left[\begin{array}{cc} F_1 & F_2 \end{array}\right], \qquad F_1 = \left[\begin{array}{cc} f_1 \ldots & f_{n_b} \end{array}\right] \tag{45}$$

$$F_2 = \left[\begin{array}{cc} f_{n_b+1} \ldots & f_{2n_b} \end{array}\right] \tag{46}$$

$$\mathcal{Q} = \left[\begin{array}{cc} B & R \end{array}\right] \tag{47}$$

$$K_0 = \left[\begin{array}{c} I \\ 0 \end{array}\right] (6HM^{-1} + V) \left[\begin{array}{cc} I & 0 \end{array}\right] \tag{48}$$

$$H = diag \left[\begin{array}{ccc} \ldots & H_i & \ldots \end{array}\right], \qquad H_i = b_i^T (f_{n_b+i} - f_i)/L_i^2 \tag{49}$$

$$V = diag \left[\begin{array}{ccc} \ldots & V_i & \ldots \end{array}\right], \qquad V_i = \|\dot{b}_i\|^2/L_i^2 \tag{50}$$

$$M = diag \left[\begin{array}{ccc} \ldots & m_i & \ldots \end{array}\right] \tag{51}$$

$$J^{-1} = \left[\begin{array}{cc} -6M^{-1} & M^{-1} \\ 6M^{-1} & M^{-1} \end{array}\right]. \tag{52}$$

Then the rigid body dynamics are given by

$$\ddot{\mathcal{Q}} + \mathcal{Q}K_0 = FJ^{-1}. \tag{53}$$

Proof: Follows trivially from (40, 41). $\Box$

4 Characterizing Bar/String connections

Definition 10 *Define the "string connectivity matrix" C by*

$$C_{ij} = \begin{cases} 1 & \textit{if string vector } s_i \textit{ terminates on node } n_j. \\ -1 & \textit{if string vector } s_i \textit{ eminates from node } n_j. \\ 0 & \textit{if string vector } s_i \textit{ does not connect with node } n_j. \end{cases} \tag{54}$$

$$C = \left[\begin{array}{cc} C_1 & C_2 \end{array}\right], \quad C_1 \in \mathbb{R}^{n_s \times n_b}, \quad C_2 \in \mathbb{R}^{n_s \times n_b}$$

Definition 11 *Define the "disturbance connectivity matrix" D by*

$$D_{ij} = \begin{cases} 1 & \textit{if disturbance vector } w_i \textit{ terminates on node } n_j. \\ -1 & \textit{if disturbance vector } w_i \textit{ eminates from node } n_j. \\ 0 & \textit{if disturbance vector } w_i \textit{ does not connect with node } n_j. \end{cases} \tag{55}$$

$$D = \left[\begin{array}{cc} D_1 & D_2 \end{array}\right], \quad D_1 \in \mathbb{R}^{n_w \times n_b}, \quad D_2 \in \mathbb{R}^{n_w \times n_b}$$

For n_w disturbance vectors applied at nodes selected by the matrix D,

$$W = \begin{bmatrix} w_1 & w_2 \cdots & w_{n_w} \end{bmatrix} \tag{56}$$

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \quad D_1 \in \mathbb{R}^{n_w \times n_b}, \quad D_2 \in \mathbb{R}^{n_w \times n_b}. \tag{57}$$

Theorem 2

$$\begin{aligned} F &= -(TC + WD) \\ S &= NC^T \end{aligned}$$

.

Proof: The result $S = NC^T$ follows immediately from the definition of the connectivity matric C. The force vectors f_i have direction defined as terminating on a node n_i. The force vectors contained in WD and TC have also been defined (by the connectivity matrices C and D) as positive when terminating on the nodes $n_i, n_2, ...n_{n_b}$. Hence, by sign convention, F must be defined such that all force vectors entering (*terminating*) on the nodes must add to zero. Hence,

$$F + TC + WD = 0. \tag{58}$$

□

Lemma 9

$$F = -\mathcal{Q}\Phi^T C^T \Gamma C - WD, \qquad \Phi^T = \begin{bmatrix} -\frac{1}{2}I & \frac{1}{2}I \\ I & I \end{bmatrix}. \tag{59}$$

Proof: From Theorem 2 and (20),

$$TC = S\Gamma C = NC^T \Gamma C$$

.

However,

$$N = \begin{bmatrix} N_1 & N_2 \end{bmatrix}, \qquad B = N_2 - N_1, \qquad \mathcal{Q} = \begin{bmatrix} B & R \end{bmatrix}, \qquad R = N_1 + \frac{1}{2}B$$

.

Hence,

$$N = \mathcal{Q}\Phi^T \tag{60}$$

$$F = -\mathcal{Q}\Phi^T C^T \Gamma C - WD. \tag{61}$$

□

Define $\Theta = 6HM^{-1} + V$, and define $\lfloor J \rfloor = \text{diag}\begin{bmatrix} \dots & J_{ii} & \dots \end{bmatrix}$. Then, noting that Θ and M are diagonal matrices, one has the following result.

Corollary 1

$$\Theta M = \left\lfloor L^{-2} \begin{bmatrix} I & 0 \end{bmatrix} \dot{\mathcal{Q}}^T \dot{\mathcal{Q}} \begin{bmatrix} I \\ 0 \end{bmatrix} M + 6L^{-2} \begin{bmatrix} I & 0 \end{bmatrix} \mathcal{Q}^T (\mathcal{Q}\Phi^T C^T \Gamma C + WD) \begin{bmatrix} I \\ -I \end{bmatrix} \right\rfloor$$

Proof: From Theorem 1,

$$\Theta = (6HM^{-1} + V),$$

where

$$H = \left\lfloor L^{-2} B^T (F_2 - F_1) \right\rfloor,$$

$$V = \left\lfloor L^{-2} \dot{B}^T \dot{B} \right\rfloor = \left\lfloor L^{-2} \begin{bmatrix} I & 0 \end{bmatrix} \dot{\mathcal{Q}}^T \dot{\mathcal{Q}} \begin{bmatrix} I \\ 0 \end{bmatrix} \right\rfloor.$$

Noting that

$$\begin{aligned} F_2 - F_1 &= F \begin{bmatrix} -I \\ I \end{bmatrix} \\ &= (-\mathcal{Q}\Phi^T C^T \Gamma C - WD) \begin{bmatrix} -I \\ I \end{bmatrix}, \end{aligned}$$

then,

$$H = \left\lfloor L^{-2} \begin{bmatrix} I & 0 \end{bmatrix} \mathcal{Q}^T (\mathcal{Q}\Phi^T C^T \Gamma C + WD) \begin{bmatrix} I \\ -I \end{bmatrix} \right\rfloor.$$

□

Theorem 3 *The dynamics of all Class 1 tensegrity systems with rigid, fixed length bars are described by*

$$\ddot{\mathcal{Q}}\mathcal{M} + \mathcal{Q}\mathcal{K} = -WD\Phi, \tag{62}$$

$$\mathcal{Q} = \begin{bmatrix} B & R \end{bmatrix}, \tag{63}$$

$$\mathcal{M} = \begin{bmatrix} \frac{1}{12}M & 0 \\ 0 & M \end{bmatrix}, \tag{64}$$

$$\mathcal{K} = \begin{bmatrix} \hat{\theta} & 0 \\ 0 & 0 \end{bmatrix} + \Phi^T C^T \Gamma C \Phi, \tag{65}$$

$$\Phi^T = \begin{bmatrix} -\frac{1}{2}I & \frac{1}{2}I \\ I & I \end{bmatrix} \tag{66}$$

$$\hat{\theta} = \frac{1}{12} L^{-2} \left\lfloor 6 \begin{bmatrix} I & 0 \end{bmatrix} \mathcal{Q}^T (\mathcal{Q}\Phi^T C^T \Gamma C + WD) \begin{bmatrix} I \\ -I \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} \dot{\mathcal{Q}}^T \dot{\mathcal{Q}} \begin{bmatrix} I \\ 0 \end{bmatrix} M \right\rfloor. \tag{67}$$

Proof: From lemma 9 and theorem 1,

$$\ddot{\mathcal{Q}} + \mathcal{Q}K_0 - FJ^{-1} = 0, \tag{68}$$

where

$$K_0 = \begin{bmatrix} I \\ 0 \end{bmatrix} \Theta \begin{bmatrix} I & 0 \end{bmatrix}. \tag{69}$$

$$J^{-1} = \begin{bmatrix} -6M^{-1} & M^{-1} \\ 6M^{-1} & M^{-1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}I & I \\ \frac{1}{2}I & I \end{bmatrix} \begin{bmatrix} 12M^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix} = \Phi\mathcal{M}^{-1}, \tag{70}$$

and note from (61) that these expressions for F and J^{-1} allow (68) to be multiplied from the right by $\mathcal{M}$ to get the result (62). $\Box$

From (67), the i^{th} element of the diagonal matrix $\hat{\theta}$ is given by

$$\theta_i = \frac{1}{2}L_i^{-2}b_i^T(\mathcal{Q}\Phi^T C^T(\hat{C}_{1i} - \hat{C}_{2i})\gamma - W(D_{2_i} - D_{1_i})) + \frac{m_i}{12L_i^2}||\dot{b}_i||^2. \tag{71}$$

5 Class k Tensegrity Systems

This section deals with constraints of the form

$$\mathcal{Q}\mathcal{C} = 0, \tag{72}$$

for some specified matrix $\mathcal{C}$. We give two examples of this type of constraint: (i) class 2 structures formed by pinning nodes together, (ii) fixed boundary conditions, where some nodes are pinned to ground, and (iii) constraints that a selected set of nodes lie in a plane (such that a flat surface is maintained).

5.1 Class 2 Tensegrity

Suppose we wish to pin nodes n_i and n_j together. This creates a class 2 tensegrity structure, since two bars are pinned together. (The end of one bar is already at each of the nodes n_i and n_j, prior to pinning these two nodes together). First note that,

$$n_i = r_i - \frac{1}{2}b_i \qquad if \qquad i \leq n_b, \tag{73}$$

$$n_i = r_{i-n_b} + \frac{1}{2}b_{i-n_b} \qquad if \qquad i > n_b. \tag{74}$$

Now the constraints $n_i = n_j$ reduce to

$$r_\alpha = r_\beta + \frac{1}{2}(\psi_i b_i - \psi_j b_j), \tag{75}$$

where,

$$\alpha = i \qquad \psi_i = +1 \qquad if \qquad i \leq n_b \tag{76}$$

$$\alpha = i - n_b \qquad \psi_i = -1 \qquad if \qquad i > n_b \tag{77}$$

$$\beta = j \qquad \psi_j = +1 \qquad if \qquad j \leq n_b \tag{78}$$

$$\beta = j - n_b \qquad \psi_j = -1 \qquad if \qquad j > n_b. \tag{79}$$

In matrix form this yields the reduced coordinates ($\Lambda \in \mathbb{R}^{2n_b - 1\times 2n_b}$)

$$\mathcal{Q} = Q\Lambda, \qquad \Lambda = \begin{bmatrix} I_{n_b} & 0 & 0 & 0 & \mu_i & 0 & 0 \\ 0_{n_b-1\times n_b} & e_1 & e_2 & ... & \eta_i & ... & e_{n_b} \end{bmatrix}, \tag{80}$$

where,

$$e_k = \begin{bmatrix} 0 & \dots & 1 & \dots & 0 \end{bmatrix}^T \tag{81}$$

$$\mu_i = \begin{bmatrix} 0 & \dots & \frac{1}{2}\psi_i & \dots & -\frac{1}{2}\psi_j & \dots & 0 \end{bmatrix}^T \tag{82}$$

$$\eta_i = \begin{bmatrix} 0 & \dots\dots & 0 & \dots\dots\dots & 1 & \dots & 0 \end{bmatrix}^T. \tag{83}$$

These discussions lead to the following result

Theorem 4 *Let the n_b bar tensegrity system have the constraint $n_i = n_j$. Then the dynamics of the reduced order system are described by*

$$\ddot{Q} + QK = -WD\Phi^T\mathcal{M}^{-1}\Lambda^+, \tag{84}$$

where,

$$K = \Lambda\mathcal{K}\mathcal{M}^{-1}\Lambda^+, \tag{85}$$

where

$$\Lambda^+ = \Lambda^T(\Lambda\Lambda^T)^{-1}, \tag{86}$$

and

$$\mathcal{Q} = Q\Lambda. \tag{87}$$

5.2 Fixed Boundary Conditions

Now consider that nodes n_i and n_j are fixed in inertial space. The dynamics are modified as follows.

$$n_i = \check{n}_i \tag{88}$$

$$n_j = \check{n}_j, \tag{89}$$

where, $\check{n}_i$ and $\check{n}_j$ are specified constant vectors. Hence,

$$r_\alpha = \frac{1}{2}\psi_i b_i + \check{n}_i, \tag{90}$$

$$r_\beta = \frac{1}{2}\psi_j b_j + \check{n}_j. \tag{91}$$

This leads to

$$\mathcal{Q} = Q\Lambda^{BC} + \check{N}, \tag{92}$$

$$\check{N} = \begin{bmatrix} 0_{3\times n_b} & 0 & \dots & \check{n}_i & \dots & 0 \end{bmatrix}, \tag{93}$$

where, Λ^{BC} is the same as Λ in(80) except that $\eta_i = 0$ in Λ^{BC}, and the constant term $\check{N}$ has been added.

Theorem 5 *Suppose constraints (boundary conditions) in (88) apply. Then the reduced order dynamics are*

$$\ddot{Q} + QK^{BC} = -(WD\Phi^T + \check{N}\mathcal{K})\mathcal{M}^{-1}\Lambda^{BC^+}, \tag{94}$$

where

$$K = \Lambda^{BC}\mathcal{K}\mathcal{M}^{-1}\Lambda^{BC^+} \tag{95}$$

5.3 Flat Shape Constraint

Suppose we require a subset Ω_F of nodes n_i, $i \in \Omega_F$ to lie in a specified plane. Then all vectors connecting nodes in this plane are perpendicular to the vector normal to this plane. That is, all vectors parallel to the specified plane (with specified normal v), satisfy,

$$v^T(n_i - n_j) = v^T N C_{ij} \qquad i, j \in \Omega_F \tag{96}$$

$$v^T N \begin{bmatrix} C_{ij} & C_{ik} & C_{jk} & ... \end{bmatrix} = v^T N C_F = 0, \tag{97}$$

where $(i, j, k, ...) \in \Omega_F$. Hence,

$$v^T N C_F = v^T \mathcal{Q} \Phi^T C_F = 0. \tag{98}$$

The standard way of handling the above constraint is to use Lagrange multipliers. If we add a matrix of forces and torques that can enforce the constraint (98), then the system dynamics are described by

$$\ddot{\mathcal{Q}}\mathcal{M} + \mathcal{Q}\mathcal{K} = -WD\Phi^T + F_\lambda \tag{99}$$

$$v^T N C_F = v^T \mathcal{Q} \Phi^T C_F = 0. \tag{100}$$

As before, from (29-33), the Lagrange multiplier must lie in the range space of the first variation of the constraint. Let $F_{\lambda_{ij}}$ denote the nonworking constraint force associated with constraint (96).

6 Statics of Class 1 Tensegrity Systems

Define

$$\bar{\mathcal{Q}} = lim_{t\to\infty} \mathcal{Q}(t). \tag{101}$$

Lemma 10 *All stable equilibria of class 1 tensegrity structures satisfy*

$$-\bar{f}_i = \bar{f}_{i+n_2} = \bar{b}_i \bar{\theta}_i, \tag{102}$$

for some scalar $\bar{\theta}_i$.

Proof: All static equilibria of (68) satisfy

$$\bar{\mathcal{Q}}\bar{K}_0\mathcal{M} = \bar{F}\Phi, \tag{103}$$

where,

$$\bar{F}\Phi = \begin{bmatrix} \frac{1}{2}(\bar{F}_2 - \bar{F}_1) & \bar{F}_1 + \bar{F}_2 \end{bmatrix}. \tag{104}$$

Hence the last n_b columns of (103) yield

$$\bar{F}\begin{bmatrix} I \\ I \end{bmatrix} = \bar{F}_1 + \bar{F}_2 = 0, \tag{105}$$

and the first n_b columns of (103) yield,

$$\bar{B}\hat{\bar{\theta}} = \frac{1}{2}(\bar{F}_2 - \bar{F}_1) \qquad\qquad = \bar{F}_2. \tag{106}$$

The i^{th} columns of (105) and (106) yield the result (102). □

Lemma 11 *If $\bar{\theta}_i > 0$, in a class 1 tensegrity structure, the bar b_i can be replaced by a string, without changing the equilibrium, where*

$$\bar{\theta}_i = \frac{1}{2} L_i^{-2} \bar{b}_i^T (\bar{\mathcal{Q}} \Phi^T C^T (\hat{C}_{1i} - \hat{C}_{2i}) \bar{\gamma} - \bar{W}(D_{2_i} - D_{1_i})) + \frac{m_i}{12 L_i^2} ||\dot{\bar{b}}_i||^2. \quad (107)$$

where, $(C_1)_i$ and $(C_2)_i$ denote the i^{th} column of matrices C_1 and C_2.

Proof: The result (107) follows from (71), noting that $\hat{x} v = \hat{v} x$, for any two vectors x and v. □

Note that if $\bar{\theta}_i > 0$ the bar b_i is in tension, and if $\bar{\theta}_i < 0$, the bar b_i is in compression. In class 1 tensegrity, the bars should all be in compression. Otherwise, we would replace the *bar* with a *string* to provide a more efficient tensile member. We note from above that in the equilibrium, $\bar{\theta}_i$ must be a negative for all i to have all bars in compression.

Lemma 12 *All class 1 tensegrity equilibria satisfy*

$$\bar{\mathcal{Q}} \Phi^T C^T \bar{\Gamma} C \begin{bmatrix} I \\ I \end{bmatrix} = -\bar{W}(D_1 + D_2). \quad (108)$$

Proof: From (104) and (105) observe that

$$\bar{F} \begin{bmatrix} I \\ I \end{bmatrix} = (-\bar{\mathcal{Q}} \Phi^T C^T \bar{\Gamma} C - \bar{W} D) \begin{bmatrix} I \\ I \end{bmatrix} = 0. \quad (109)$$

□

All stable equilibria of (62) satisfy

$$\bar{\mathcal{Q}} \bar{\mathcal{K}} = -\bar{W} D \Phi, \quad (110)$$

where,

$$\bar{\mathcal{K}} = \begin{bmatrix} \hat{\bar{\theta}} & 0 \\ 0 & 0 \end{bmatrix} + \Phi^T C^T \bar{\Gamma} C \Phi, \quad (111)$$

where, for $i \leqslant n_b$, θ_i is given by lemma 11 above.

Furthermore we note the fact that

$$C\Phi = \begin{bmatrix} \frac{1}{2}(C_2 - C_1) & C_1 + C_2 \end{bmatrix}. \quad (112)$$

The i^{th} column of (110) yields, using (111), and (112),

$$(I - L_i^{-2} \bar{b}_i \bar{b}_i^T)(\bar{\mathcal{Q}} \Phi^T C^T (\hat{C}_{2i} - \hat{C}_{1i}) \bar{\gamma} + \bar{W}(D_{2_i} - D_{1_i})) + \frac{m_i}{6 L_i^2} ||\dot{\bar{b}}_i||^2 \bar{b}_i = 0. \quad (113)$$

For $i \geqslant n_b$, the i^{th} column of (110) yields,

$$\bar{\mathcal{Q}} \Phi^T C^T (\hat{C}_{1i} + \hat{C}_{2i}) \bar{\gamma} = 0. \quad (114)$$

Assembly of equations (114) and (113) yields

$$A\bar{\gamma} = y, \tag{115}$$

where, the i^{th} row block $(3 \times n_s)$ of A is

$$A_i = (I - L_i^{-2}\bar{b}_i\bar{b}_i^T)\bar{\mathcal{Q}}\Phi^T C^T(\hat{C}_{2i} - \hat{C}_{1i}), \tag{116}$$

and

$$y_i = -(I - L_i^{-2}\bar{b}_i\bar{b}_i{}^T)\bar{W}(D_{2_i} - D_{1_i}) - \frac{m_i}{6L_i^2}||\dot{\bar{b}}_i||^2 b_i, \quad i \le n_b. \tag{117}$$

For $i = 1, 2...n_b$, the $i + n_b$ row blocks of A (also $3 \times n_s$),

$$A_{i+n_b} = \bar{\mathcal{Q}}\Phi^T C^T(\hat{C}_{1i} + \hat{C}_{2i}), \tag{118}$$

and for $i = 1, 2...n_b$,

$$y_{i+n_b} = -\bar{W}(D_{1_i} + D_{2_i}). \tag{119}$$

Lemma 13 *Any static equilibrium ($\dot{\bar{\mathcal{Q}}} = 0$), for class 1 tensegrity structures satisfy: column rank of $A < n_s$, where matrix $A(\bar{\mathcal{Q}})$ is given by (116) and (118).*

Proof: Note that $y = 0$ for any static equilibrium. □

Note the equivalence between (105), (108), and (115)/ (118), and the equivalence between (106) and (115)/(116).

6.1 Buckling Load Constraints

From (105) and (106), the static loads in the bars are described by $\bar{F}_2$. The following characterizes the Euler buckling load constraint for all bars.

$$F_2^T F_2 < ML^{-2}\Lambda L^{-2}M, \tag{120}$$

where, Λ is diagonal with elements $\frac{E_i\pi}{4\rho_i^2}$, and ρ_i is mass density of the bar material, and E_i is Young's modulus of elasticity.

Lemma 14 *The Euler buckling of the bars is prevented by satisfying the matrix inequality*

$$\begin{bmatrix} ML^{-2}\Lambda L^{-2}M & F_2^T \\ F_2 & I \end{bmatrix} > 0. \tag{121}$$

Noting that $N = \mathcal{Q}\Phi^T$ and that $F = -\mathcal{Q}\Phi^T C^T \Gamma C$, this lemma is equivalent to the inequality,

$$\begin{bmatrix} ML^{-2}\Lambda L^{-2}M & \begin{bmatrix} 0 & I \end{bmatrix} C^T\Gamma C N^T \\ NC^T\Gamma C \begin{bmatrix} 0 \\ I \end{bmatrix} & I \end{bmatrix} > 0, \tag{122}$$

which is easy to test if the nodal positions of interest, $\bar{N}$, are specified. These two above matrix inequalities describe a condition that is conservative, for two different reasons. The actual buckling constraint limits the force directed along the bar, whereas, the force $F_2(t)$ is not directed along the bar, accept at $t \to \infty$. Secondly, the actual buckling constraints limits the diagonal elements of the matrix $F_2^T F_2$, and the matrix inequality is conservative because $F_2^T F_2$ is not diagonal. If only the diagonal elements of the matrix must be diagonal, then the "matrix positive definite" condition is conservative by the size of the Schur compliments. So, the off-diagonal elements of $F_2^T F_2$ relate to the orthogonality of the bar forces. That is, $F_2^T F_2$ is diagonal if f_{i+n_b} is orthogonal to f_{j+n_b} for $j > i$. Such conservatism is trivial to accommodate in practice, since the safety margins always involve an inflation in the values used for Λ (Λ selected "twice" the actual buckling load, for a factor of "two" margin, etc). Hence, (120),(121), or (122) are practical buckling constraints, considering that Λ might be scaled for safety margins.

7 Material Selections

The previous section helps one to design the geometry and the force distribution among the members to get a stable equilibrium, ($\bar{Q}$ and $\bar{\Gamma}$). This section must choose material properties to synthesize the decisions made in the previous design of $\bar{Q}$ and $\bar{\Gamma}$.

Let the following definitions apply:

- σ_{i_y} = yield strength of string s_i
- E_i = Young's modulus of the bar material
- A_i = cross-sectional area of string s_i
- $s_{i_0} = \frac{s_i(0)}{||s_i||}$, where $s_i(0)$ represents the unstretched length (rest-length) of the string, and $||s_i||$ denotes the actual length
- k_i= spring constant of string s_i

For linear elastic material,

$$||t_i|| = k_i(||s_i|| - s_i(0)). \tag{123}$$

From the definition of the string force density γ_i, and the above definitions it follows that

$$\Gamma = \hat{\gamma} = \hat{k}(I - \hat{s_0}), \tag{124}$$

where, $\hat{\gamma}$ is diagonal with diagonal elements γ_i, $\hat{k}$ is diagonal with diagonal elements k_i, and $\hat{s_0}$ is diagonal with elements $s_{i_0} = \frac{s_i(0)}{||s_i||}$. Note that $\frac{s_i(0)}{||s_i||} < 1$ when the rest length does not exceed the actual length (hence $\gamma_i > 0$), and $\frac{s_i(0)}{||s_i||} > 1$ if the rest length exceeds the actual length (yielding $\gamma_i < 0$). This latter event cannot happen with strings that cannot take compression, but this may happen if strings are replaced by springs which can be in compression or in tension.

8 Control Problems

This section treats γ as a control variable, where we state a variety of control problems. We shall describe the imprecision in control actuation by uncorrelated zero-mean white noise sources. We shall assume that the state is measured without error for feedback control, but noisy sensors can be added in future work. Let (110) and (111) characterize all equilibria. Let the expectation operator be written $\mathcal{E}$. Let Y, defined by,

$$Y = \mathcal{QP} \tag{125}$$

for a given matrix $\mathcal{P}$, characterize the particular subset of the generalized coordinates $\mathcal{Q}$ that we wish to control to some specified accuracy Ω. That is, we require the dynamic response to satisfy the following performance constraint,

$$\mathcal{E}(Y - \bar{Y})^T (Y - \bar{Y}) < \Omega, \tag{126}$$

where the output Y at equilibrium is $\bar{Y} = \bar{\mathcal{Q}}\mathcal{P}$. Suppose we further constrain the control effort by covariance upper bound U,

$$\mathcal{E}(\gamma - \bar{\gamma})(\gamma - \bar{\gamma})^T < U. \tag{127}$$

The first problem we seek to solve is to find $(\bar{\mathcal{Q}}, \dot{\bar{\mathcal{Q}}}, \bar{\gamma}, \gamma)$ such that the following constraints are satisfied (126), (127), (110),(111), and (62).

8.1 Linearization

The nonlinear stochastic problem stated above is difficult and we shall not attempt to solve it directly. However, in nonlinear systems theory it is well known that a solution $(\bar{\mathcal{Q}}, \dot{\bar{\mathcal{Q}}}, \bar{\gamma}, \bar{W}))$ of the nonlinear system is locally stable, if the null solution of the system linearized about $(\bar{\mathcal{Q}}, \dot{\bar{\mathcal{Q}}}, \bar{\gamma}, \bar{W}))$ is asymptotically stable. Therefore, to guarantee at least a local stability of the solution of the nonlinear system, we shall linearize (62) about solution $(\bar{\mathcal{Q}}, \dot{\bar{\mathcal{Q}}}, \bar{\gamma}, \bar{W}))$ satisfying (110),(111).

Theorem 6 *A linearization of (62) about the solution $(\bar{\mathcal{Q}}, \dot{\bar{\mathcal{Q}}}, \bar{\gamma}, \bar{W}))$, satisfying (110),(111), satisfies*

$$\ddot{\tilde{\mathcal{Q}}}\mathcal{M} + \tilde{\mathcal{Q}}\bar{\mathcal{K}} + \bar{\mathcal{Q}}\tilde{\mathcal{K}} = -\tilde{W}D\Phi, \tag{128}$$

where, $(\bar{\mathcal{Q}}, \dot{\bar{\mathcal{Q}}}, \bar{\gamma}, \bar{W}))$ represents any chosen solution of

$$\ddot{\bar{\mathcal{Q}}}\mathcal{M} + \bar{\mathcal{Q}}\bar{K} = -\bar{W}D\Phi, \tag{129}$$

and the $\tilde{\mathcal{Q}}$ notation denotes the small difference between the actual solution $\mathcal{Q}$ and the desired solution $\bar{\mathcal{Q}}$, ($\tilde{\mathcal{Q}} = \mathcal{Q} - \bar{\mathcal{Q}}$). The matrix $\bar{\mathcal{K}}$ is given by

$$\bar{\mathcal{K}} = \begin{bmatrix} \hat{\bar{\theta}} & 0 \\ 0 & 0 \end{bmatrix} + \Phi^T C^T \hat{\bar{\gamma}} C \Phi, \tag{130}$$

and the matrix $\tilde{\mathcal{K}}$ is given by

$$\tilde{\mathcal{K}} = \begin{bmatrix} \hat{\tilde{\theta}} & 0 \\ 0 & 0 \end{bmatrix} + \Phi^T C^T \hat{\tilde{\gamma}} C \Phi, \tag{131}$$

where

$$\bar{\theta}_i = \frac{1}{2} L_i^{-2} \bar{b}_i^T (\bar{\mathcal{Q}} \Phi^T C^T (\hat{C}_{1i} - \hat{C}_{2i}) \bar{\gamma} - \bar{W}(D_{2_i} - D_{1_i})) + \frac{m_i}{12 L_i^2} ||\dot{\bar{b}}_i||^2, \tag{132}$$

and,

$$\tilde{\theta}_i = \frac{1}{2} L_i^{-2} \bar{\gamma}^T (\hat{C}_{1_i} - \hat{C}_{2_i}) C \Phi \bar{\mathcal{Q}}^T \tilde{b}_i + \frac{1}{2} L_i^{-2} \bar{b_i}^T (\bar{\mathcal{Q}} \Phi^T C^T (\hat{C}_{1_i} - \hat{C}_{2_i}) \tilde{\gamma} \tag{133}$$

$$+ \tilde{\mathcal{Q}} \Phi^T C^T (\hat{C}_{1i} - \hat{C}_{2i}) \bar{\gamma}) - \frac{1}{2} L_i^{-2} (\bar{b_i}^T \tilde{W} + \tilde{b_i}^T \bar{W})(D_{2_i} - D_{1_i}) + \frac{m_i}{6 L_i^2} \dot{\bar{b}}_i^T \dot{\tilde{b}}_i, \tag{134}$$

9 Conclusions

We have derived the dynamics of a system of n_b disconnected rigid bodies in the form of a second order differential equation of a $3 \times 2n_b$ *configuration matrix*. These equations contain no trigonometric nonlinearities, and require no inversion of a mass matrix containing configuration variables. All equilibria are characterized, and for any given configuration the equilibria equations reduce to linear algebra in the steady state control variables, the string force densities.

The nonlinear equations have been linearized about unspecified equilibria, in order to state control problems that allow freedom in the equilibria to be determined at the same time the controls are determined. This allows smaller energy to control to a specified error bound on the configuration.