PERGAMON Journal of the Franklin Institute 338 (2001) 255–320

Journal of The Franklin Institute

www.elsevier.nl/locate/jfranklin

Dynamics of the shell class of tensegrity structures

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Abstract

A tensegrity structure is a special truss structure in a stable equilibrium, with selected members designated for only tension loading, and the members in tension form a continuous network of cables separated by a set of compressive members. This paper develops an explicit analytical model of the nonlinear dynamics of a large class of tensegrity structures, constructed of rigid rods connected by a continuous network of elastic cables. The kinematics are described by positions and velocities of the ends of the rigid rods, hence, the use of angular velocities of each rod is avoided. The model yields an analytical expression for accelerations of all rods, making the model efficient for simulation, since the update and inversion of a nonlinear mass matrix is not required. The model is intended for shape control and design of deployable structures. Indeed, the explicit analytical expressions are provided herein for the study of stable equilibria and controllability, but the control issues are not treated in this paper.

Keywords: Tensegrity; Rigid body; Dynamics; Flexible structures

1. Introduction

The history of structural design can be divided into four eras classified by design objectives: in the *prehistoric era* which produced such structures as Stonehenge, the

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PII: S0016-0032(00)00078-8

objective was simply to oppose gravity, to take the *static* loads. The *classical era*, considered the *dynamic* response and placed design constraints on the eigenvectors as well as eigenvalues. In the *modern era*, design constraints could be so demanding that the dynamic response objectives require feedback control. In this era the control discipline *followed* the classical structure design, where the structure and control disciplines were ingredients in a *multidisciplinary* system design, but no *interdisciplinary* tools were developed to integrate the design of the structure and the control. Hence, in this modern era, the dynamics of the structure and control were not cooperating to the fullest extent possible. The *post-modern era* of structural systems is identified by attempts to unify the structure and control design to a common objective.

The ultimate performance capability of many new products and systems cannot be achieved until mathematical tools exist that can extract that full measure of cooperation possible between the dynamics of all components (structural components, controls, sensors, actuators, etc.) This requires new research. Control theory describes how the design of one component (the controller) should be influenced by the (given) dynamics of all other components. However, in *systems design*, where *more than one* component remains to be designed, there is inadequate theory to suggest how the dynamics of two or more components should influence each other at the design stage. In the future, controlled structures will not be conceived merely as *multi*disciplinary design steps, where a plate, beam or shell is first designed, followed by the addition of control actuation. Rather, controlled structures will be conceived as an *inter*disciplinary process in which both material architecture and feedback information architecture will be jointly determined. New paradigms for material and structure design might be found to help unify the disciplines. Such a search motivates this work. Preliminary work on the integration of structure and control design appears in [1–3].

Bendsoe and others [4–7] optimize structures by beginning with a solid brick and deleting finite elements until minimal mass or other objective functions are extremized. But, a very important factor in determining performance is the paradigm used for structure design. This paper describes the dynamics of a structural system composed of axially loaded compression members and tendon members that easily allow the unification of structure and control functions. Sensing and actuating functions can sense or control the tension or the length of tension members. Under the assumption that the axial loads are much smaller than the buckling loads, we treat the rods as rigid bodies. Since all members experience only axial loads the mathematical model is more accurate than models of systems with members in bending. This unidirectional loading of members is a distinct advantage of our paradigm, since this eliminates many nonlinearities that plague other controlled structural concepts: hysterisis, friction, deadzones, backlash.

It has been known since the middle of the 20th century that continua cannot explain the strength of materials. While science can now *observe* at the nanoscale, to witness the architecture of materials preferred by nature, we cannot yet *design* or *manufacture* man-made materials that duplicate the incredible structural efficiencies of natural systems. Nature's strongest fiber, the spider fiber, arranges simple non-toxic materials (amino acids) into a microstructure that contains a continuous network of members in tension (amorphous strains) and a discontinuous set of members in compression (the β-pleated sheets in Fig. 1 [8,9]).

This class of structure, with a continuous network of tension members and discontinuous network of compression members, will be called a Class 1 *tensegrity* structure. The important lessons learned from the tensegrity structure of the spider fiber is that:

(i) Structural members never reverse their role. The compressive members never take tension, and of course, tension members never take compression.
(ii) The compressive members do not touch (there are no joints in the structure).
(iii) The tensile strength is largely determined by the local topology of tension and compressive members.

Another example from nature, with important lessons for our new paradigms is the carbon nanotube often called the Fullerene (or Buckytube), which is a derivative of the Buckyball. Imagine a 1-atom thick sheet of a graphene, which has hexagonal holes, due to the arrangements of material at the atomic level (see Fig. 2). Now imagine that the flat sheet is closed into a tube by choosing an axis about which the sheet is closed to form a tube. A specific set of rules must define this closure which

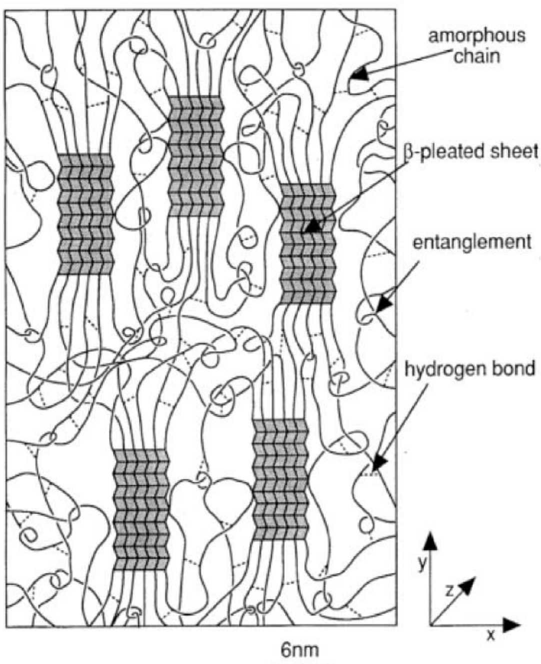

Fig. 1. Nature's strongest fiber: the spider fiber [9].

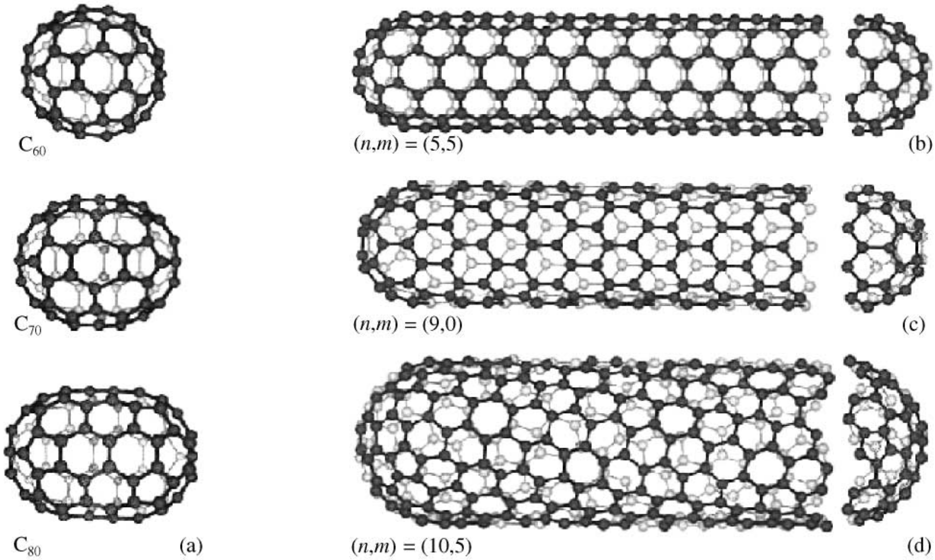

Fig. 2. Buckytubes [27].

takes the sheet to a tube, and the electrical and mechanical properties of the resulting tube depends on the *rules of closure* (axis of wrap, relative to the local hexagonal topology) [27]. Smalley won the Nobel Prize in 1996 for these insights into the Fullerenes. The spider fiber and the Fullerene provide motivation to construct man-made materials whose overall mechanical, thermal, and electrical properties can be predetermined by choice of the local topology and the rules of closure which generate the three-dimensional structure from a given local topology. By combining these motivations from Fullerenes with the tensegrity architecture of the spider fiber, this paper will derive the static and dynamic models of a shell class of tensegrity structures. Future papers will exploit the control advantages of such structures. The existing literature on tensegrity deals mainly with statics [10–22], with some elementary work on the dynamics in [23–25].

2. Tensegrity definitions

Kenneth Snelson built the first tensegrity structure in 1948 (Fig. 3) and Buckminster Fuller coined the word "tensegrity". For 50 years tensegrity has existed as an art form with some architectural appeal, but the engineering use has been hampered by the lack of models for the dynamics. In fact, the engineering use of tensegrity has been doubted by the inventor himself, Kenneth Snelson, "As I see it, this type of structure, at least in its purest form, is not likely to prove highly efficient or utilitarian," [letter to R. Motro, International Journal of Space Structures]. This

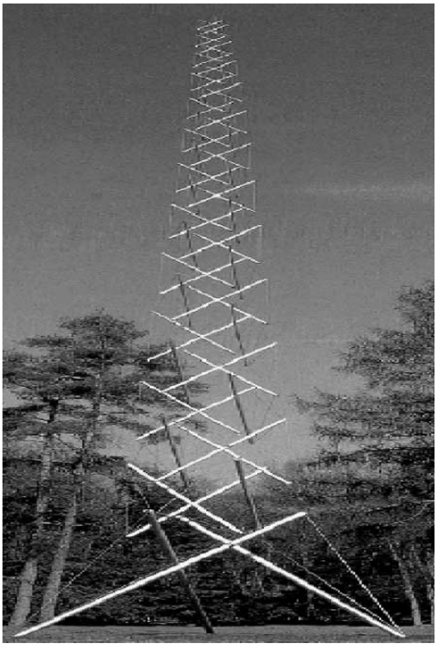

Fig. 3. Needle Tower of Kenneth Snelson, Class 1 tensegrity. Krueller-Mueller Museum, Otterlo, Netherlands.

statement might partially explain why no one bothered to develop math models to convert the artform into an engineering practice. We seek to use science to prove the artist wrong, that his invention is indeed more valuable than the artistic scope that he ascribed to it. Mathematical models are essential design tools to make engineered products. This paper provides a dynamical model of a class of tensegrity structures that is appropriate for space structures.

This paper derives the nonlinear equations of motion for space structures that can be deployed or held to precise shape by feedback control, although control is beyond the scope of this paper. For engineering purposes, more precise definitions of tensegrity are needed.

One can imagine a truss as a structure whose compressive members are all connected with ball joints so that no torques can be transmitted. Of course, tension members connected to compressive members do not transmit torques, so that our truss is composed of members experiencing no moments. The following definitions are useful.

Definition 2.1. A given configuration of a structure is in a *stable equilibrium* if, in the absence of external forces, an arbitrarily small initial deformation returns to the given configuration.

Definition 2.2. A tensegrity structure is a stable system of axially loaded-members.

Definition 2.3. A stable structure is said to be a "Class 1" tensegrity structure if the members in tension form a continuous network, and the members in compression form a discontinuous set of members.

Definition 2.4. A stable structure is said to be a "Class 2" tensegrity structure if the members in tension form a continuous set of members, and there are at most two members in compression connected to each node.

Fig. 4 illustrates a Class 1 and a Class 2 tensegrity structure.

Consider the topology of structural members given in Fig. 5, where thick lines indicate rigid rods which take compressive loads and the thin lines represent tendons. This is a Class 1 tensegrity structure.

Definition 2.5. Let the topology of Fig. 5 describe a 3-dimensional structure by connecting points A to A, B to B, C to $C, \ldots, I$ to I. This constitutes a "Class 1 Tensegrity Shell" if there exists a set of tensions in all tendons $t_{\alpha\beta\gamma}$ $(\alpha = 1 \to 10,\ \beta = 1 \to n,\ \gamma = 1 \to m)$ such that the structure is in a stable equilibrium.

2.1. A typical element

The axial members in Fig. 5 illustrate only the pattern of member connections, and not the actual loaded configuration. The purpose of this section is two-fold: (i) to define a typical "element" which can be repeated to generate all elements, (ii) to define rules of closure that will generate a "shell" type of structure.

Consider the members which make the typical ij element where $i = 1, 2, \ldots, n$ indexes the element to the left, and $j = 1, 2, \ldots, m$ indexes the element up the page in Fig. 5. We will describe the axial elements by vectors. That is, the vectors describing the ij element, are $\mathbf{t}_{1ij}, \mathbf{t}_{2ij}, \ldots, \mathbf{t}_{10ij}$ and $\mathbf{r}_{1ij}, \mathbf{r}_{2ij}$, where, within the ij element, $\mathbf{t}_{\alpha ij}$ is a vector whose tail is fixed at the specified end of tendon number α, and the head of the vector is fixed at the other end of tendon number α as shown in Fig. 6 where $\alpha = 1, 2, \ldots, 10$. The ij element has two compressive members which we will call "rods", shaded in Fig. 6. Within the ij element the vector $\mathbf{r}_{1ij}$ lies along the rod r_{1ij} and the vector $\mathbf{r}_{2ij}$ lies along the rod r_{2ij}. The first goal of this paper is to derive the equations of motion for the dynamics of the two rods in the ij element. The second goal is to write the dynamics for the entire system composed of nm elements.

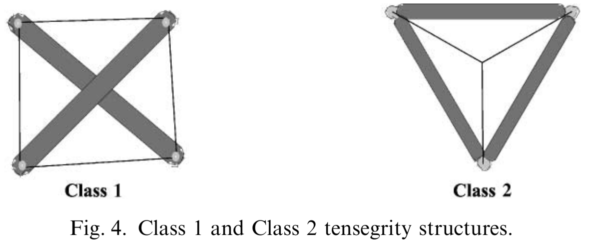

Fig. 4. Class 1 and Class 2 tensegrity structures.

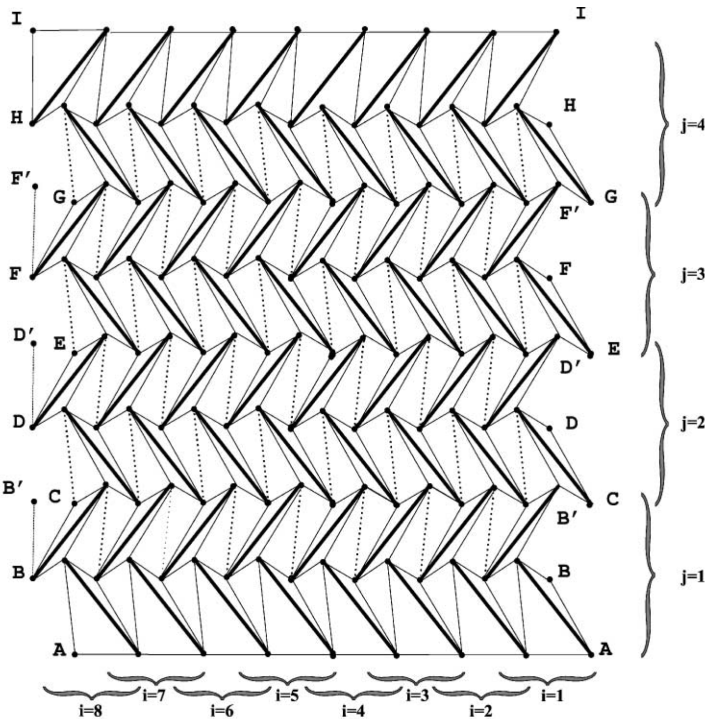

Fig. 5. Topology of an (8,4) Class 1 tensegrity shell.

Figs. 5 and 7 illustrate these closure rules for the case $(n, m) = (8, 4)$ and $(n, m) = (3, 1)$.

Lemma. *Consider the structure of Fig. 5 with elements defined by Fig. 6. A Class 2 tensegrity shell is formed by adding constraints such that for all $i = 1, 2, \ldots, n$, and for $m > j > 1$,*

$$\begin{aligned} -\mathbf{t}_{1ij} + \mathbf{t}_{4ij} &= \mathbf{0}, \\ \mathbf{t}_{2ij} + \mathbf{t}_{3ij} &= \mathbf{0}, \\ \mathbf{t}_{5ij} + \mathbf{t}_{6ij} &= \mathbf{0}, \\ \mathbf{t}_{7ij} + \mathbf{t}_{8ij} &= \mathbf{0}. \end{aligned} \tag{2.1}$$

This closes nodes n_{2ij} and $n_{1(i+1)(j+1)}$ to a single node, and closes nodes $n_{3(i-1)j}$ and $n_{4i(j-1)}$ to a single node (with ball joints). The nodes are closed outside the rod, so that all tension elements are on the exterior of the tensegrity structure and the rods are in the interior.

The point here is that a Class 2 shell can be obtained as a special case of the Class 1 shell, by imposing constraints (2.1). To create a tensegrity structure, not all tendons

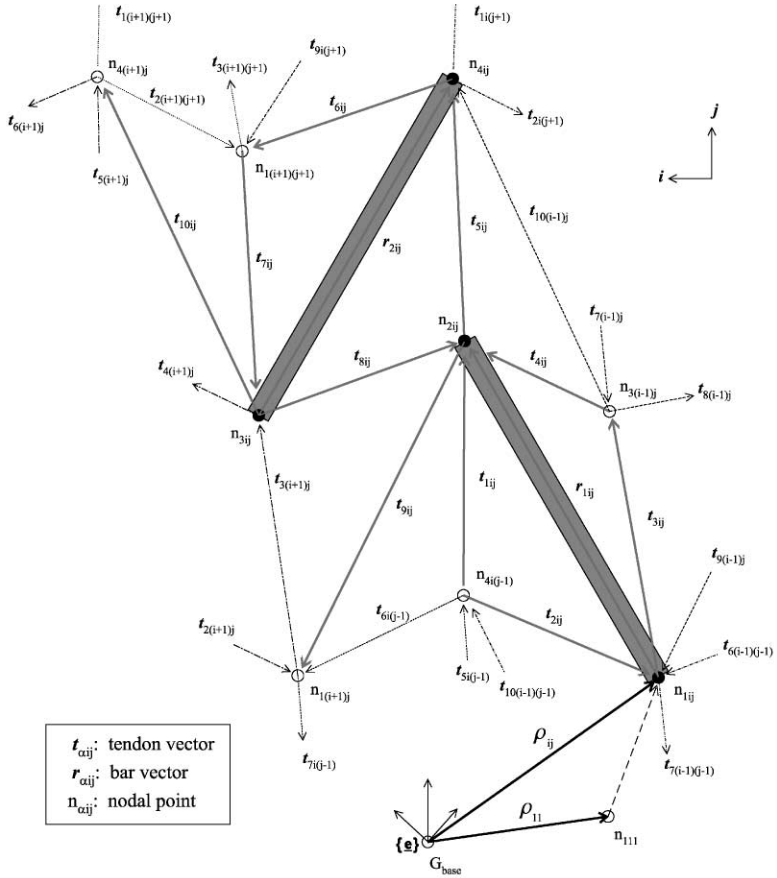

Fig. 6. A typical *ij* element.

in Fig. 5 are necessary. The following definition eliminates tendons $\mathbf{t}_{9ij}$ and $\mathbf{t}_{10ij}$ $(i = 1 \to n,\ j = 1 \to m)$.

Definition 2.6. Consider the shell of Figs. 5 and 6, which may be Class 1 or Class 2 depending on whether constraints (2.1) are applied. In the absence of dotted tendons (labelled t_9 and t_{10}), this is called a Primal Tensegrity Shell. When all tendons t_9, t_{10} are present in Fig. 5, it is called simply Class 1 or Class 2 tensegrity shell.

The remainder of this paper focuses on the general Class 1 shell of Figs. 5 and 6.

2.2. Rules of closure for the shell class

Each tendon exerts a positive force away from a node and $\mathbf{f}_{\alpha\beta\gamma}$ is the force exerted by tendon $\mathbf{t}_{\alpha\beta\gamma}$ and $\hat{\mathbf{f}}_{\alpha ij}$ denotes the force vector acting on the node $n_{\alpha ij}$. All $\mathbf{f}_{\alpha ij}$ forces

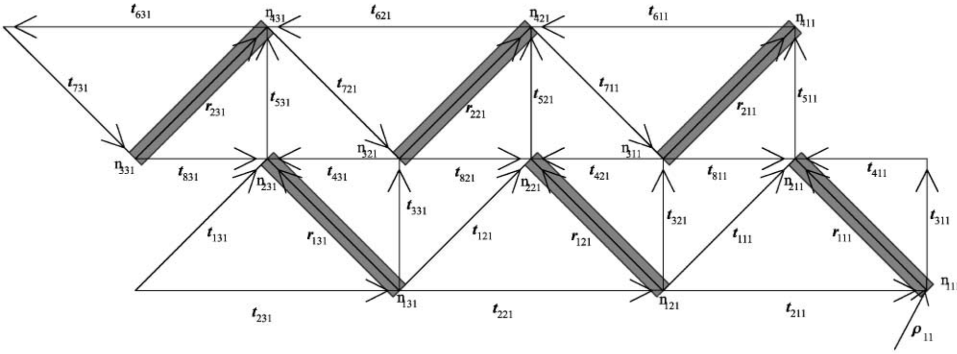

Fig. 7. Class 1 shell: $(n,m)=(3,1)$.

are *positive in the direction of the arrows* in Fig. 6, where $\mathbf{w}_{\alpha ij}$ is the external applied force at node $n_{\alpha ij}$, $\alpha = 1,2,3,4$. At the base, the rules of closure, from Figs. 5 and 6, are

$$\mathbf{t}_{9i1} = -\mathbf{t}_{1i1}, \quad i = 1,2,\ldots,n, \tag{2.2}$$

$$\mathbf{t}_{6i0} = \mathbf{0}, \tag{2.3}$$

$$\mathbf{t}_{600} = -\mathbf{t}_{2n1}, \tag{2.4}$$

$$\mathbf{t}_{901} = \mathbf{t}_{9n1} = -\mathbf{t}_{1n1}, \tag{2.5}$$

$$\mathbf{0} = \mathbf{t}_{10(i-1)0} = \mathbf{t}_{5i0} = \mathbf{t}_{7i0} = \mathbf{t}_{7(i-1)0}, \quad i = 1,2,\ldots,n. \tag{2.6}$$

At the top, the closure rules are

$$\mathbf{t}_{10im} = -\mathbf{t}_{7im}, \tag{2.7}$$

$$\mathbf{t}_{100m} = -\mathbf{t}_{70m} = -\mathbf{t}_{7nm}, \tag{2.8}$$

$$\mathbf{t}_{2i(m+1)} = \mathbf{0}, \tag{2.9}$$

$$\begin{aligned}\mathbf{0} &= \mathbf{t}_{1i(m+1)} = \mathbf{t}_{9i(m+1)} = \mathbf{t}_{3(i+1)(m+1)} \\ &= \mathbf{t}_{1(i+1)(m+1)} = \mathbf{t}_{2(i+1)(m+1)}.\end{aligned} \tag{2.10}$$

At the closure of the circumference (where $i = 1$):

$$\mathbf{t}_{90j} = \mathbf{t}_{9nj}, \quad \mathbf{t}_{60(j-1)} = \mathbf{t}_{6n(j-1)}, \quad \mathbf{t}_{70(j-1)} = \mathbf{t}_{7n(j-1)}, \tag{2.11}$$

$$\mathbf{t}_{80j} = \mathbf{t}_{8nj}, \quad \mathbf{t}_{70j} = \mathbf{t}_{7nj}, \quad \mathbf{t}_{100(j-1)} = \mathbf{t}_{10n(j-1)}. \tag{2.12}$$

From Figs. 5 and 6, when $j = 1$, then

$$\mathbf{0} = \mathbf{f}_{7i(j-1)} = \mathbf{f}_{7(i-1)(j-1)} = \mathbf{f}_{5i(j-1)} = \mathbf{f}_{10(i-1)(j-1)} \tag{2.13}$$

and for $j = m$ where, in Figs. 5 and 6,

$$\mathbf{0} = \mathbf{f}_{1i(m+1)} = \mathbf{f}_{9i(m+1)} = \mathbf{f}_{3(i+1)(m+1)} = \mathbf{f}_{1(i+1)(m+1)}. \tag{2.14}$$

Nodes $n_{11j}, n_{3nj}, n_{41j}$ for $j = 1, 2, \ldots, m$ are involved in the longitudinal "zipper" that closes the structure in circumference. The forces at these nodes are written explicitly to illustrate the closure rules.

In Section 4, the rod dynamics will be expressed in terms of sums and differences of the nodal forces, so the forces acting on each node is presented in the following form, convenient for later use. The definitions of the matrices $\mathbf{B}_i$ are found in Appendix E.

The forces acting on the nodes can be written in vector form:

$$\mathbf{f} = \mathbf{B}^{\mathrm{d}}\mathbf{f}^{\mathrm{d}} + \mathbf{B}^0\mathbf{f}^0 + \mathbf{W}^0\mathbf{w}, \tag{2.15}$$

where

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_m \end{bmatrix}, \quad \mathbf{f}^{\mathrm{d}} = \begin{bmatrix} \bar{\mathbf{f}}_1^{\mathrm{d}} \\ \mathbf{f}_2^{\mathrm{d}} \\ \vdots \\ \mathbf{f}_m^{\mathrm{d}} \end{bmatrix}, \quad \mathbf{f}^0 = \begin{bmatrix} \mathbf{f}_1^0 \\ \vdots \\ \mathbf{f}_m^0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_m \end{bmatrix},$$

$$\mathbf{W}^0 = BlockDiag[\cdots, \mathbf{W}_1, \mathbf{W}_1, \cdots],$$

$$\mathbf{B}^{\mathrm{d}} = \begin{bmatrix} \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_5 & \mathbf{B}_6 & \ddots & \ddots & \vdots \\ \mathbf{0} & \bar{\mathbf{B}}_5 & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \mathbf{B}_6 & \mathbf{B}_4 \\ \mathbf{0} & \cdots & \mathbf{0} & \bar{\mathbf{B}}_5 & \mathbf{B}_8 \end{bmatrix}, \quad \mathbf{B}^0 = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_7 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \mathbf{B}_2 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{B}_7 \end{bmatrix}$$

and

$$\mathbf{f}_{ij}^0 = \begin{bmatrix} \mathbf{f}_5 \\ \mathbf{f}_1 \end{bmatrix}_{ij}, \quad \mathbf{f}_{ij}^{\mathrm{d}} = \begin{bmatrix} \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_6 \\ \mathbf{f}_7 \\ \mathbf{f}_8 \\ \mathbf{f}_9 \\ \mathbf{f}_{10} \end{bmatrix}_{ij}, \quad \mathbf{w}_{ij} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \mathbf{w}_4 \end{bmatrix}_{ij}.$$

Now that we have an expression for the forces, let us write the dynamics.

3. Dynamics of a 2 rod element

Any discussion of rigid-body dynamics should properly begin with some decision on how the motion of each body is to be described. A common way to describe rigid-body orientation is to use three successive angular rotations to define the orientation of three mutually orthogonal axes fixed in the body. The measure numbers of the angular velocity of the body may then be expressed in terms of these angles and their time derivatives.

This approach must be reconsidered when the body of interest is idealized as a rod. The reason is that the concept of "body fixed axes" becomes ambiguous. Two different sets of axes with a common axis along the rod can be considered equally "body fixed" in the sense that all mass particles of the rod have zero velocity in both sets. This remains true even if relative rotation is allowed along the common axis. The angular velocity of the rod is also ill defined since the component of angular velocity along the rod axis is arbitrary. For these reasons, we are motivated to seek a kinematical description which avoids introducing "body fixed" reference frames and angular velocity. This objective may be accomplished by describing the configuration of the system in terms of vectors which locate only the end points of the rods. In this case, no angles are used.

We will use the following notational conventions. Lower case, bold faced symbols with an underline will indicate vector quantities with magnitude and direction in three-dimensional space. These are the usual vector quantities we are familiar with from elementary dynamics. The same bold face symbols without an underline will indicate a matrix whose elements are scalars. Sometimes we will also need to introduce matrices whose elements are vectors. These quantities will be indicated with an upper case symbol that is both bold faced and underlined.

As an example of this notation, a position vector $\underline{\mathbf{p}}_i$ can be expressed as

$$\underline{\mathbf{p}}_i = \begin{bmatrix} \underline{\mathbf{e}}_1 & \underline{\mathbf{e}}_2 & \underline{\mathbf{e}}_3 \end{bmatrix} \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{bmatrix} = \underline{\mathbf{E}}\mathbf{p}_i.$$

In this expression, $\mathbf{p}_i$ is a column matrix whose elements are the measure numbers of $\underline{\mathbf{p}}_i$ for the mutually orthogonal inertial unit vectors $\underline{\mathbf{e}}_1$, $\underline{\mathbf{e}}_2$, $\underline{\mathbf{e}}_3$. Similarly, we may represent a force vector $\hat{\mathbf{f}}_i$ as

$$\hat{\underline{\mathbf{f}}}_i = \underline{\mathbf{E}}\hat{\mathbf{f}}_i.$$

Matrix notation will be used in most of the development to follow.

We now consider a single rod as shown in Fig. 8 with nodal forces $\hat{\underline{\mathbf{f}}}_1$ and $\hat{\underline{\mathbf{f}}}_2$ applied to the ends of the rod.

The following theorem will be fundamental to our development:

Theorem 3.1. *Given a rigid rod of constant mass m and constant length L, the governing equations may be described as*

$$\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{H}\tilde{\mathbf{f}}, \tag{3.1}$$

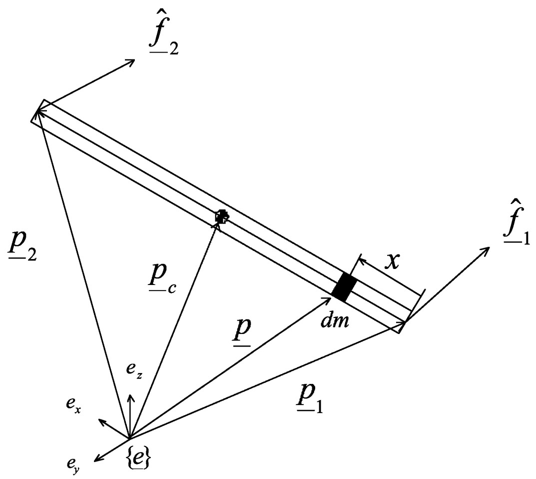

Fig. 8. A single rigid rod.

where

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 + \mathbf{p}_2 \\ \mathbf{p}_2 - \mathbf{p}_1 \end{bmatrix},$$

$$\tilde{\mathbf{f}} = \begin{bmatrix} \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2 \\ \hat{\mathbf{f}}_1 - \hat{\mathbf{f}}_2 \end{bmatrix}, \quad \mathbf{H} = \frac{2}{m}\begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \frac{3}{L^2}\tilde{\mathbf{q}}_2^2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L^{-2}\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2\mathbf{I}_3 \end{bmatrix}.$$

The notation $\tilde{\mathbf{r}}$ *denotes the skew symmetric matrix formed from the elements of* $\mathbf{r}$:

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

and the square of this matrix is

$$\tilde{\mathbf{r}}^2 = \begin{bmatrix} -r_2^2 - r_3^2 & r_1 r_2 & r_1 r_3 \\ r_2 r_1 & -r_1^2 - r_3^2 & r_2 r_3 \\ r_3 r_1 & r_3 r_2 & -r_1^2 - r_2^2 \end{bmatrix}.$$

The matrix elements $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, etc. are to be interpreted as the measure numbers of the corresponding vectors for an orthogonal set of inertially fixed unit vectors $\underline{\mathbf{e}}_1$, $\underline{\mathbf{e}}_2$, $\underline{\mathbf{e}}_3$. Thus, using the convention introduced earlier,

$$\underline{\mathbf{r}} = \underline{\mathbf{E}}\mathbf{r}, \quad \underline{\mathbf{q}} = \underline{\mathbf{E}}\mathbf{q}, \ \text{etc.}$$

The proof of Theorem 3.1 is given in Appendix A. This theorem will provide the basis of our dynamic model for the shell class of tensegrity structures.

Now consider the dynamics of the 2-rod element of the Class 1 tensegrity shell in Fig. 5. Here, we assume the lengths of the rods are constant. From Theorem 3.1 and Appendix A, the motion equations for the ij unit can be described as follows. For rod 1,

$$\begin{aligned} &\frac{m_{1ij}}{2}\ddot{\mathbf{q}}_{1ij} = \hat{\mathbf{f}}_{1ij} + \hat{\mathbf{f}}_{2ij}, \\ &\frac{m_{1ij}}{6}(\mathbf{q}_{2ij} \times \ddot{\mathbf{q}}_{2ij}) = \mathbf{q}_{2ij} \times (\hat{\mathbf{f}}_{2ij} - \hat{\mathbf{f}}_{1ij}), \\ &\dot{\mathbf{q}}_{2ij} \cdot \dot{\mathbf{q}}_{2ij} + \mathbf{q}_{2ij} \cdot \ddot{\mathbf{q}}_{2ij} = 0, \\ &\mathbf{q}_{2ij} \cdot \mathbf{q}_{2ij} = L_{1ij}^2, \end{aligned} \tag{3.2}$$

and for rod 2,

$$\begin{aligned} &\frac{m_{2ij}}{2}\ddot{\mathbf{q}}_{3ij} = \hat{\mathbf{f}}_{3ij} + \hat{\mathbf{f}}_{4ij}, \\ &\frac{m_{2ij}}{6}(\mathbf{q}_{4ij} \times \ddot{\mathbf{q}}_{4ij}) = \mathbf{q}_{4ij} \times (\hat{\mathbf{f}}_{4ij} - \hat{\mathbf{f}}_{3ij}), \\ &\dot{\mathbf{q}}_{4ij} \cdot \dot{\mathbf{q}}_{4ij} + \mathbf{q}_{4ij} \cdot \ddot{\mathbf{q}}_{4ij} = 0, \\ &\mathbf{q}_{4ij} \cdot \mathbf{q}_{4ij} = L_{2ij}^2, \end{aligned} \tag{3.3}$$

where the mass of the rod αij is $m_{\alpha ij}$ and $||\mathbf{r}_{\alpha ij}|| = L_{\alpha ij}$. As before, we refer everything to a common inertial reference frame ($\underline{\mathbf{E}}$). Hence,

$$\mathbf{q}_{1ij} \triangleq \begin{bmatrix} q_{11ij} \\ q_{12ij} \\ q_{13ij} \end{bmatrix}, \quad \mathbf{q}_{2ij} \triangleq \begin{bmatrix} q_{21ij} \\ q_{22ij} \\ q_{23ij} \end{bmatrix}, \quad \mathbf{q}_{3ij} \triangleq \begin{bmatrix} q_{31ij} \\ q_{32ij} \\ q_{33ij} \end{bmatrix}, \quad \mathbf{q}_{4ij} \triangleq \begin{bmatrix} q_{41ij} \\ q_{42ij} \\ q_{43ij} \end{bmatrix},$$

$$\mathbf{q}_{ij} \triangleq \begin{bmatrix} \mathbf{q}_{1ij}^{\mathrm{T}}, & \mathbf{q}_{2ij}^{\mathrm{T}}, & \mathbf{q}_{3ij}^{\mathrm{T}}, & \mathbf{q}_{4ij}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

and the force vectors appear in the form

$$\mathbf{H}_{1ij} = \frac{2}{m_{1ij}}\begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \frac{3}{L_{1ij}^2}\tilde{\mathbf{q}}_{2ij}^2 \end{bmatrix}, \quad \mathbf{H}_{2ij} = \frac{2}{m_{2ij}}\begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \frac{3}{L_{2ij}^2}\tilde{\mathbf{q}}_{4ij}^2 \end{bmatrix}, \quad \mathbf{H}_{ij} = \begin{bmatrix} \mathbf{H}_{1ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{2ij} \end{bmatrix},$$

$$\mathbf{f}_{ij} \triangleq \begin{bmatrix} \hat{\mathbf{f}}_{1ij} + \hat{\mathbf{f}}_{2ij} \\ \hat{\mathbf{f}}_{1ij} - \hat{\mathbf{f}}_{2ij} \\ \hat{\mathbf{f}}_{3ij} + \hat{\mathbf{f}}_{4ij} \\ \hat{\mathbf{f}}_{3ij} - \hat{\mathbf{f}}_{4ij} \end{bmatrix}.$$

Using Theorem 3.1, the dynamics for the ij unit can be expressed as follows:

$$\ddot{\mathbf{q}}_{ij} + \mathbf{\Omega}_{ij}\mathbf{q}_{ij} = \mathbf{H}_{ij}\mathbf{f}_{ij},$$

where

$$\boldsymbol{\Omega}_{ij} = \begin{bmatrix} \boldsymbol{\Omega}_{1ij} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{2ij} \end{bmatrix}, \quad \boldsymbol{\Omega}_{1ij} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_{1ij}^{-2}\dot{\mathbf{q}}_{2ij}^{\mathrm{T}}\dot{\mathbf{q}}_{2ij}\mathbf{I}_3 \end{bmatrix}, \quad \boldsymbol{\Omega}_{2ij} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_{2ij}^{-2}\dot{\mathbf{q}}_{4ij}^{\mathrm{T}}\dot{\mathbf{q}}_{4ij}\mathbf{I}_3 \end{bmatrix},$$

$$\mathbf{q} = [\mathbf{q}_{11}^{\mathrm{T}}, \ldots, \mathbf{q}_{n1}^{\mathrm{T}}, \mathbf{q}_{12}^{\mathrm{T}}, \ldots, \mathbf{q}_{n2}^{\mathrm{T}}, \ldots, \mathbf{q}_{1m}^{\mathrm{T}}, \ldots, \mathbf{q}_{nm}^{\mathrm{T}}]^{\mathrm{T}}.$$

The shell system dynamics are given by

$$\ddot{\mathbf{q}} + \mathbf{K}_r\mathbf{q} = \mathbf{H}\mathbf{f}, \tag{3.4}$$

where $\mathbf{f}$ is defined in (2.15) and

$$\mathbf{q} = [\mathbf{q}_{11}^{\mathrm{T}}, \ldots, \mathbf{q}_{n1}^{\mathrm{T}}, \mathbf{q}_{12}^{\mathrm{T}}, \ldots, \mathbf{q}_{n2}^{\mathrm{T}}, \ldots, \mathbf{q}_{1m}^{\mathrm{T}}, \ldots, \mathbf{q}_{nm}^{\mathrm{T}}]^{\mathrm{T}},$$

$$\mathbf{K}_r = BlockDiag[\boldsymbol{\Omega}_{11}, \ldots, \boldsymbol{\Omega}_{n1}, \boldsymbol{\Omega}_{12}, \ldots, \boldsymbol{\Omega}_{n2}, \ldots, \boldsymbol{\Omega}_{1m}, \ldots, \boldsymbol{\Omega}_{nm}],$$

$$\mathbf{H} = BlockDiag[\mathbf{H}_{11}, \ldots, \mathbf{H}_{n1}, \mathbf{H}_{12}, \ldots, \mathbf{H}_{n2}, \ldots, \mathbf{H}_{1m}, \ldots, \mathbf{H}_{nm}].$$

4. Choice of independent variables and coordinate transformations

The tendon vectors $t_{\alpha\beta\gamma}$ are needed to express the forces. Hence, the dynamical model will be completed by expressing the tendon forces, $\mathbf{f}$, in terms of variables $\mathbf{q}$. From Figs. 6 and 9, it follows that vectors $\hat{\boldsymbol{\rho}}_{ij}$ and $\boldsymbol{\rho}_{ij}$ can be

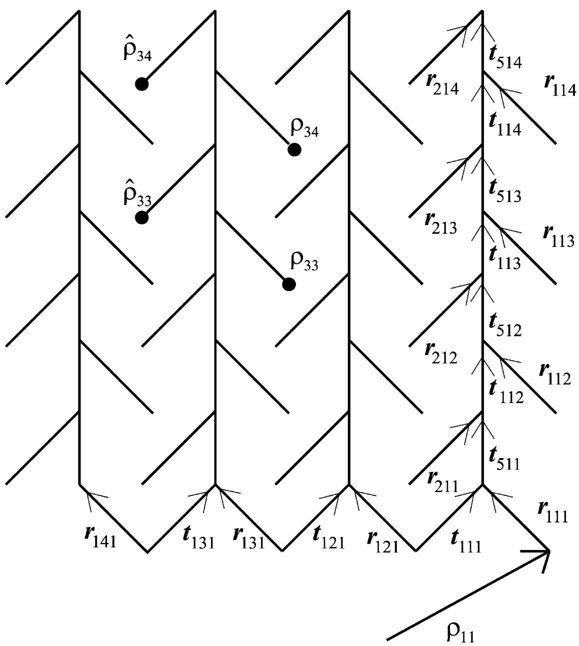

Fig. 9. Choice of independent variables.

described by

$$\boldsymbol{\rho}_{ij} = \boldsymbol{\rho}_{11} + \sum_{k=1}^{i} \mathbf{r}_{1k1} - \sum_{k=1}^{i-1} \mathbf{t}_{1k1} + \sum_{k=2}^{j} \mathbf{t}_{1ik} + \sum_{k=1}^{j-1} \mathbf{t}_{5ik} - \mathbf{r}_{1ij}, \tag{4.1}$$

$$\hat{\boldsymbol{\rho}}_{ij} = \boldsymbol{\rho}_{ij} + \mathbf{r}_{1ij} + \mathbf{t}_{5ij} - \mathbf{r}_{2ij}. \tag{4.2}$$

To describe the geometry, we choose the independent vectors $\{\mathbf{r}_{1ij},\ \mathbf{r}_{2ij},\ \mathbf{t}_{5ij}$, for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\}$ and $\{\boldsymbol{\rho}_{11},\ \mathbf{t}_{1ij}$, for $i = 1, 2, \ldots,\ n, j = 1, 2, \ldots, m$, and $i < n$ when $j = 1\}$.

This section writes the relationship between the $\mathbf{q}$ variables and the string and rod vectors $\mathbf{t}_{\alpha\beta\gamma}$ and $\mathbf{r}_{\beta ij}$. From Figs. 5 and 6, the position vectors from the origin of the reference frame, E, to the nodal points, $\mathbf{p}_{1ij},\ \mathbf{p}_{2ij},\ \mathbf{p}_{3ij}$, and $\mathbf{p}_{4ij}$, can be described as follows:

$$\begin{aligned}
\mathbf{p}_{1ij} &= \boldsymbol{\rho}_{ij},\\
\mathbf{p}_{2ij} &= \boldsymbol{\rho}_{ij} + \mathbf{r}_{1ij},\\
\mathbf{p}_{3ij} &= \hat{\boldsymbol{\rho}}_{ij},\\
\mathbf{p}_{4ij} &= \hat{\boldsymbol{\rho}}_{ij} + \mathbf{r}_{2ij}.
\end{aligned} \tag{4.3}$$

We define

$$\begin{aligned}
\mathbf{q}_{1ij} &\triangleq \mathbf{p}_{2ij} + \mathbf{p}_{1ij} = 2\boldsymbol{\rho}_{ij} + \mathbf{r}_{1ij},\\
\mathbf{q}_{2ij} &\triangleq \mathbf{p}_{2ij} - \mathbf{p}_{1ij} = \mathbf{r}_{1ij},\\
\mathbf{q}_{3ij} &\triangleq \mathbf{p}_{4ij} + \mathbf{p}_{3ij} = 2\hat{\boldsymbol{\rho}}_{1ij} + \mathbf{r}_{2ij},\\
\mathbf{q}_{4ij} &\triangleq \mathbf{p}_{4ij} - \mathbf{p}_{3ij} = \mathbf{r}_{2ij}.
\end{aligned} \tag{4.4}$$

Then,

$$\mathbf{q}_{ij} \triangleq \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{bmatrix}_{ij} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{bmatrix}_{ij} = \begin{bmatrix} 2\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}_{ij} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{ij}. \tag{4.5}$$

In shape control, we will later be interested in the $\mathbf{p}$ vector to describe all nodal points of the structure. This relation is

$$\mathbf{p} = \mathbf{P}\mathbf{q}, \quad \mathbf{P} = BlockDiag[\ldots, \mathbf{P}_1, \ldots, \mathbf{P}_1, \ldots],$$

$$\mathbf{P}_1 \triangleq \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{4.6}$$

The equations of motion will be written in the $\mathbf{q}$ coordinates. Substitution of (4.1) and (4.2) into (4.4) yields the relationship between $\mathbf{q}$ and the independent variables $\mathbf{t}_5,\ \mathbf{t}_1,\ \mathbf{r}_1,\ \mathbf{r}_2$ as follows:

$$\begin{aligned}
\mathbf{q}_{1ij} &= 2\left[\boldsymbol{\rho}_{11} + \sum_{k=1}^{i} \mathbf{r}_{1k1} - \sum_{k=1}^{i-1} \mathbf{t}_{1k1} + \sum_{k=2}^{j} \mathbf{t}_{1ik} + \sum_{k=1}^{j-1} \mathbf{t}_{5ik}\right] - \mathbf{r}_{1ij}, \\
\mathbf{q}_{2ij} &= \mathbf{r}_{1ij}, \\
\mathbf{q}_{3ij} &= 2\left[\boldsymbol{\rho}_{11} + \sum_{k=1}^{i} \mathbf{r}_{1k1} - \sum_{k=1}^{i-1} \mathbf{t}_{1k1} + \sum_{k=2}^{j} \mathbf{t}_{1ik} + \sum_{k=1}^{j} \mathbf{t}_{5ik}\right] - \mathbf{r}_{2ij}, \\
\mathbf{q}_{4ij} &= \mathbf{r}_{2ij}.
\end{aligned} \tag{4.7}$$

To put (4.7) in a matrix form, define these matrices:

$$\mathbf{l}_{ij} = \begin{bmatrix} \mathbf{r}_{1ij} \\ \mathbf{r}_{2ij} \\ \mathbf{t}_{5ij} \\ \mathbf{t}_{1ij} \end{bmatrix} \quad \text{for } j = 2, 3, \ldots, m, \quad \mathbf{l}_{11} = \begin{bmatrix} \boldsymbol{\rho}_{11} \\ \mathbf{r}_{111} \\ \mathbf{r}_{211} \\ \mathbf{t}_{511} \end{bmatrix}, \quad \mathbf{l}_{i1} = \begin{bmatrix} \mathbf{t}_{1(i-1)1} \\ \mathbf{r}_{1i1} \\ \mathbf{r}_{2i1} \\ \mathbf{t}_{5i1} \end{bmatrix},$$

for $i = 2, \ldots, n$

and

$$\mathbf{l} = [\mathbf{l}_{11}^{\mathrm{T}}, \mathbf{l}_{21}^{\mathrm{T}}, \ldots, \mathbf{l}_{n1}^{\mathrm{T}}, \mathbf{l}_{12}^{\mathrm{T}}, \ldots, \mathbf{l}_{n2}^{\mathrm{T}}, \ldots, \mathbf{l}_{1m}^{\mathrm{T}}, \ldots, \mathbf{l}_{nm}^{\mathrm{T}}]^{\mathrm{T}},$$

$$\mathbf{A} = \begin{bmatrix} 2\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ 2\mathbf{I}_3 & 2\mathbf{I}_3 & -\mathbf{I}_3 & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -2\mathbf{I}_3 & 2\mathbf{I}_3 & -\mathbf{I}_3 & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & 2\mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & 2\mathbf{I}_3 & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} -2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 2\mathbf{I}_3 & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\mathbf{I}_3 & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2\mathbf{I}_3 & 2\mathbf{I}_3 & \mathbf{0} & 2\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then (4.7) can be written simply

$$\mathbf{q} = \mathbf{Q}\mathbf{l}, \tag{4.8}$$

where the $12nm \times 12nm$ matrix $\mathbf{Q}$ is composed of the 12×12 matrices A–H as follows:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \ddots & & \vdots & \\ \mathbf{Q}_{21} & \mathbf{Q}_{32} & \mathbf{Q}_{22} & \ddots & \vdots & \\ \mathbf{Q}_{21} & \mathbf{Q}_{32} & \mathbf{Q}_{32} & \mathbf{Q}_{22} & \mathbf{0} & \\ \mathbf{Q}_{21} & \mathbf{Q}_{32} & \mathbf{Q}_{32} & \mathbf{Q}_{32} & \mathbf{Q}_{22} & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \tag{4.9}$$

$$\mathbf{Q}_{11} = \left.\begin{bmatrix} \mathbf{A} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{D} & \mathbf{B} & \ddots & & & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{B} & \ddots & & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{E} & \mathbf{B} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} & \mathbf{B} \end{bmatrix}\right\} n \times n \text{ blocks of } 12 \times 12 \text{ matrices},$$

$$\mathbf{Q}_{21} = \left.\begin{bmatrix} \mathbf{F} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{D} & \mathbf{G} & \ddots & & & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{G} & \ddots & & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{E} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{E} & \mathbf{E} & \mathbf{E} & \mathbf{G} \end{bmatrix}\right\} 12n \times 12n \text{ matrix},$$

$$\mathbf{Q}_{22} = BlockDiag[\ldots, \mathbf{C}, \ldots, \mathbf{C}],$$

$$\mathbf{Q}_{32} = BlockDiag[\ldots, \mathbf{J}, \ldots, \mathbf{J}],$$

where each $\mathbf{Q}_{ij}$ is $12n \times 12n$ and there are m row blocks and m column blocks in $\mathbf{Q}$. Appendix B provides an explicit expression for the inverse matrix $\mathbf{Q}$, which will be needed later to express the tendon forces in terms of $\mathbf{q}$.

Eq. (4.8) provides the relationship between the selected generalized coordinates and an independent set of the tendon and rod vectors forming $\mathbf{l}$. All remaining tendon vectors may be written as a linear combination of $\mathbf{l}$. This relation will now be established. The following equations are written by inspection of Figs. 5–7 where

$$\mathbf{t}_{1n1} = \boldsymbol{\rho}_{n1} + \mathbf{r}_{1n1} - \boldsymbol{\rho}_{11} \tag{4.10}$$

and for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$ we have,

$$\begin{aligned}
\mathbf{t}_{2ij} &= \boldsymbol{\rho}_{ij} - (\hat{\boldsymbol{\rho}}_{i(j-1)} + \mathbf{r}_{2i(j-1)}) \quad (j > 1), \\
\mathbf{t}_{3ij} &= \hat{\boldsymbol{\rho}}_{(i-1)j} - \boldsymbol{\rho}_{ij}, \\
\mathbf{t}_{4ij} &= -\mathbf{t}_{3ij} + \mathbf{r}_{1ij} = \boldsymbol{\rho}_{ij} + \mathbf{r}_{1ij} - \hat{\boldsymbol{\rho}}_{(i-1)j}, \\
\mathbf{t}_{6ij} &= \boldsymbol{\rho}_{(i+1)(j+1)} - (\hat{\boldsymbol{\rho}}_{ij} + \mathbf{r}_{2ij}) \quad (j<m), \\
\mathbf{t}_{7ij} &= \hat{\boldsymbol{\rho}}_{ij} - \boldsymbol{\rho}_{(i+1)(j+1)} \quad (j<m), \\
\mathbf{t}_{8ij} &= \boldsymbol{\rho}_{ij} + \mathbf{r}_{1ij} - \hat{\boldsymbol{\rho}}_{ij} = -\mathbf{r}_{1ij} - \mathbf{t}_{5ij} + \mathbf{r}_{2ij}, \\
\mathbf{t}_{9ij} &= \boldsymbol{\rho}_{(i+1)j} - (\boldsymbol{\rho}_{ij} + \mathbf{r}_{1ij}), \\
\mathbf{t}_{10ij} &= \hat{\boldsymbol{\rho}}_{(i+1)j} + \mathbf{r}_{2(i+1)j} - \hat{\boldsymbol{\rho}}_{ij}.
\end{aligned} \tag{4.11}$$

For $j = 1$ we replace $\mathbf{t}_{2ij}$ with

$$\mathbf{t}_{2i1} = \boldsymbol{\rho}_{i1} - \boldsymbol{\rho}_{(i+1)1}.$$

For $j = m$ we replace $\mathbf{t}_{6ij}$ and $\mathbf{t}_{7ij}$ with

$$\mathbf{t}_{6im} = \hat{\boldsymbol{\rho}}_{(i+1)m} + \mathbf{r}_{2(i+1)m} - (\hat{\boldsymbol{\rho}}_{im} + \mathbf{r}_{2im}),$$

$$\mathbf{t}_{7im} = \hat{\boldsymbol{\rho}}_{im} - (\hat{\boldsymbol{\rho}}_{(i+1)m} + \mathbf{r}_{2(i+1)m}).$$

where $\boldsymbol{\rho}_{0j} \triangleq \boldsymbol{\rho}_{nj}$, $\hat{\boldsymbol{\rho}}_{0j} \triangleq \hat{\boldsymbol{\rho}}_{nj}$, and $i + n = i$. Eq. (4.11) has the matrix form,

$$\mathbf{t}_{ij}^{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \\ \mathbf{t}_6 \\ \mathbf{t}_7 \\ \mathbf{t}_8 \\ \mathbf{t}_9 \\ \mathbf{t}_{10} \end{bmatrix}_{ij} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{i(j-1)} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i-1)j}$$

$$+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{ij} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i+1)j}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i+1)(j+1)},$$

$$
\mathbf{t}_{i1}^{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \\ \mathbf{t}_6 \\ \mathbf{t}_7 \\ \mathbf{t}_8 \\ \mathbf{t}_9 \\ \mathbf{t}_{10} \end{bmatrix}_{i1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i-1)1} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{i1}
$$

$$
+ \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i+1)1} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i+1)2},
$$

$$
\mathbf{t}_{im}^{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \\ \mathbf{t}_6 \\ \mathbf{t}_7 \\ \mathbf{t}_8 \\ \mathbf{t}_9 \\ \mathbf{t}_{10} \end{bmatrix}_{im} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{i(m-1)} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i-1)m}
$$

$$
+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{im} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{(i+1)m}. \tag{4.12}
$$

Eq. (4.5) yields

$$\begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \hat{\boldsymbol{\rho}} \\ \mathbf{r}_2 \end{bmatrix}_{ij} = \begin{bmatrix} \frac{1}{2}\mathbf{I}_3 & -\frac{1}{2}\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{I}_3 & -\frac{1}{2}\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{q}_{ij}. \tag{4.13}$$

Hence, (4.12) and (4.13) yield

$$\mathbf{t}^{\mathrm{d}}_{ij} \triangleq \begin{bmatrix} \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \\ \mathbf{t}_6 \\ \mathbf{t}_7 \\ \mathbf{t}_8 \\ \mathbf{t}_9 \\ \mathbf{t}_{10} \end{bmatrix}_{ij} = \frac{1}{2} \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{q}_{i(j-1)} + \frac{1}{2} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{q}_{(i-1)j}$$

$$+ \frac{1}{2} \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \mathbf{q}_{ij} + \frac{1}{2} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \mathbf{q}_{(i+1)j}$$

$$+ \frac{1}{2} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{q}_{(i+1)(j+1)},$$

$$\mathbf{t}_{i1}^{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \\ \mathbf{t}_6 \\ \mathbf{t}_7 \\ \mathbf{t}_8 \\ \mathbf{t}_9 \\ \mathbf{t}_{10} \end{bmatrix}_{i1} = \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{q}_{(i-1)1} + \frac{1}{2}\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}\mathbf{q}_{i1}$$

$$+\frac{1}{2}\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}\mathbf{q}_{(i+1)1} + \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{q}_{(i+1)2},$$

$$\mathbf{t}_{im}^{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \\ \mathbf{t}_6 \\ \mathbf{t}_7 \\ \mathbf{t}_8 \\ \mathbf{t}_9 \\ \mathbf{t}_{10} \end{bmatrix}_{im} = \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{q}_{i(m-1)} + \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{q}_{(i-1)m}$$

$$+\frac{1}{2}\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}\mathbf{q}_{im} + \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}\mathbf{q}_{(i+1)m}. \tag{4.14}$$

Also, from (4.10) and (4.12)

$$\mathbf{t}_{1n1} = [\,-\mathbf{I}_3,\quad \mathbf{0},\quad \mathbf{0},\quad \mathbf{0}\,]\begin{bmatrix}\boldsymbol{\rho}\\ \mathbf{r}_1\\ \hat{\boldsymbol{\rho}}\\ \mathbf{r}_2\end{bmatrix}_{11} + [\,\mathbf{I}_3,\quad \mathbf{I}_3,\quad \mathbf{0},\quad \mathbf{0}\,]\begin{bmatrix}\boldsymbol{\rho}\\ \mathbf{r}_1\\ \hat{\boldsymbol{\rho}}\\ \mathbf{r}_2\end{bmatrix}_{n1},$$

$$\begin{aligned}\mathbf{t}_{1n1} &= [\,-\tfrac{1}{2}\mathbf{I}_3,\quad \tfrac{1}{2}\mathbf{I}_3,\quad \mathbf{0},\quad \mathbf{0}\,]\mathbf{q}_{11} + [\,\tfrac{1}{2}\mathbf{I}_3,\quad \tfrac{1}{2}\mathbf{I}_3,\quad \mathbf{0},\quad \mathbf{0}\,]\mathbf{q}_{n1}\\ &= \mathbf{E}_6\mathbf{q}_{11} + \mathbf{E}_7\mathbf{q}_{n1}\\ &= [\,\mathbf{E}_6,\quad \mathbf{0},\quad \ldots,\quad \mathbf{0},\quad \mathbf{E}_7\,]\begin{bmatrix}\mathbf{q}_{11}\\ \mathbf{q}_{21}\\ \vdots\\ \mathbf{q}_{n1}\end{bmatrix},\quad \mathbf{E}_6 \in \mathbb{R}^{3\times 12},\quad \mathbf{E}_7 \in \mathbb{R}^{3\times 12},\end{aligned}$$

$$\mathbf{t}_{1n1} = \mathbf{R}_0\mathbf{q}_1 = [\,\mathbf{R}_0,\quad \mathbf{0}\,]\mathbf{q},\quad \mathbf{R}_0 \in \mathbb{R}^{3\times 12n}. \tag{4.15}$$

With the obvious definitions of the 24×12 matrices $\mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3,\ \mathbf{E}_4,\ \hat{\mathbf{E}}_4,\ \bar{\mathbf{E}}_4,\ \mathbf{E}_5$, equations in (4.14) are written in the form, where $\mathbf{q}_{01} = \mathbf{q}_{n1},\ \mathbf{q}_{(n+1)j} = \mathbf{q}_{ij}$,

$$\begin{aligned}\mathbf{t}^{\mathrm{d}}_{i1} &= \mathbf{E}_2\mathbf{q}_{(i-1)1} + \mathbf{E}_3\mathbf{q}_{i1} + \hat{\mathbf{E}}_4\mathbf{q}_{(i+1)1} + \mathbf{E}_5\mathbf{q}_{(i+1)2},\\ \mathbf{t}^{\mathrm{d}}_{ij} &= \mathbf{E}_1\mathbf{q}_{i(j-1)} + \mathbf{E}_2\mathbf{q}_{(i-1)j} + \mathbf{E}_3\mathbf{q}_{ij} + \mathbf{E}_4\mathbf{q}_{(i+1)j} + \mathbf{E}_5\mathbf{q}_{(i+1)(j+1)},\\ \mathbf{t}^{\mathrm{d}}_{im} &= \mathbf{E}_1\mathbf{q}_{i(m-1)} + \mathbf{E}_2\mathbf{q}_{(i-1)m} + \mathbf{E}_3\mathbf{q}_{im} + \bar{\mathbf{E}}_4\mathbf{q}_{(i+1)m}.\end{aligned} \tag{4.16}$$

Now from (4.14) and (4.15), define

$$\begin{aligned}\mathbf{l}^{\mathrm{d}} &= [\,\mathbf{t}^{\mathrm{dT}}_{1n1},\quad \mathbf{t}^{\mathrm{dT}}_{11},\quad \mathbf{t}^{\mathrm{dT}}_{21},\quad \ldots,\quad \mathbf{t}^{\mathrm{dT}}_{n1}|\quad \mathbf{t}^{\mathrm{dT}}_{12},\quad \ldots,\quad \mathbf{t}^{\mathrm{dT}}_{n2}|\quad \ldots,\quad \mathbf{t}^{\mathrm{dT}}_{nm}\,]^{\mathrm{T}}\\ &= [\,\mathbf{t}^{\mathrm{dT}}_{1n1},\quad \mathbf{t}^{\mathrm{dT}}_{1},\quad \mathbf{t}^{\mathrm{dT}}_{2},\quad \ldots,\quad \mathbf{t}^{\mathrm{dT}}_{n}\,]^{\mathrm{T}}\end{aligned}$$

to get

$$\mathbf{l}^{\mathrm{d}} = \mathbf{R}\mathbf{q},\quad \mathbf{R} \in \mathbb{R}^{(24nm+3)\times 12nm},\quad \mathbf{q} \in \mathbb{R}^{12nm},\quad \mathbf{l}^{\mathrm{d}} \in \mathbb{R}^{(24nm+3)}, \tag{4.17}$$

$$\mathbf{R} = \begin{bmatrix}\mathbf{R}_0 & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0}\\ \hat{\mathbf{R}}_{11} & \mathbf{R}_{12} & \ddots & & & \vdots\\ \mathbf{R}_{21} & \mathbf{R}_{11} & \mathbf{R}_{12} & \ddots & & \vdots\\ \mathbf{0} & \mathbf{R}_{21} & \mathbf{R}_{11} & \mathbf{R}_{12} & \ddots & \vdots\\ \vdots & \ddots & \mathbf{R}_{21} & \mathbf{R}_{11} & \ddots & \mathbf{0}\\ \vdots & & \ddots & \ddots & \ddots & \mathbf{R}_{12}\\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{R}_{21} & \bar{\mathbf{R}}_{11}\end{bmatrix},\quad \mathbf{R}_{ij} \in \mathbb{R}^{24n\times 12n},\quad \mathbf{R}_0 \in \mathbb{R}^{3\times 12n},$$

$$\mathbf{R}_{11}=\begin{bmatrix}\mathbf{E}_3 & \mathbf{E}_4 & \mathbf{0} & \cdots & \cdots & \mathbf{E}_2\\ \mathbf{E}_2 & \mathbf{E}_3 & \mathbf{E}_4 & \ddots & & \vdots\\ \mathbf{0} & \mathbf{E}_2 & \mathbf{E}_3 & \mathbf{E}_4 & \ddots & \vdots\\ \vdots & \ddots & \mathbf{E}_2 & \mathbf{E}_3 & \mathbf{E}_4 & \mathbf{0}\\ \mathbf{0} & & \ddots & \ddots & \ddots & \mathbf{E}_4\\ \mathbf{E}_4 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{E}_2 & \mathbf{E}_3\end{bmatrix},\quad \mathbf{R}_{12}=\begin{bmatrix}\mathbf{0} & \mathbf{E}_5 & \mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{E}_5 & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \mathbf{0}\\ \mathbf{0} & & & \ddots & \ddots & \mathbf{E}_5\\ \mathbf{E}_5 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\mathbf{R}_{i(i+k)}=\mathbf{0}\quad \text{if } k>1,\quad \mathbf{R}_{(i+k)i}=\mathbf{0}\quad \text{if } k>1,$$

$$\mathbf{R}_{21}=BlockDiag[\cdots,\quad \mathbf{E}_1,\quad \mathbf{E}_1,\cdots],\qquad \mathbf{E}_i\in\mathbb{R}^{24\times12},\quad i=1\to5,$$

$$\mathbf{R}_0=[\mathbf{E}_6,\quad \mathbf{0},\quad \cdots,\quad \mathbf{0},\quad \mathbf{E}_7],\quad \mathbf{E}_6=\tfrac{1}{2}[-\mathbf{I}_3,\quad \mathbf{I}_3,\quad \mathbf{0},\quad \mathbf{0}],$$
$$\mathbf{E}_7=\tfrac{1}{2}[\mathbf{I}_3,\quad \mathbf{I}_3,\quad \mathbf{0},\quad \mathbf{0}].$$

$\hat{\mathbf{R}}_{11}$ and $\bar{\mathbf{R}}_{11}$ have the same structure as $\mathbf{R}_{11}$ except $\mathbf{E}_4$ is replaced by $\hat{\mathbf{E}}_4$ and $\bar{\mathbf{E}}_4$, respectively. Eq. (4.17) will be needed to express the tendon forces in terms of $\mathbf{q}$. Eqs. (4.8) and (4.17) yield the dependent vectors $(\mathbf{t}_{1n1},\ \mathbf{t}_2,\ \mathbf{t}_3,\ \mathbf{t}_4,\ \mathbf{t}_6,\ \mathbf{t}_7,\ \mathbf{t}_9,\ \mathbf{t}_{10})$ in terms of the independent vectors $(\mathbf{t}_5,\ \mathbf{t}_1,\ \mathbf{r}_1,\ \mathbf{r}_2)$ Therefore,

$$\mathbf{l}^{\mathrm{d}}=\mathbf{RQl}. \tag{4.18}$$

5. Tendon forces

Let the tendon forces be described by

$$\mathbf{f}_{\alpha ij}=F_{\alpha ij}\frac{\mathbf{t}_{\alpha ij}}{||\mathbf{t}_{\alpha ij}||}. \tag{5.1}$$

For tensegrity structures with some slack strings, the magnitude of the force $F_{\alpha ij}$ can be zero, for taut strings $F_{\alpha ij}>0$. Since tendons cannot compress, $F_{\alpha ij}$ cannot be negative. Hence, the magnitude of the force is

$$F_{\alpha ij}=k_{\alpha ij}(||\mathbf{t}_{\alpha ij}||-\bar{L}_{\alpha ij}), \tag{5.2}$$

where

$$k_{\alpha ij}\triangleq\begin{cases}0 & \text{if } \bar{L}_{\alpha ij}>||\mathbf{t}_{\alpha ij}||,\\ \bar{k}_{\alpha ij}>0 & \text{if } \bar{L}_{\alpha ij}\leqslant||\mathbf{t}_{\alpha ij}||,\end{cases}$$

$$\bar{L}_{\alpha ij}\triangleq-u_{\alpha ij}+L^0_{\alpha ij}\geqslant0, \tag{5.3}$$

where $L^0_{\alpha ij}>0$ is the rest length of tendon $t_{\alpha ij}$ before any control is applied, and the control is $u_{\alpha ij}$, the change in the rest length. The control shortens or lengthens the tendon, so $u_{\alpha ij}$ can be positive or negative, but $L^0_{\alpha ij}>0$. So $u_{\alpha ij}$ must obey constraint

(5.3), and

$$u_{\alpha ij} \leqslant L^0_{\alpha ij} > 0. \tag{5.4}$$

Note that for t_{1n1} and for $\alpha = 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 10$ the vectors $\mathbf{t}_{\alpha ij}$ appear in the vector $\mathbf{l}^{\mathrm{d}}$ related to $\mathbf{q}$ from Eq. (4.7) by $\mathbf{l}^{\mathrm{d}} = \mathbf{R}\mathbf{q}$, and for $\alpha = 5, 1$ the vectors $\mathbf{t}_{\alpha ij}$ appear in the vector $\mathbf{l}$ related to $\mathbf{q}$ from Eq. (4.8), by $\mathbf{l} = \mathbf{Q}^{-1}\mathbf{q}$. Let $\mathscr{R}_{\alpha ij}$ denote the selected row of $\mathbf{R}$ associated with $\mathbf{t}_{\alpha ij}$ for $\alpha ij = 1n1$ and for $\alpha = 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 10$. Let $\mathscr{R}_{\alpha ij}$ also denote the selected row of $\mathbf{Q}^{-1}$ when $\alpha = 5,\ 1$. Then,

$$\mathbf{t}_{\alpha ij} = \mathscr{R}_{\alpha ij}\mathbf{q}, \quad \mathscr{R}_{\alpha ij} \in \mathbb{R}^{3\times 12nm}, \tag{5.5}$$

$$||\mathbf{t}_{\alpha ij}||^2 = \mathbf{q}^{\mathrm{T}}\mathscr{R}^{\mathrm{T}}_{\alpha ij}\mathscr{R}_{\alpha ij}\mathbf{q}. \tag{5.6}$$

From (5.1) and (5.5),

$$\mathbf{f}_{\alpha ij} = -\mathbf{K}_{\alpha ij}(\mathbf{q})\mathbf{q} + \mathbf{b}_{\alpha ij}(\mathbf{q})u_{\alpha ij},$$

where

$$\mathbf{K}_{\alpha ij}(\mathbf{q}) \triangleq k_{\alpha ij}(L^0_{\alpha ij}(\mathbf{q}^{\mathrm{T}}\mathscr{R}^{\mathrm{T}}_{\alpha ij}\mathscr{R}_{\alpha ij}\mathbf{q})^{-1/2} - 1)\mathscr{R}_{\alpha ij}, \quad \mathbf{K}_{\alpha ij} \in \mathbb{R}^{3\times 12nm}, \tag{5.7}$$

$$\mathbf{b}_{\alpha ij}(\mathbf{q}) \triangleq k_{\alpha ij}(\mathbf{q}^{\mathrm{T}}\mathscr{R}^{\mathrm{T}}_{\alpha ij}\mathscr{R}_{\alpha ij}\mathbf{q})^{-1/2}\mathscr{R}_{\alpha ij}\mathbf{q}, \quad \mathbf{b}_{\alpha ij} \in \mathbb{R}^{3\times 1}. \tag{5.8}$$

Hence,

$$\mathbf{f}^{\mathrm{d}}_{ij} = \begin{bmatrix} \mathbf{f}_{2ij} \\ \mathbf{f}_{3ij} \\ \mathbf{f}_{4ij} \\ \mathbf{f}_{6ij} \\ \mathbf{f}_{7ij} \\ \mathbf{f}_{8ij} \\ \mathbf{f}_{9ij} \\ \mathbf{f}_{10ij} \end{bmatrix} = - \begin{bmatrix} \mathbf{K}_{2ij} \\ \mathbf{K}_{3ij} \\ \mathbf{K}_{4ij} \\ \mathbf{K}_{6ij} \\ \mathbf{K}_{7ij} \\ \mathbf{K}_{8ij} \\ \mathbf{K}_{9ij} \\ \mathbf{K}_{10ij} \end{bmatrix} \mathbf{q} + \begin{bmatrix} \mathbf{b}_{2ij} & & & & & & & \\ & \mathbf{b}_{3ij} & & & & & & \\ & & \mathbf{b}_{4ij} & & & & & \\ & & & \mathbf{b}_{6ij} & & & & \\ & & & & \mathbf{b}_{7ij} & & & \\ & & & & & \mathbf{b}_{8ij} & & \\ & & & & & & \mathbf{b}_{9ij} & \\ & & & & & & & \mathbf{b}_{10ij} \end{bmatrix} \begin{bmatrix} u_{2ij} \\ u_{3ij} \\ u_{4ij} \\ u_{6ij} \\ u_{7ij} \\ u_{8ij} \\ u_{9ij} \\ u_{10ij} \end{bmatrix}$$

or

$$\mathbf{f}^{\mathrm{d}}_{ij} = -\mathbf{K}^{\mathrm{d}}_{ij}\mathbf{q} + \mathbf{P}^{\mathrm{d}}_{ij}\mathbf{u}^{\mathrm{d}}_{ij} \tag{5.9}$$

and

$$\mathbf{f}^0_{ij} = \begin{bmatrix} \mathbf{f}_{5ij} \\ \mathbf{f}_{1ij} \end{bmatrix} = - \begin{bmatrix} \mathbf{K}_{5ij} \\ \mathbf{K}_{1ij} \end{bmatrix} \mathbf{q} + \begin{bmatrix} \mathbf{b}_{5ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_{1ij} \end{bmatrix} \begin{bmatrix} u_{5ij} \\ u_{1ij} \end{bmatrix}$$

or

$$\mathbf{f}_{ij}^{0} = -\mathbf{K}_{ij}^{0}\mathbf{q} + \mathbf{P}_{ij}^{0}\mathbf{u}_{ij}^{0}. \tag{5.10}$$

Now substitute (5.9) and (5.10) into

$$\bar{\mathbf{f}}_1^{\mathrm{d}} = \begin{bmatrix} \mathbf{f}_{1n1} \\ \mathbf{f}_{11}^{\mathrm{d}} \\ \mathbf{f}_{21}^{\mathrm{d}} \\ \vdots \\ \mathbf{f}_{n1}^{\mathrm{d}} \end{bmatrix} = -\begin{bmatrix} \mathbf{K}_{1n1} \\ \mathbf{K}_{11}^{\mathrm{d}} \\ \mathbf{K}_{21}^{\mathrm{d}} \\ \vdots \\ \mathbf{K}_{n1}^{\mathrm{d}} \end{bmatrix}\mathbf{q} + \begin{bmatrix} \mathbf{P}_{1n1} & & & & \\ & \mathbf{P}_{11}^{\mathrm{d}} & & & \\ & & \mathbf{P}_{21}^{\mathrm{d}} & & \\ & & & \ddots & \\ & & & & \mathbf{P}_{n1}^{\mathrm{d}} \end{bmatrix} \begin{bmatrix} u_{1n1} \\ \mathbf{u}_{11}^{\mathrm{d}} \\ \mathbf{u}_{21}^{\mathrm{d}} \\ \vdots \\ \mathbf{u}_{n1}^{\mathrm{d}} \end{bmatrix}$$

$$= -\bar{\mathbf{K}}_1^{\mathrm{d}}\mathbf{q} + \bar{\mathbf{P}}_1^{\mathrm{d}}\bar{\mathbf{u}}_1^{\mathrm{d}}$$

$$\mathbf{f}_2^{\mathrm{d}} = \begin{bmatrix} \mathbf{f}_{12}^{\mathrm{d}} \\ \mathbf{f}_{22}^{\mathrm{d}} \\ \vdots \\ \mathbf{f}_{n2}^{\mathrm{d}} \end{bmatrix} = -\begin{bmatrix} \mathbf{K}_{12}^{\mathrm{d}} \\ \mathbf{K}_{22}^{\mathrm{d}} \\ \vdots \\ \mathbf{K}_{n2}^{\mathrm{d}} \end{bmatrix}\mathbf{q} + \begin{bmatrix} \mathbf{P}_{12}^{\mathrm{d}} & & & \\ & \mathbf{P}_{22}^{\mathrm{d}} & & \\ & & \ddots & \\ & & & \mathbf{P}_{n2}^{\mathrm{d}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{12}^{\mathrm{d}} \\ \mathbf{u}_{22}^{\mathrm{d}} \\ \vdots \\ \mathbf{u}_{n2}^{\mathrm{d}} \end{bmatrix} = -\mathbf{K}_2^{\mathrm{d}}\mathbf{q}_2 + \mathbf{P}_2^{\mathrm{d}}\mathbf{u}_2^{\mathrm{d}}.$$

Hence, in general,

$$\mathbf{f}_j^{\mathrm{d}} = -\mathbf{K}_j^{\mathrm{d}}\mathbf{q} + \mathbf{P}_j^{\mathrm{d}}\mathbf{u}_j^{\mathrm{d}}$$

or by defining

$$\mathbf{K}^{\mathrm{d}} = \begin{bmatrix} \bar{\mathbf{K}}_1^{\mathrm{d}} \\ \mathbf{K}_2^{\mathrm{d}} \\ \vdots \\ \mathbf{K}_m^{\mathrm{d}} \end{bmatrix}, \quad \mathbf{P}^{\mathrm{d}} = \begin{bmatrix} \bar{\mathbf{P}}_1^{\mathrm{d}} \\ \mathbf{P}_2^{\mathrm{d}} \\ \vdots \\ \mathbf{P}_m^{\mathrm{d}} \end{bmatrix}, \tag{5.11}$$

$$\mathbf{f}^{\mathrm{d}} = -\mathbf{K}^{\mathrm{d}}\mathbf{q} + \mathbf{P}^{\mathrm{d}}\mathbf{u}^{\mathrm{d}}.$$

Likewise for $\mathbf{f}_{ij}^{0}$ forces (5.10),

$$\mathbf{f}_1^{0} = \begin{bmatrix} \mathbf{f}_{11}^{0} \\ \mathbf{f}_{21}^{0} \\ \vdots \\ \mathbf{f}_{n1}^{0} \end{bmatrix} = -\begin{bmatrix} \mathbf{K}_{11}^{0} \\ \mathbf{K}_{21}^{0} \\ \vdots \\ \mathbf{K}_{n1}^{0} \end{bmatrix}\mathbf{q} + \begin{bmatrix} \mathbf{P}_{11}^{0} & & & \\ & \mathbf{P}_{21}^{0} & & \\ & & \ddots & \\ & & & \mathbf{P}_{n1}^{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{11}^{0} \\ \mathbf{u}_{21}^{0} \\ \vdots \\ \mathbf{u}_{n1}^{0} \end{bmatrix},$$

$$\mathbf{f}_j^0 = \begin{bmatrix} \mathbf{f}_{1j}^0 \\ \mathbf{f}_{2j}^0 \\ \vdots \\ \mathbf{f}_{nj}^0 \end{bmatrix} = -\begin{bmatrix} \mathbf{K}_{1j}^0 \\ \mathbf{K}_{2j}^0 \\ \vdots \\ \mathbf{K}_{nj}^0 \end{bmatrix}\mathbf{q} + \begin{bmatrix} \mathbf{P}_{1j}^0 & & & \\ & \mathbf{P}_{2j}^0 & & \\ & & \ddots & \\ & & & \mathbf{P}_{nj}^0 \end{bmatrix}\begin{bmatrix} \mathbf{u}_{1j}^0 \\ \mathbf{u}_{2j}^0 \\ \vdots \\ \mathbf{u}_{nj}^0 \end{bmatrix},$$

$$\mathbf{f}_j^0 = -\mathbf{K}_j^0\mathbf{q} + \mathbf{P}_j^0\mathbf{u}_j^0,$$

$$\mathbf{f}^0 = -\mathbf{K}^0\mathbf{q} + \mathbf{P}^0\mathbf{u}^0. \tag{5.12}$$

Substituting (5.11) and (5.12) into (E.21) yields

$$\mathbf{f} = -(\mathbf{B}^{\mathrm{d}}\mathbf{K}^{\mathrm{d}} + \mathbf{B}^0\mathbf{K}^0)\mathbf{q} + \mathbf{B}^{\mathrm{d}}\mathbf{P}^{\mathrm{d}}\mathbf{u}^{\mathrm{d}} + \mathbf{B}^0\mathbf{P}^0\mathbf{u}^0 + \mathbf{W}^0\mathbf{w}, \tag{5.13}$$

which is written simply as

$$\mathbf{f} = -\tilde{\mathbf{K}}\mathbf{q} + \tilde{\mathbf{B}}\mathbf{u} + \mathbf{W}^0\mathbf{w} \tag{5.14}$$

by defining,

$$\tilde{\mathbf{K}} \triangleq \mathbf{B}^{\mathrm{d}}\mathbf{K}^{\mathrm{d}} + \mathbf{B}^0\mathbf{K}^0,$$

$$\tilde{\mathbf{B}} \triangleq [\mathbf{B}^{\mathrm{d}}\mathbf{P}^{\mathrm{d}}, \mathbf{B}^0\mathbf{P}^0],$$

$$\mathbf{B}^{\mathrm{d}}\mathbf{P}^{\mathrm{d}} = \begin{bmatrix} \mathbf{B}_3\bar{\mathbf{P}}_1^{\mathrm{d}} & \mathbf{B}_4\mathbf{P}_2^{\mathrm{d}} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{B}_5\bar{\mathbf{P}}_1^{\mathrm{d}} & \mathbf{B}_6\mathbf{P}_2^{\mathrm{d}} & \mathbf{B}_4\mathbf{P}_3^{\mathrm{d}} & \ddots & & \vdots \\ \mathbf{0} & \bar{\mathbf{B}}_5\mathbf{P}_2^{\mathrm{d}} & \mathbf{B}_6\mathbf{P}_3^{\mathrm{d}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \bar{\mathbf{B}}_5\mathbf{P}_3^{\mathrm{d}} & \ddots & \ddots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \ddots & \mathbf{B}_4\mathbf{P}_m^{\mathrm{d}} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \bar{\mathbf{B}}_5\mathbf{P}_{m-1}^{\mathrm{d}} & \mathbf{B}_8\mathbf{P}_m^{\mathrm{d}} \end{bmatrix},$$

$$\mathbf{B}^0\mathbf{P}^0 = \begin{bmatrix} \mathbf{B}_1\mathbf{P}_1^0 & \mathbf{B}_2\mathbf{P}_2^0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_7\mathbf{P}_2^0 & \mathbf{B}_2\mathbf{P}_3^0 & \ddots & \vdots \\ \vdots & \ddots & \mathbf{B}_7\mathbf{P}_2^0 & \ddots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \mathbf{B}_2\mathbf{P}_m^0 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{B}_7\mathbf{P}_m^0 \end{bmatrix},$$

$$\tilde{\mathbf{K}} = \mathbf{B}^\mathrm{d}\mathbf{K}^\mathrm{d} + \mathbf{B}^0\mathbf{K}^0 = \begin{bmatrix} \mathbf{B}_3\bar{\mathbf{K}}_1^\mathrm{d} + \mathbf{B}_1\mathbf{K}_1^0 + \mathbf{B}_4\mathbf{K}_2^\mathrm{d} + \mathbf{B}_2\mathbf{K}_2^0 \\ \mathbf{B}_5\bar{\mathbf{K}}_1^\mathrm{d} + \mathbf{B}_6\mathbf{K}_2^\mathrm{d} + \mathbf{B}_4\mathbf{K}_3^\mathrm{d} + \mathbf{B}_7\mathbf{K}_2^0 + \mathbf{B}_2\mathbf{K}_3^0 \\ \bar{\mathbf{B}}_5\mathbf{K}_2^\mathrm{d} + \mathbf{B}_6\mathbf{K}_3^\mathrm{d} + \mathbf{B}_4\mathbf{K}_4^\mathrm{d} + \mathbf{B}_7\mathbf{K}_3^0 + \mathbf{B}_2\mathbf{K}_4^0 \\ \bar{\mathbf{B}}_5\mathbf{K}_3^\mathrm{d} + \mathbf{B}_6\mathbf{K}_4^\mathrm{d} + \mathbf{B}_4\mathbf{K}_5^\mathrm{d} + \mathbf{B}_7\mathbf{K}_4^0 + \mathbf{B}_2\mathbf{K}_5^0 \\ \vdots \\ \bar{\mathbf{B}}_5\mathbf{K}_{m-2}^\mathrm{d} + \mathbf{B}_6\mathbf{K}_{m-1}^\mathrm{d} + \mathbf{B}_4\mathbf{K}_m^\mathrm{d} + \mathbf{B}_7\mathbf{K}_{m-1}^0 + \mathbf{B}_2\mathbf{K}_m^0 \\ \bar{\mathbf{B}}_5\mathbf{K}_{m-1}^\mathrm{d} + \mathbf{B}_6\mathbf{K}_m^\mathrm{d} + \mathbf{B}_7\mathbf{K}_m^0 \end{bmatrix}, \quad (5.15)$$

$$\tilde{\mathbf{u}} = \begin{bmatrix} \bar{\mathbf{u}}_1^\mathrm{d} \\ \mathbf{u}_2^\mathrm{d} \\ \mathbf{u}_3^\mathrm{d} \\ \mathbf{u}_4^\mathrm{d} \\ \vdots \\ \mathbf{u}_m^\mathrm{d} \\ \mathbf{u}_1^0 \\ \mathbf{u}_2^0 \\ \mathbf{u}_3^0 \\ \mathbf{u}_4^0 \\ \vdots \\ \mathbf{u}_m^0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \hat{\mathbf{u}}_1^\mathrm{d} \\ \mathbf{u}_2^\mathrm{d} \\ \mathbf{u}_3^\mathrm{d} \\ \mathbf{u}_4^\mathrm{d} \\ \vdots \\ \hat{\mathbf{u}}_m^\mathrm{d} \\ \mathbf{u}_1^0 \\ \mathbf{u}_2^0 \\ \mathbf{u}_3^0 \\ \mathbf{u}_4^0 \\ \vdots \\ \mathbf{u}_m^0 \end{bmatrix}. \quad (5.16)$$

In vector $\tilde{\mathbf{u}}$ in (5.16), u_{1n1} appears twice (for notational convenience u_{1n1} appears in $\bar{\mathbf{u}}_1^\mathrm{d}$ and in $\mathbf{u}_1^0$). From the rules of closure, $\mathbf{t}_{9i1} = -\mathbf{t}_{1i1}$ and $\mathbf{t}_{7im} = -\mathbf{t}_{10im}$, $i = 1, 2, \ldots, n$, but $\mathbf{t}_{1i1}$, $\mathbf{t}_{7im}$, $\mathbf{t}_{9i1}$, $\mathbf{t}_{10im}$ all appear in (5.16). Hence, the rules of closure leave only $n(10m-2)$ tendons in the structure, but (5.16) contains $10nm+1$ tendons. To eliminate the redundant variables in (5.16) define $\tilde{\mathbf{u}} = \mathbf{T}\mathbf{u}$, where $\mathbf{u}$ is the independent set $\mathbf{u} \in \mathbb{R}^{n(10m-2)}$, and $\tilde{\mathbf{u}} \in \mathbb{R}^{10nm+1}$ is given by (5.16). We choose to keep $\mathbf{t}_{7im}$ in $\mathbf{u}$ and delete $\mathbf{t}_{10im}$ by setting $\mathbf{t}_{10im} = -\mathbf{t}_{7im}$. We choose to keep $\mathbf{t}_{1i1}$ and delete $\mathbf{t}_{9i1}$ by setting

$\mathbf{t}_{9i1} = -\mathbf{t}_{1i1},\ i = 1, 2, \ldots, n$. This requires new definitions of certain subvectors as follows in (5.19), (5.20). The vector $\mathbf{u}$ is now defined in (5.16). We have reduced the $\tilde{\mathbf{u}}$ vector by $2n+1$ scalars to $\mathbf{u}$. The $\mathbf{T}$ matrix is formed by the following blocks,

$$\mathbf{T} = \left(\begin{array}{c|c|c|c|c}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\ 0\ 1 & \mathbf{0} \\ \hline
\begin{matrix} \mathbf{T}_1 & & \\ & \ddots & \\ & & \mathbf{T}_1 \end{matrix} & & & \begin{matrix} \mathbf{S} & & \\ & \ddots & \\ & & \mathbf{S} \end{matrix} & \\ \hline
& \begin{matrix} \mathbf{I}_8 & & \\ & \ddots & \\ & & \mathbf{I}_8 \end{matrix} & & & \\ \hline
& & \begin{matrix} \mathbf{T}_2 & & \\ & \ddots & \\ & & \mathbf{T}_2 \end{matrix} & & \\ \hline
& & & \begin{matrix} \mathbf{I}_2 & & \\ & \ddots & \\ & & \mathbf{I}_2 \end{matrix} & \\ \hline
& & & & \begin{matrix} \mathbf{I}_2 & & \\ & \ddots & \\ & & \mathbf{I}_2 \end{matrix}
\end{array}\right)$$

$$\in \mathbb{R}^{(10nm+1)\times(n(10m-2))}, \tag{5.17}$$

where

$$\mathbf{T}_1 = \left(\begin{array}{c|c} \mathbf{I}_6 & \mathbf{0}_{6\times 1} \\ \hline \mathbf{0}_{1\times 6} & 0 \\ \mathbf{0}_{1\times 6} & 1 \end{array}\right) \in \mathbb{R}^{8\times 7},$$

$$\mathrm{T}_2 = \left(\begin{array}{ccccccc} \multicolumn{7}{c}{\mathbf{I}_7} \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right) \in \mathbb{R}^{8\times 7}, \tag{5.18}$$

$$\mathrm{S} = \left(\begin{array}{cc} \multicolumn{2}{c}{\mathbf{0}_{6\times 2}} \\ \hline 0 & -1 \\ 0 & 0 \end{array}\right) \in \mathbb{R}^{8\times 2}.$$

There are n blocks labelled $\mathbf{T}_1$, $n(m-2)$ blocks labelled $\mathbf{I}_8$ (for $m \leqslant 2$ no $\mathbf{I}_8$ blocks needed, see Appendix D), n blocks labelled $\mathbf{T}_2$, nm blocks labelled $\mathbf{I}_2$ blocks, and n blocks labelled $\mathbf{S}$.

The $\mathbf{u}_1^{\mathrm{d}}$ block becomes

$$\hat{\mathbf{u}}_1^{\mathrm{d}} \triangleq \begin{bmatrix} \hat{\mathbf{u}}_{11}^{\mathrm{d}} \\ \hat{\mathbf{u}}_{21}^{\mathrm{d}} \\ \hat{\mathbf{u}}_{31}^{\mathrm{d}} \\ \vdots \\ \hat{\mathbf{u}}_{n1}^{\mathrm{d}} \end{bmatrix}, \quad \hat{\mathbf{u}}_{i1}^{\mathrm{d}} = \begin{bmatrix} u_{2n1} \\ u_{3n1} \\ u_{4n1} \\ u_{6n1} \\ u_{7n1} \\ u_{8n1} \\ u_{10n1} \end{bmatrix} \in \mathbb{R}^{7\times 1}, \quad i = 1, 2, 3, \ldots n, \;\; j = 1. \tag{5.19}$$

The $\mathbf{u}_m^{\mathrm{d}}$ block becomes

$$\hat{\mathbf{u}}_m^{\mathrm{d}} \triangleq \begin{bmatrix} \hat{\mathbf{u}}_{1m}^{\mathrm{d}} \\ \hat{\mathbf{u}}_{2m}^{\mathrm{d}} \\ \hat{\mathbf{u}}_{3m}^{\mathrm{d}} \\ \vdots \\ \hat{\mathbf{u}}_{nm}^{\mathrm{d}} \end{bmatrix}, \quad \hat{\mathbf{u}}_{im}^{\mathrm{d}} = \begin{bmatrix} u_{2nm} \\ u_{3nm} \\ u_{4nm} \\ u_{6nm} \\ u_{7nm} \\ u_{8nm} \\ u_{9nm} \end{bmatrix} \in \mathbb{R}^{7\times 1}, \quad i = 1, 2, 3, \ldots, n, \;\; j = m. \tag{5.20}$$

The $\mathbf{u}_1^{\mathrm{d}}$ block is the $\bar{\mathbf{u}}_1^{\mathrm{d}}$ block with the first element u_{1n1} removed, since it is included in $\mathbf{u}_{n1}^0$. From (3.1) and (5.14)

$$\ddot{\mathbf{q}} + (\mathbf{K}_r(\dot{\mathbf{q}}) + \mathbf{K}_p(\mathbf{q}))\mathbf{q} = \mathbf{B}(\mathbf{q})\mathbf{u} + \mathbf{D}(\mathbf{q})\mathbf{w}, \tag{5.21}$$

where

$$\mathbf{K}_p = \mathbf{H}(\mathbf{q})\tilde{\mathbf{K}}(\mathbf{q}),$$

$$\mathbf{B} = \mathbf{H}(\mathbf{q})\tilde{\mathbf{B}}(\mathbf{q})\mathbf{T},$$

$$\mathbf{D} = \mathbf{H}(\mathbf{q})\mathbf{W}^0.$$

The nodal points of the structure are located by the vector $\mathbf{p}$. Suppose that a selected set of nodal points are chosen as outputs of interest. Then

$$\mathbf{y}_p = \mathbf{C}\mathbf{p} = \mathbf{C}\mathbf{P}\mathbf{q}, \tag{5.22}$$

where $\mathbf{P}$ is defined by (4.6). The length of tendon vector $\mathbf{t}_{\alpha ij} = \mathscr{R}_{\alpha ij}\mathbf{q}$ is given from (5.6). Therefore, the output vector $\mathbf{y}_l$ describing all tendon lengths, is

$$\mathbf{y}_l = \begin{bmatrix} \vdots \\ \mathbf{y}_{\alpha ij} \\ \vdots \end{bmatrix}, \quad \mathbf{y}_{\alpha ij} = (\mathbf{q}^{\mathrm{T}}\mathscr{R}_{\alpha ij}^{\mathrm{T}}\mathscr{R}_{\alpha ij}\mathbf{q})^{1/2}.$$

Another output of interest might be tension, so from (5.2) and (5.6)

$$\mathbf{y}_f = \begin{bmatrix} \vdots \\ F_{\alpha ij} \\ \vdots \end{bmatrix}, \quad F_{\alpha ij} = k_{\alpha ij}(\mathbf{y}_{\alpha ij} - \bar{L}_{\alpha ij}).$$

The static equilibria can be studied from the equations

$$\mathbf{K}_p(\mathbf{q})\mathbf{q} = \mathbf{B}(\mathbf{q})\mathbf{u} + \mathbf{D}(\mathbf{q})\mathbf{w}, \quad \mathbf{y}_p = \mathbf{CPq}. \tag{5.23}$$

Of course, one way to generate equilibria is by simulation from arbitrary initial conditions and record the steady-state value of $\mathbf{q}$. The exhaustive definitive study of the stable equilibria follows in a separate paper [26].

Damping strategies for controlled tensegrity structures is a subject of further research. The example case given in Appendix D was coded in Matlab and simulated. Artificial critical damping was included in the simulation below. The simulation does not include external disturbances or control inputs. All nodes of the structure were placed symmetrically around the surface of a cylinder, as seen in Fig. 10. Spring constants and natural rest lengths were specified equally for all

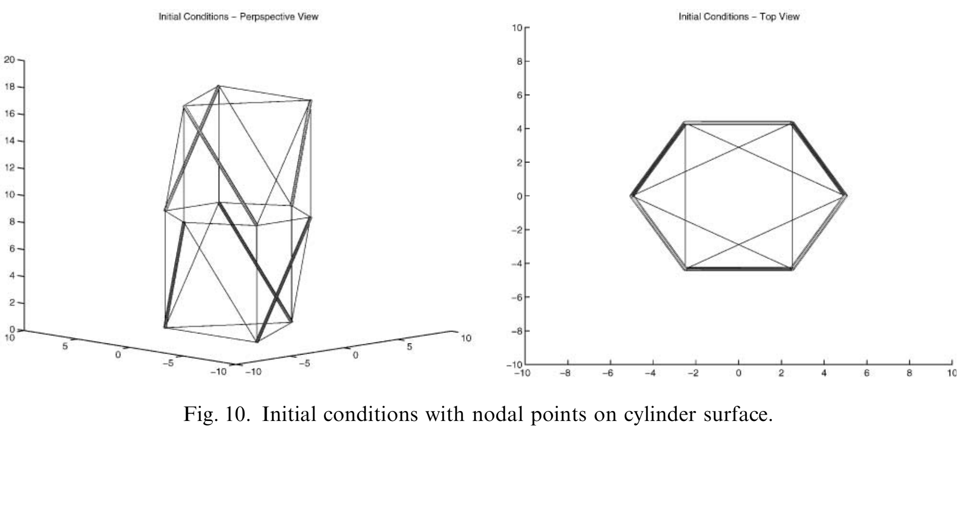

Fig. 10. Initial conditions with nodal points on cylinder surface.

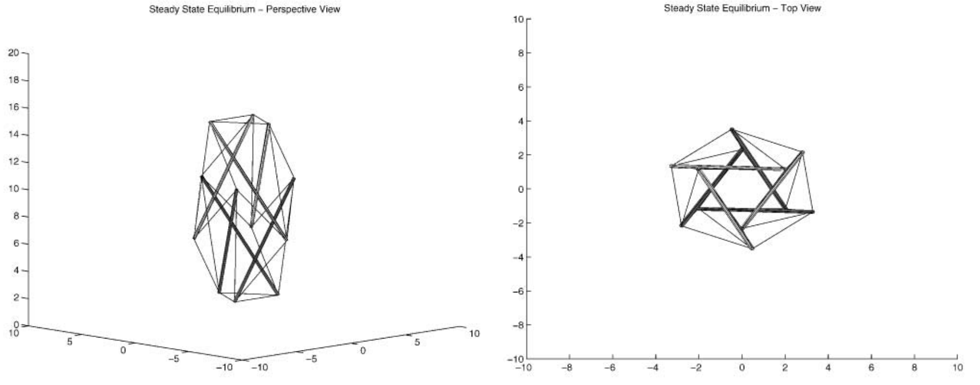

Fig. 11. Steady-state equilibrium.

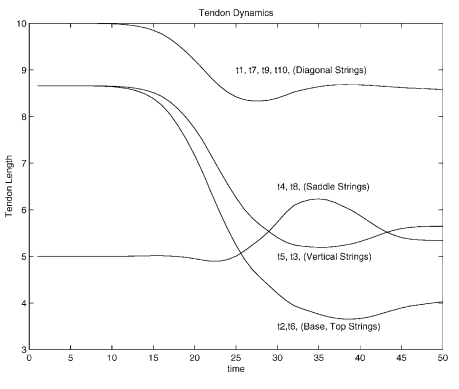

Fig. 12. Tendon dynamics.

tendons in the structure. One would expect the structure to collapse in on itself with this given initial condition. A plot of steady-state equilibrium is given in Fig. 11 and string lengths in Fig. 12.

6. Conclusion

This paper develops the exact nonlinear equations for a Class 1 tensegrity shell, having nm rigid rods and $n(10m - 2)$ tendons, subject to the assumption that the tendons are linear-elastic, the rods are rigid rods of constant length. The equations are described in terms of $6nm$ degrees of freedom, and the accelerations are given explicitly. Hence, no inversion of the mass matrix is required. For large systems this greatly improves accuracy of simulations.

Tensegrity systems of four classes are characterized by these models. Class 2 includes rods that are in contact at nodal points, with a ball joint, transmitting no torques. In Class 1, the rods do not touch and a stable equilibrium must be achieved by pretension in the tendons. The Primal Shell Class contains the minimum number of tendons ($8nm$) for which stability is possible.

Tensegrity structures offer some potential advantages over classical structural systems composed of continua (such as columns, beams, plates, and shells): the overall structure can bend but all elements of the structure experience only axial loads, so no member bending, the absence of bending in the members promises more precise models (and hopefully more precise control). Prestress allows members to be uni-directionally loaded, meaning that no member experiences reversal in the direction of the load carried by the member. This eliminates a host of nonlinear problems known to create difficulties in control (hysterisis, deadzones, friction).

Acknowledgements

The authors recognize the valuable efforts of T. Yamashita in the first draft of this paper.

Appendix A. Proof of Theorem 3.1

Refer to Fig. 8 and define

$$\underline{\mathbf{q}}_1 = \underline{\mathbf{p}}_2 + \underline{\mathbf{p}}_1, \quad \underline{\mathbf{q}}_2 = \underline{\mathbf{p}}_2 - \underline{\mathbf{p}}_1$$

using the vectors $\underline{\mathbf{p}}_1$ and $\underline{\mathbf{p}}_2$ which locate the end points of the rod. The rod mass center is located by the vector,

$$\underline{\mathbf{p}}_c = \tfrac{1}{2}\underline{\mathbf{q}}_1. \tag{A.1}$$

Hence, the translation equation of motion for the mass center of the rod is

$$m\ddot{\underline{\mathbf{p}}}_c = \frac{m}{2}\ddot{\underline{\mathbf{q}}}_1 = (\hat{\underline{\mathbf{f}}}_1 + \hat{\underline{\mathbf{f}}}_2), \tag{A.2}$$

where a dot over a vector is a time derivative with respect to the inertial reference frame. A vector $\underline{\mathbf{p}}$ locating a mass element, dm, along the centerline of the rod (Fig. 13) is

$$\underline{\mathbf{p}} = \underline{\mathbf{p}}_1 + \rho(\underline{\mathbf{p}}_2 - \underline{\mathbf{p}}_1) = \tfrac{1}{2}\underline{\mathbf{q}}_1 + (\rho - \tfrac{1}{2})\underline{\mathbf{q}}_2, \quad 0 \leqslant \rho \leqslant 1, \quad \rho = \frac{x}{L} \tag{A.3}$$

and the velocity of the mass dm, $\underline{\mathbf{v}}$, is

$$\underline{\mathbf{v}} = \dot{\underline{\mathbf{p}}} = \tfrac{1}{2}\dot{\underline{\mathbf{q}}}_1 + (\rho - \tfrac{1}{2})\dot{\underline{\mathbf{q}}}_2. \tag{A.4}$$

The angular momentum for the rod about the mass center, $\underline{\mathbf{h}}_c$, is

$$\underline{\mathbf{h}}_c = \int_m (\underline{\mathbf{p}} - \underline{\mathbf{p}}_c) \times \dot{\underline{\mathbf{p}}}\,\mathrm{d}m, \tag{A.5}$$

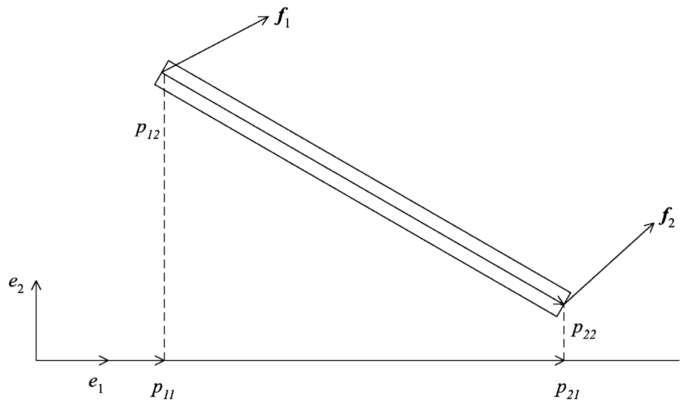

Fig. 13. A rigid bar of the length L and mass m.

where the mass $\mathrm{d}m$ can be described using $\mathrm{d}x = L\,\mathrm{d}\rho$ as

$$\mathrm{d}m = \left(\frac{m}{L}\right)(L\,\mathrm{d}\rho) = m\,\mathrm{d}\rho. \tag{A.6}$$

Hence, (A.2) can be re-written as follows:

$$\underline{\mathbf{h}}_c = \int_0^1 (\underline{\mathbf{p}} - \underline{\mathbf{p}}_c) \times \dot{\underline{\mathbf{p}}}(m\,\mathrm{d}\rho), \tag{A.7}$$

where (A.1) and (A.3) yield

$$\underline{\mathbf{p}} - \underline{\mathbf{p}}_c = (\rho - \tfrac{1}{2})\underline{\mathbf{q}}_2. \tag{A.8}$$

(A.4)–(A.8) yield

$$\begin{aligned}
\underline{\mathbf{h}}_c &= m\int_0^1 \left(\rho - \frac{1}{2}\right)\underline{\mathbf{q}}_2 \times \left\{\frac{1}{2}\dot{\underline{\mathbf{q}}}_1 + \left(\rho - \frac{1}{2}\right)\dot{\underline{\mathbf{q}}}_2\right\}\mathrm{d}\rho \\
&= m\underline{\mathbf{q}}_2 \times \left\{\dot{\underline{\mathbf{q}}}_1 \int_0^1 \frac{1}{2}\left(\rho - \frac{1}{2}\right)\mathrm{d}\rho + \dot{\underline{\mathbf{q}}}_2 \int_0^1 \left(\rho - \frac{1}{2}\right)^2 \mathrm{d}\rho\right\} \\
&= m\underline{\mathbf{q}}_2 \times \left(\frac{1}{2}\left[\frac{1}{2}\rho^2 - \frac{1}{2}\rho\right]_0^1 \dot{\underline{\mathbf{q}}}_1 + \left[\frac{1}{3}\left(\rho - \frac{1}{2}\right)^3\right]_0^1 \dot{\underline{\mathbf{q}}}_2\right) \\
&= \frac{m}{12}\underline{\mathbf{q}}_2 \times \dot{\underline{\mathbf{q}}}_2.
\end{aligned} \tag{A.9}$$

The applied torque about the mass center, $\underline{\tau}_c$, is

$$\underline{\tau}_c = \tfrac{1}{2}\underline{\mathbf{q}}_2 \times (\hat{\underline{\mathbf{f}}}_2 - \hat{\underline{\mathbf{f}}}_1).$$

Then, substituting $\underline{\mathbf{h}}_c$ and $\underline{\tau}_c$ from (A.9) into Euler's equations, we obtain

$$\dot{\underline{\mathbf{h}}}_c = \underline{\tau}_c$$

or

$$\begin{aligned}\dot{\underline{\mathbf{h}}}_c &= \frac{m}{12}(\dot{\underline{\mathbf{q}}}_2 \times \dot{\underline{\mathbf{q}}}_2 + \underline{\mathbf{q}}_2 \times \ddot{\underline{\mathbf{q}}}_2)\\ &= \frac{m}{12}\underline{\mathbf{q}}_2 \times \ddot{\underline{\mathbf{q}}}_2 = \frac{1}{2}\underline{\mathbf{q}}_2 \times (\hat{\underline{\mathbf{f}}}_2 - \hat{\underline{\mathbf{f}}}_1).\end{aligned} \tag{A.10}$$

Hence, (A.2) and (A.10) yield the motion equations for the rod:

$$\begin{aligned}&\frac{m}{2}\ddot{\underline{\mathbf{q}}}_1 = \hat{\underline{\mathbf{f}}}_1 + \hat{\underline{\mathbf{f}}}_2,\\ &\frac{m}{6}(\underline{\mathbf{q}}_2 \times \ddot{\underline{\mathbf{q}}}_2) = \underline{\mathbf{q}}_2 \times (\hat{\underline{\mathbf{f}}}_2 - \hat{\underline{\mathbf{f}}}_1).\end{aligned} \tag{A.11}$$

We have assumed that the rod length L is constant. Hence, the following constraints for $\underline{\mathbf{q}}_2$ hold:

$$\begin{aligned}&\underline{\mathbf{q}}_2 \cdot \underline{\mathbf{q}}_2 = L^2,\\ &\frac{\mathrm{d}}{\mathrm{d}t}(\underline{\mathbf{q}}_2 \cdot \underline{\mathbf{q}}_2) = \dot{\underline{\mathbf{q}}}_2 \cdot \underline{\mathbf{q}}_2 + \underline{\mathbf{q}}_2 \cdot \dot{\underline{\mathbf{q}}}_2 = 2\underline{\mathbf{q}}_2 \cdot \dot{\underline{\mathbf{q}}}_2 = 0,\\ &\frac{\mathrm{d}}{\mathrm{d}t}(\underline{\mathbf{q}}_2 \cdot \dot{\underline{\mathbf{q}}}_2) = \dot{\underline{\mathbf{q}}}_2 \cdot \dot{\underline{\mathbf{q}}}_2 + \underline{\mathbf{q}}_2 \cdot \ddot{\underline{\mathbf{q}}}_2 = 0.\end{aligned}$$

Collecting (A.11) and the constraint equations we have

$$\begin{aligned}&\frac{m}{2}\ddot{\underline{\mathbf{q}}}_1 = \hat{\underline{\mathbf{f}}}_1 + \hat{\underline{\mathbf{f}}}_2,\\ &\frac{m}{6}(\underline{\mathbf{q}}_2 \times \ddot{\underline{\mathbf{q}}}_2) = \underline{\mathbf{q}}_2 \times (\hat{\underline{\mathbf{f}}}_2 - \hat{\underline{\mathbf{f}}}_1),\\ &\dot{\underline{\mathbf{q}}}_2 \cdot \dot{\underline{\mathbf{q}}}_2 + \underline{\mathbf{q}}_2 \cdot \ddot{\underline{\mathbf{q}}}_2 = 0,\\ &\underline{\mathbf{q}}_2 \cdot \underline{\mathbf{q}}_2 = L^2.\end{aligned} \tag{A.12}$$

We now develop the matrix version of (A.12). Recall that

$$\underline{\mathbf{q}}_i = \underline{\mathbf{E}}\mathbf{q}_i, \quad \hat{\underline{\mathbf{f}}}_i = \underline{\mathbf{E}}\hat{\mathbf{f}}_i.$$

Also note that $\underline{\mathbf{E}}^{\mathrm{T}} \cdot \underline{\mathbf{E}} = 3 \times 3$ identity. After some manipulation, (A.12) can be written as

$$\begin{aligned}&\frac{m}{2}\ddot{\mathbf{q}}_1 = \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2,\\ &\frac{m}{6}\tilde{\mathbf{q}}_2\ddot{\mathbf{q}}_2 = \tilde{\mathbf{q}}_2(\hat{\mathbf{f}}_2 - \hat{\mathbf{f}}_1),\\ &\mathbf{q}_2^{\mathrm{T}}\ddot{\mathbf{q}}_2 = -\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2,\\ &\mathbf{q}_2^{\mathrm{T}}\mathbf{q}_2 = L^2.\end{aligned} \tag{A.13}$$

Introduce scaled force vectors by dividing the applied forces by m and mL

$$\mathbf{g}_1 \triangleq (\hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2)\frac{2}{m}, \quad \mathbf{g}_2 \triangleq (\hat{\mathbf{f}}_1 - \hat{\mathbf{f}}_2)\frac{6}{mL^2}.$$

Then, (A.13) can be re-written as

$$\begin{aligned} \ddot{\mathbf{q}}_1 &= \mathbf{g}_1, \\ \tilde{\mathbf{q}}_2\ddot{\mathbf{q}}_2 &= \tilde{\mathbf{q}}_2(-\mathbf{g}_2 L^2), \\ \mathbf{q}_2^{\mathrm{T}}\ddot{\mathbf{q}}_2 &= -\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2, \\ \mathbf{q}_2^{\mathrm{T}}\mathbf{q}_2 &= L^2. \end{aligned} \tag{A.14}$$

Solving for $\ddot{\mathbf{q}}_2$ requires,

$$\begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix} \ddot{\mathbf{q}}_2 = \begin{bmatrix} \mathbf{0} \\ -\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2 \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{0} \end{bmatrix} \mathbf{g}_2 L^2. \tag{A.15}$$

Lemma. *For any vector* $\mathbf{q}$, *such that* $\mathbf{q}^{\mathrm{T}}\mathbf{q} = L^2$,

$$\begin{bmatrix} \tilde{\mathbf{q}} \\ \mathbf{q}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{\mathbf{q}} \\ \mathbf{q}^{\mathrm{T}} \end{bmatrix} = L^2\mathbf{I}_3.$$

Proof.

$$\begin{bmatrix} 0 & q_3 & -q_2 & q_1 \\ -q_3 & 0 & q_1 & q_2 \\ q_2 & -q_1 & 0 & q_3 \end{bmatrix} \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \\ q_1 & q_2 & q_3 \end{bmatrix} = L^2\mathbf{I}_3.$$

Since the coefficient of $\ddot{\mathbf{q}}_2$ in (A.15) has linearly independent columns by virtue of the lemma, the unique solution for $\ddot{\mathbf{q}}_2$ is

$$\ddot{\mathbf{q}}_2 = \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix}^{+} \left(\begin{bmatrix} \mathbf{0} \\ -\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2 \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{0} \end{bmatrix} \mathbf{g}_2 L^2 \right), \tag{A.16}$$

where the pseudoinverse is uniquely given by

$$\begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix}^{+} = \left(\begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix} \right)^{-1} \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = L^{-2} \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

It is easily verified that the existence condition for $\ddot{\mathbf{q}}_2$ in (A.15) is satisfied since

$$\left(\mathbf{I} - \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{q}_2^{\mathrm{T}} \end{bmatrix}^{+} \right) \left(\begin{bmatrix} \mathbf{0} \\ -\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2 \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{q}}_2 \\ \mathbf{0} \end{bmatrix} \mathbf{g}_2 L^2 \right) = \mathbf{0}.$$

Hence, (A.16) yields

$$\ddot{\mathbf{q}}_2 = -\frac{\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2}{L^2}\mathbf{q}_2 + \tilde{\mathbf{q}}_2^2\mathbf{g}_2. \tag{A.17}$$

Bringing the first equation of (A.14) together with (A.17) leads to

$$\begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2}{L^2}\mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{q}}_2^2 \end{bmatrix} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix}. \tag{A.18}$$

Recalling the definition of $\mathbf{g}_1$ and $\mathbf{g}_2$ we obtain

$$\begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2}{L^2}\mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \frac{2}{m}\begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \frac{3}{L^2}\tilde{\mathbf{q}}_2^2 \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \tag{A.19}$$

where we clarify

$$\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2 \\ \hat{\mathbf{f}}_1 - \hat{\mathbf{f}}_2 \end{bmatrix}.$$

Eq. (A.19) is identical to (3.1), so this completes the proof of Theorem 3.1. □

Example 1. In this example we derive the dynamics of a single rod, as shown in Fig. 13.

$$\mathbf{q}_1 = \mathbf{p}_1 + \mathbf{p}_2 = \begin{bmatrix} p_{11} + p_{21} \\ p_{12} + p_{22} \end{bmatrix} = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix},$$

$$\mathbf{q}_2 = \mathbf{p}_2 - \mathbf{p}_1 = \begin{bmatrix} p_{21} - p_{11} \\ p_{22} - p_{12} \end{bmatrix} = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix}.$$

The generalized forces are now defined as

$$\mathbf{g}_1 = \frac{2}{m}(\hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2) = \frac{2}{m}\begin{bmatrix} f_{11} + f_{21} \\ f_{12} + f_{22} \end{bmatrix}, \quad \mathbf{g}_2 = \frac{6}{mL^2}(\hat{\mathbf{f}}_2 - \hat{\mathbf{f}}_1) = \frac{6}{mL^2}\begin{bmatrix} f_{21} - f_{11} \\ f_{22} - f_{12} \end{bmatrix}.$$

From (A.16),

$$\begin{bmatrix} \ddot{q}_{11} \\ \ddot{q}_{12} \\ \ddot{q}_{21} \\ \ddot{q}_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{\dot{q}_{21}^2 + \dot{q}_{22}^2}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{\dot{q}_{21}^2 + \dot{q}_{22}^2}{L^2} \end{array}\right] \begin{bmatrix} q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{bmatrix} + \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & q_{22}^2 & -q_{21}q_{22} \\ 0 & 0 & -q_{21}q_{22} & q_{21}^2 \end{array}\right] \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix}.$$

Example 2. Using the formulation developed in Sections 3–5 we derive the dynamics of a planar tensegrity. The rules of closure become

$$\mathbf{t}_5 = -\mathbf{t}_4,$$

$$\mathbf{t}_8 = \mathbf{t}_1,$$

$$\mathbf{t}_7 = -\mathbf{t}_2,$$

$$\mathbf{t}_6 = -\mathbf{t}_3.$$

We define the independent vectors $\mathbf{l}^0$ and $\mathbf{l}^d$:

$$\mathbf{l}^0 = \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{t}_5 \end{bmatrix}, \quad \mathbf{l}^d = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix}.$$

The nodal forces are

$$\bar{\mathbf{f}} = \begin{bmatrix} \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2 \\ \hat{\mathbf{f}}_1 - \hat{\mathbf{f}}_2 \\ \hat{\mathbf{f}}_3 + \hat{\mathbf{f}}_4 \\ \hat{\mathbf{f}}_3 - \hat{\mathbf{f}}_4 \end{bmatrix} = \begin{bmatrix} (\mathbf{f}_3 - \mathbf{f}_2 + \mathbf{w}_1) + (\mathbf{f}_5 - \mathbf{f}_1 + \mathbf{w}_2) \\ (\mathbf{f}_3 - \mathbf{f}_2 + \mathbf{w}_1) - (\mathbf{f}_5 - \mathbf{f}_1 + \mathbf{w}_2) \\ (\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{w}_3) + (-\mathbf{f}_3 - \mathbf{f}_5 + \mathbf{w}_4) \\ (\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{w}_3) - (-\mathbf{f}_3 - \mathbf{f}_5 + \mathbf{w}_4) \end{bmatrix}.$$

We can write

$$\bar{\mathbf{f}} = \begin{bmatrix} \mathbf{I}_2 \\ -\mathbf{I}_2 \\ -\mathbf{I}_2 \\ \mathbf{I}_2 \end{bmatrix} \mathbf{f}^0 + \begin{bmatrix} -\mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{I}_2 & -\mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix} \mathbf{f}^d + \begin{bmatrix} \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & -\mathbf{I}_2 \end{bmatrix} \mathbf{w},$$

where

$$\mathbf{f}^0 = [\mathbf{f}_5], \quad \mathbf{f}^d = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \mathbf{w}_4 \end{bmatrix}.$$

Or, with the obvious definitions for $\mathbf{B}^0$, $\mathbf{B}^d$, and $\mathbf{W}_1$, in matrix notation:

$$\bar{\mathbf{f}} = \mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^d\mathbf{f}^d + \mathbf{W}_1\mathbf{w}. \tag{A.20}$$

The nodal vectors are defined as follows:

$$\mathbf{p}_1 = \boldsymbol{\rho},$$

$$\mathbf{p}_2 = \boldsymbol{\rho} + \mathbf{r}_1,$$

$$\mathbf{p}_3 = \hat{\boldsymbol{\rho}},$$

$$\mathbf{p}_4 = \hat{\boldsymbol{\rho}} + \mathbf{r}_2$$

and

$$\hat{\boldsymbol{\rho}} = \boldsymbol{\rho} + \mathbf{r}_1 + \mathbf{t}_5 - \mathbf{r}_2.$$

We define

$$\mathbf{q}_1 \triangleq \mathbf{p}_2 + \mathbf{p}_1 = 2\boldsymbol{\rho} + \mathbf{r}_1,$$

$$\mathbf{q}_2 \triangleq \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{r}_1,$$

$$\mathbf{q}_3 \triangleq \mathbf{p}_4 + \mathbf{p}_3 = 2\hat{\boldsymbol{\rho}} + \mathbf{r}_2 = 2(\boldsymbol{\rho} + \mathbf{r}_1 + \mathbf{t}_5) - \mathbf{r}_2,$$

$$\mathbf{q}_4 \triangleq \mathbf{p}_4 - \mathbf{p}_3 = \mathbf{r}_2.$$

The relation between $\mathbf{q}$ and $\mathbf{p}$ can be written as follows:

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_2 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{bmatrix}$$

and

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{bmatrix} = \begin{bmatrix} 2\mathbf{I}_2 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ 2\mathbf{I}_2 & 2\mathbf{I}_2 & -\mathbf{I}_2 & 2\mathbf{I}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{t}_5 \end{bmatrix} = \mathbf{Q}\mathbf{l}^0. \tag{A.21}$$

We can now write the dependent variables $\mathbf{l}^{\mathrm{d}}$ in terms of independent variables $\mathbf{l}^0$. From (4.10) and (4.11):

$$\mathbf{t}_1 = \boldsymbol{\rho} + \mathbf{r}_1 - (\boldsymbol{\rho} + \mathbf{r}_1 + \mathbf{t}_5 - \mathbf{r}_2),$$

$$\mathbf{t}_2 = \boldsymbol{\rho} - (\boldsymbol{\rho} + \mathbf{r}_1 + \mathbf{t}_5 - \mathbf{r}_2).$$

By inspection of Fig. 14, (4.10) and (4.11) reduce to

$$\mathbf{t}_1 = \mathbf{r}_2 - \mathbf{t}_5,$$

$$\mathbf{t}_2 = -\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{t}_5,$$

$$\mathbf{t}_3 = \mathbf{r}_1 + \mathbf{t}_5 \tag{A.22}$$

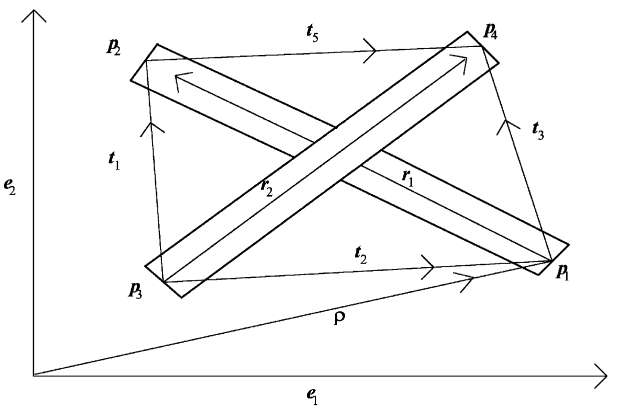

Fig. 14. A planar tensegrity.

or,

$$\mathbf{l}^{\mathrm{d}} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & -\mathbf{I}_2 \\ \mathbf{0} & -\mathbf{I}_2 & \mathbf{I}_2 & -\mathbf{I}_2 \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{t}_5 \end{bmatrix}.$$

Eq. (A.21) yields

$$\mathbf{l}^{0} = \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{t}_5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2\mathbf{I}_2 \\ -\mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{bmatrix} = \mathbf{Q}^{-1}\mathbf{q}.$$

Hence,

$$\mathbf{l}^{\mathrm{d}} = \frac{1}{2} \begin{bmatrix} \mathbf{I}_2 & \mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{I}_2 & -\mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix} \mathbf{q} = \mathbf{R}\mathbf{q}.$$

We can now write out the tendon forces as follows:

$$\mathbf{f}^{\mathrm{d}} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix} = -\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \\ \mathbf{K}_3 \end{bmatrix} \mathbf{q} + \begin{bmatrix} \mathbf{b}_1 & & \\ & \mathbf{b}_2 & \\ & & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

or

$$\mathbf{f}^{\mathrm{d}} = -\mathbf{K}^{\mathrm{d}}\mathbf{q} + \mathbf{P}^{\mathrm{d}}\mathbf{u}^{\mathrm{d}}$$

and

$$\mathbf{f}^{0} = [\mathbf{f}_5] = -[\mathbf{K}_5]\mathbf{q} + [\mathbf{b}_5][u_5]$$

or

$$\mathbf{f}^{0} = -\mathbf{K}^{0}\mathbf{q} + \mathbf{P}^{0}\mathbf{u}^{0}$$

using the same definitions for $\mathbf{K}$ and $\mathbf{b}$ as found in (5.7) and (5.8), simply removing the ij element indices. Substitution into (A.20) yields

$$\begin{aligned} \bar{\mathbf{f}} &= \mathbf{B}^{0}(-\mathbf{K}^{0}\mathbf{q} + \mathbf{P}^{0}\mathbf{u}^{0}) + \mathbf{B}^{\mathrm{d}}(-\mathbf{K}^{\mathrm{d}}\mathbf{q} + \mathbf{P}^{\mathrm{d}}\mathbf{u}^{\mathrm{d}}) \\ &= -(\mathbf{B}^{0}\mathbf{K}^{0} + \mathbf{B}^{\mathrm{d}}\mathbf{K}^{\mathrm{d}})\mathbf{q} + (\mathbf{B}^{0}\mathbf{P}^{0}\mathbf{u}^{0} + \mathbf{B}^{\mathrm{d}}\mathbf{P}^{\mathrm{d}}\mathbf{u}^{\mathrm{d}}). \end{aligned}$$

With the matrices derived in this section, we can express the dynamics in the form of (3.4)

$$\ddot{\mathbf{q}} + (\mathbf{K}_r + \mathbf{K}_p)\mathbf{q} = \mathbf{B}\mathbf{u} + \mathbf{D}\mathbf{w},$$

$$\mathbf{K}_r = \boldsymbol{\Omega}_1,$$

$$\mathbf{K}_p = \mathbf{H}\tilde{\mathbf{K}},$$

$$\mathbf{B} = \mathbf{H}\tilde{\mathbf{B}},$$

$$\mathbf{D} = \mathbf{H}\mathbf{W}^0 = \mathbf{H}_1\mathbf{W}_1,$$

where

$$\mathbf{H} = \mathbf{H}_1 = \frac{2}{m}\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{3}{L_1^2}\tilde{\mathbf{q}}_2^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{3}{L_2^2}\tilde{\mathbf{q}}_4^2 \end{bmatrix},$$

$$\mathbf{K}_r = \mathbf{\Omega}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_1^{-2}\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & L_2^{-2}\dot{\mathbf{q}}_4^{\mathrm{T}}\dot{\mathbf{q}}_4\mathbf{I}_2 \end{bmatrix},$$

$$\tilde{\mathbf{K}} \triangleq \mathbf{B}^{\mathrm{d}}\mathbf{K}^{\mathrm{d}} + \mathbf{B}^0\mathbf{K}^0,$$

$$\tilde{\mathbf{B}} \triangleq [\mathbf{B}^{\mathrm{d}}\mathbf{P}^{\mathrm{d}}, \mathbf{B}^0\mathbf{P}^0]$$

and

$$\tilde{\mathbf{q}}_2^2 = \begin{bmatrix} -q_{22}^2 & q_{21}q_{22} \\ q_{21}q_{22} & -q_{21}^2 \end{bmatrix}, \quad \tilde{\mathbf{q}}_4^2 = \begin{bmatrix} -q_{42}^2 & q_{41}q_{42} \\ q_{41}q_{42} & -q_{41}^2 \end{bmatrix},$$

$$\dot{\mathbf{q}}_2^{\mathrm{T}}\dot{\mathbf{q}}_2 = \dot{q}_{21}^2 + \dot{q}_{22}^2, \quad \dot{\mathbf{q}}_4^{\mathrm{T}}\dot{\mathbf{q}}_4 = \dot{q}_{41}^2 + \dot{q}_{42}^2.$$

Appendix B. Algebraic inversion of the Q matrix

This appendix will algebraically invert a 5×5 block $\mathbf{Q}$ matrix. Given $\mathbf{Q}$ in the form:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{21} & \mathbf{Q}_{32} & \mathbf{Q}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{21} & \mathbf{Q}_{32} & \mathbf{Q}_{32} & \mathbf{Q}_{22} & \mathbf{0} \\ \mathbf{Q}_{21} & \mathbf{Q}_{32} & \mathbf{Q}_{32} & \mathbf{Q}_{32} & \mathbf{Q}_{22} \end{bmatrix}, \tag{B.1}$$

we define $\mathbf{x}$ and $\mathbf{y}$ matrices so that

$$\mathbf{Q}\mathbf{x} = \mathbf{y}, \tag{B.2}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \\ \mathbf{y}_5 \end{bmatrix}. \tag{B.3}$$

Solving (B.2) for $\mathbf{x}$ we obtain

$$\mathbf{x} = \mathbf{Q}^{-1}\mathbf{y}. \tag{B.4}$$

Substituting (B.1) and (B.3) into (B.2) and carrying out the matrix operations we obtain

$$\begin{aligned}
&\mathbf{Q}_{11}\mathbf{x}_1 = \mathbf{y}_1,\\
&\mathbf{Q}_{21}\mathbf{x}_1 + \mathbf{Q}_{22}\mathbf{x}_2 = \mathbf{y}_2,\\
&\mathbf{Q}_{21}\mathbf{x}_1 + \mathbf{Q}_{32}\mathbf{x}_2 + \mathbf{Q}_{22}\mathbf{x}_3 = \mathbf{y}_3,\\
&\mathbf{Q}_{21}\mathbf{x}_1 + \mathbf{Q}_{32}\mathbf{x}_2 + \mathbf{Q}_{32}\mathbf{x}_3 + \mathbf{Q}_{22}\mathbf{x}_4 = \mathbf{y}_4,\\
&\mathbf{Q}_{21}\mathbf{x}_1 + \mathbf{Q}_{32}\mathbf{x}_2 + \mathbf{Q}_{32}\mathbf{x}_3 + \mathbf{Q}_{32}\mathbf{x}_4 + \mathbf{Q}_{22}\mathbf{x}_5 = \mathbf{y}_5.
\end{aligned} \tag{B.5}$$

Solving this system of equations for $\mathbf{x}$ will give us the desired $\mathbf{Q}^{-1}$ matrix. Solving each equation for $\mathbf{x}$ we have

$$\begin{aligned}
&\mathbf{x}_1 = \mathbf{Q}_{11}^{-1}\mathbf{y}_1,\\
&\mathbf{x}_2 = \mathbf{Q}_{22}^{-1}(-\mathbf{Q}_{21}\mathbf{x}_1 + \mathbf{y}_2),\\
&\mathbf{x}_3 = \mathbf{Q}_{22}^{-1}(-\mathbf{Q}_{21}\mathbf{x}_1 - \mathbf{Q}_{32}\mathbf{x}_2 + \mathbf{y}_3),\\
&\mathbf{x}_4 = \mathbf{Q}_{22}^{-1}(-\mathbf{Q}_{21}\mathbf{x}_1 - \mathbf{Q}_{32}\mathbf{x}_2 - \mathbf{Q}_{32}\mathbf{x}_3 + \mathbf{y}_4),\\
&\mathbf{x}_5 = \mathbf{Q}_{22}^{-1}(-\mathbf{Q}_{21}\mathbf{x}_1 - \mathbf{Q}_{32}\mathbf{x}_2 - \mathbf{Q}_{32}\mathbf{x}_3 - \mathbf{Q}_{32}\mathbf{x}_4 + \mathbf{y}_5).
\end{aligned} \tag{B.6}$$

Elimination of $\mathbf{x}$ on the right-hand side of (B.6) by substitution yields

$$\begin{aligned}
&\mathbf{x}_1 = \mathbf{\Lambda}_{11}\mathbf{y}_1,\\
&\mathbf{x}_2 = \mathbf{\Lambda}_{21}\mathbf{y}_1 + \mathbf{\Lambda}_{22}\mathbf{y}_2,\\
&\mathbf{x}_3 = \mathbf{\Lambda}_{31}\mathbf{y}_1 + \mathbf{\Lambda}_{32}\mathbf{y}_2 + \mathbf{\Lambda}_{33}\mathbf{y}_3,\\
&\mathbf{x}_4 = \mathbf{\Lambda}_{41}\mathbf{y}_1 + \mathbf{\Lambda}_{42}\mathbf{y}_2 + \mathbf{\Lambda}_{43}\mathbf{y}_3 + \mathbf{\Lambda}_{44}\mathbf{y}_4,\\
&\mathbf{x}_5 = \mathbf{\Lambda}_{51}\mathbf{y}_1 + \mathbf{\Lambda}_{52}\mathbf{y}_2 + \mathbf{\Lambda}_{53}\mathbf{y}_3 + \mathbf{\Lambda}_{54}\mathbf{y}_4 + \mathbf{\Lambda}_{55}\mathbf{y}_5.
\end{aligned} \tag{B.7}$$

Or, in matrix form,

$$\mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{\Lambda}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Lambda}_{31} & \mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Lambda}_{41} & \mathbf{\Lambda}_{42} & \mathbf{\Lambda}_{43} & \mathbf{\Lambda}_{44} & \mathbf{0} \\ \mathbf{\Lambda}_{51} & \mathbf{\Lambda}_{52} & \mathbf{\Lambda}_{53} & \mathbf{\Lambda}_{54} & \mathbf{\Lambda}_{55} \end{bmatrix}, \tag{B.8}$$

where we have used the notation $\mathbf{\Lambda}_{\text{row,column}}$ with the following definitions:

$$\begin{aligned}
\mathbf{\Lambda}_{11} &= \mathbf{Q}_{11}^{-1},\\
\mathbf{\Lambda}_{21} &= -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} = -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{\Lambda}_{11} = -\mathbf{\Lambda}_{22}\mathbf{Q}_{21}\mathbf{\Lambda}_{11},\\
\mathbf{\Lambda}_{31} &= \mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1},\\
\mathbf{\Lambda}_{41} &= -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} + 2\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1},\\
\mathbf{\Lambda}_{51} &= -3\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} + 3\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\\
&\quad + \mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1},\\
\mathbf{\Lambda}_{22} &= \mathbf{\Lambda}_{33} = \mathbf{\Lambda}_{44} = \mathbf{\Lambda}_{55} = \mathbf{Q}_{22}^{-1},\\
\mathbf{\Lambda}_{32} &= \mathbf{\Lambda}_{43} = \mathbf{\Lambda}_{54} = -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1},\\
\mathbf{\Lambda}_{42} &= \mathbf{\Lambda}_{53} = \mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1},\\
\mathbf{\Lambda}_{52} &= -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1} + 2\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{32}\mathbf{Q}_{22}^{-1}.
\end{aligned}$$

Using repeated patterns the inverse may be computed by

$$\mathbf{Q}^{-1} = \begin{bmatrix}
\mathbf{\Lambda}_{11} & & & & & & \\
\mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{22} & & & & & \\
\mathbf{\Theta}\mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{22} & & & & \\
\mathbf{\Theta}^2\mathbf{\Lambda}_{21} & \mathbf{\Theta}\mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{22} & & & \\
\mathbf{\Theta}^3\mathbf{\Lambda}_{21} & \mathbf{\Theta}^2\mathbf{\Lambda}_{32} & \mathbf{\Theta}\mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{22} & & \\
\mathbf{\Theta}^4\mathbf{\Lambda}_{21} & \mathbf{\Theta}^3\mathbf{\Lambda}_{32} & \mathbf{\Theta}^2\mathbf{\Lambda}_{32} & \mathbf{\Theta}\mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{22} & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \tag{B.9}$$

using

$$\begin{aligned}
\mathbf{\Theta} &= (\mathbf{I} - \mathbf{\Lambda}_{22}\mathbf{Q}_{32}),\\
\mathbf{\Lambda}_{11} &= \mathbf{Q}_{11}^{-1},\\
\mathbf{\Lambda}_{21} &= -\mathbf{\Lambda}_{22}\mathbf{Q}_{21}\mathbf{\Lambda}_{11},\\
\mathbf{\Lambda}_{22} &= \mathbf{Q}_{22}^{-1},\\
\mathbf{\Lambda}_{32} &= -\mathbf{\Lambda}_{22}\mathbf{Q}_{32}\mathbf{\Lambda}_{22}.
\end{aligned}$$

Only $\mathbf{\Lambda}_{11}, \mathbf{\Lambda}_{22}, \mathbf{\Lambda}_{21}, \mathbf{\Lambda}_{32}$, and powers of $\mathbf{\Theta}$ need to be calculated to obtain $\mathbf{Q}^{-1}$ for any (n, m). The only matrix inversion that needs to be computed to obtain $\mathbf{Q}^{-1}$ is $\mathbf{Q}_{11}^{-1}$ and $\mathbf{Q}_{22}^{-1}$, substantially reducing computer processing time for computer simulations.

Appendix C. General case for $(n, m) = (i, 1)$

As done in Appendix E, the forces acting on each node is presented, making special exceptions for the case when $(j = m = 1)$. The exceptions arise for

$(j = m = 1)$ because one stage now contains both closure rules for the base and the top of the structure. In the following synthesis we use $\hat{\mathbf{f}}_{111}$ and $\hat{\mathbf{f}}_{211}$ from (E.1) and $\hat{\mathbf{f}}_{311}$ and $\hat{\mathbf{f}}_{411}$ from (E.4). At the right, where $(i = 1, j = 1)$:

$$\bar{\mathbf{f}}_{11} = \begin{bmatrix} \hat{\mathbf{f}}_{111} + \hat{\mathbf{f}}_{211} \\ \hat{\mathbf{f}}_{111} - \hat{\mathbf{f}}_{211} \\ \hat{\mathbf{f}}_{311} + \hat{\mathbf{f}}_{411} \\ \hat{\mathbf{f}}_{311} - \hat{\mathbf{f}}_{411} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{211} + \mathbf{f}_{311} + \mathbf{f}_{1n1} + \mathbf{f}_{2n1} + \mathbf{w}_{111}) + (\mathbf{f}_{511} - \mathbf{f}_{111} - \mathbf{f}_{411} - \mathbf{f}_{811} + \mathbf{w}_{211}) \\ (-\mathbf{f}_{211} + \mathbf{f}_{311} + \mathbf{f}_{1n1} + \mathbf{f}_{2n1} + \mathbf{w}_{111}) - (\mathbf{f}_{511} - \mathbf{f}_{111} - \mathbf{f}_{411} - \mathbf{f}_{811} + \mathbf{w}_{211}) \\ (\mathbf{f}_{811} - \mathbf{f}_{711} - \mathbf{f}_{321} + \mathbf{f}_{421} + \mathbf{w}_{311}) + (\mathbf{f}_{611} - \mathbf{f}_{511} - \mathbf{f}_{6n1} + \mathbf{f}_{7n1} + \mathbf{w}_{411}) \\ (\mathbf{f}_{811} - \mathbf{f}_{711} - \mathbf{f}_{321} + \mathbf{f}_{421} + \mathbf{w}_{311}) - (\mathbf{f}_{611} - \mathbf{f}_{511} - \mathbf{f}_{6n1} + \mathbf{f}_{7n1} + \mathbf{w}_{411}) \end{bmatrix}.$$

At the center, where $(1 < i < n,\ j = 1)$

$$\bar{\mathbf{f}}_{i1} = \begin{bmatrix} \hat{\mathbf{f}}_{1i1} + \hat{\mathbf{f}}_{2i1} \\ \hat{\mathbf{f}}_{1i1} - \hat{\mathbf{f}}_{2i1} \\ \hat{\mathbf{f}}_{3i1} + \hat{\mathbf{f}}_{4i1} \\ \hat{\mathbf{f}}_{3i1} - \hat{\mathbf{f}}_{4i1} \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{f}_{1(i-1)1} + \mathbf{f}_{2(i-1)1} - \mathbf{f}_{2i1} + \mathbf{f}_{3i1} + \mathbf{w}_{1i1}) + (-\mathbf{f}_{1i1} - \mathbf{f}_{4i1} + \mathbf{f}_{5i1} - \mathbf{f}_{8i1} + \mathbf{w}_{2i1}) \\ (\mathbf{f}_{1(i-1)1} + \mathbf{f}_{2(i-1)1} - \mathbf{f}_{2i1} + \mathbf{f}_{3i1} + \mathbf{w}_{1i1}) - (-\mathbf{f}_{1i1} - \mathbf{f}_{4i1} + \mathbf{f}_{5i1} - \mathbf{f}_{8i1} + \mathbf{w}_{2i1}) \\ (\mathbf{f}_{8i1} - \mathbf{f}_{7i1} - \mathbf{f}_{3(i+1)1} + \mathbf{f}_{4(i+1)1} + \mathbf{w}_{3i1}) + (-\mathbf{f}_{5i1} + \mathbf{f}_{6i1} - \mathbf{f}_{6(i-1)} + \mathbf{f}_{7(i-1)1} + \mathbf{w}_{4i1}) \\ (\mathbf{f}_{8i1} - \mathbf{f}_{7i1} - \mathbf{f}_{3(i+1)1} + \mathbf{f}_{4(i+1)1} + \mathbf{w}_{3i1}) - (-\mathbf{f}_{5i1} + \mathbf{f}_{6i1} - \mathbf{f}_{6(i-1)} + \mathbf{f}_{7(i-1)1} + \mathbf{w}_{4i1}) \end{bmatrix}.$$

At the left end of the base in Fig. 5, where $(i = n,\ j = 1)$:

$$\bar{\mathbf{f}}_{n1} = \begin{bmatrix} \hat{\mathbf{f}}_{1n1} + \hat{\mathbf{f}}_{2n1} \\ \hat{\mathbf{f}}_{1n1} - \hat{\mathbf{f}}_{2n1} \\ \hat{\mathbf{f}}_{3n1} + \hat{\mathbf{f}}_{4n1} \\ \hat{\mathbf{f}}_{3n1} - \hat{\mathbf{f}}_{4n1} \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{f}_{1(n-1)1} + \mathbf{f}_{2(n-1)1} + \mathbf{f}_{3n1} - \mathbf{f}_{2n1} + \mathbf{w}_{1n1}) + (-\mathbf{f}_{1n1} - \mathbf{f}_{4n1} + \mathbf{f}_{5n1} - \mathbf{f}_{8n1} + \mathbf{w}_{2n1}) \\ (\mathbf{f}_{1(n-1)1} + \mathbf{f}_{2(n-1)1} + \mathbf{f}_{3n1} - \mathbf{f}_{2n1} + \mathbf{w}_{1n1}) - (-\mathbf{f}_{1n1} - \mathbf{f}_{4n1} + \mathbf{f}_{5n1} - \mathbf{f}_{8n1} + \mathbf{w}_{2n1}) \\ (-\mathbf{f}_{7n1} + \mathbf{f}_{8n1} - \mathbf{f}_{311} + \mathbf{f}_{411} + \mathbf{w}_{3n1}) + (-\mathbf{f}_{6(n-1)1} - \mathbf{f}_{5n1} + \mathbf{f}_{6n1} + \mathbf{f}_{7(n-1)1} + \mathbf{w}_{4n1}) \\ (-\mathbf{f}_{7n1} + \mathbf{f}_{8n1} - \mathbf{f}_{311} + \mathbf{f}_{411} + \mathbf{w}_{3n1}) - (-\mathbf{f}_{6(n-1)1} - \mathbf{f}_{5n1} + \mathbf{f}_{6n1} + \mathbf{f}_{7(n-1)1} + \mathbf{w}_{4n1}) \end{bmatrix}.$$

(C.1)

Using,

$$\mathbf{f}^0_{ij} = \begin{bmatrix} \mathbf{f}_5 \\ \mathbf{f}_1 \end{bmatrix}_{ij}, \quad \mathbf{f}^{\mathrm{d}}_{ij} = \begin{bmatrix} \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_6 \\ \mathbf{f}_7 \\ \mathbf{f}_8 \\ \mathbf{f}_9 \\ \mathbf{f}_{10} \end{bmatrix}_{ij}, \quad \mathbf{w}_{ij} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \mathbf{w}_4 \end{bmatrix}_{ij},$$

$$\begin{aligned}
\bar{\mathbf{f}}_{11} = {} & \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{n1} \\
& + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{11} \\
& + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{21} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^0_{11} + \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{f}_{1n1} \\
& + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{11}.
\end{aligned}$$

Or, in matrix notation, with the implied matrix definitions:

$$\bar{\mathbf{f}}_{11} = \hat{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{n1} + \mathbf{B}^{\mathrm{d}}_{11}\mathbf{f}^{\mathrm{d}}_{11} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{21} + \mathbf{B}^0_{11}\mathbf{f}^0_{11} + \mathbf{B}_{1n1}\mathbf{f}_{1n1} + \mathbf{W}\mathbf{w}_{11}, \tag{C.2}$$

$$\bar{\mathbf{f}}_{i1} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(i-1)1}$$

$$+\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\rm d}_{i1}$$

$$+\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\rm d}_{(i+1)1} + \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{0}_{(i-1)1}$$

$$+\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix}\mathbf{f}^{0}_{i1} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix}\mathbf{w}_{i1}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{i1} = \hat{\mathbf{B}}^{\rm d}_{01}\mathbf{f}^{\rm d}_{(i-1)1} + \mathbf{B}^{\rm d}_{11}\mathbf{f}^{\rm d}_{i1} + \mathbf{B}^{\rm d}_{21}\mathbf{f}^{\rm d}_{(i+1)1} + \mathbf{B}^{0}_{01}\mathbf{f}^{0}_{(i-1)1} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{i1} + \mathbf{W}\mathbf{w}_{i1}, \tag{C.3}$$

$$\bar{\mathbf{f}}_{n1} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\rm d}_{(n-1)1}$$

$$+\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\rm d}_{n1}$$

$$+\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\rm d}_{11} + \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{0}_{(n-1)1} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix}\mathbf{f}^{0}_{n1}$$

$$+\begin{bmatrix} -\mathbf{I}_3 \\ \mathbf{I}_3 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}\mathbf{f}_{1n1} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix}\mathbf{w}_{n1}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{n1} = \hat{\mathbf{B}}^{\rm d}_{01}\mathbf{f}^{\rm d}_{(n-1)1} + \mathbf{B}^{\rm d}_{11}\mathbf{f}^{\rm d}_{n1} + \mathbf{B}^{\rm d}_{21}\mathbf{f}^{\rm d}_{11} + \mathbf{B}^{0}_{01}\mathbf{f}^{0}_{(n-1)1} + \mathbf{B}^{0}_{n1}\mathbf{f}^{0}_{n1} + \bar{\mathbf{B}}_{1n1}\mathbf{f}_{1n1} + \mathbf{W}\mathbf{w}_{n1}. \tag{C.4}$$

Now, assemble (C.2)–(C.4) into the form

$$\mathbf{f}_1 = \mathbf{B}_3\bar{\mathbf{f}}_1^{\mathrm{d}} + \mathbf{B}_1\mathbf{f}_1^0 + \mathbf{W}_1\mathbf{w}_1, \tag{C.5}$$

where

$$\mathbf{f}_j = \begin{bmatrix} \bar{\mathbf{f}}_{1j} \\ \bar{\mathbf{f}}_{2j} \\ \vdots \\ \bar{\mathbf{f}}_{nj} \end{bmatrix}, \quad \bar{\mathbf{f}}_1^{\mathrm{d}} = \begin{bmatrix} \mathbf{f}_{1n1} \\ \mathbf{f}_1^{\mathrm{d}} \end{bmatrix}, \quad \mathbf{f}_j^0 = \begin{bmatrix} \mathbf{f}_{1j}^0 \\ \mathbf{f}_{2j}^0 \\ \vdots \\ \mathbf{f}_{nj}^0 \end{bmatrix}, \quad \mathbf{w}_j = \begin{bmatrix} \mathbf{w}_{1j} \\ \mathbf{w}_{2j} \\ \vdots \\ \mathbf{w}_{nj} \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} \mathbf{B}_{1n1} & \mathbf{B}_{11}^{\mathrm{d}} & \mathbf{B}_{21}^{\mathrm{d}} & \mathbf{0} & \cdots & \mathbf{0} & \hat{\mathbf{B}}_{n1}^{\mathrm{d}} \\ \mathbf{0} & \hat{\mathbf{B}}_{01}^{\mathrm{d}} & \ddots & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & & \ddots & \ddots & \ddots & \mathbf{B}_{21}^{\mathrm{d}} \\ \bar{\mathbf{B}}_{1n1} & \mathbf{B}_{21}^{\mathrm{d}} & \mathbf{0} & \cdots & \mathbf{0} & \hat{\mathbf{B}}_{01}^{\mathrm{d}} & \mathbf{B}_{11}^{\mathrm{d}} \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \mathbf{B}_{11}^0 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{B}_{01}^0 & \ddots & \ddots & & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{B}_{11}^0 & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_{01}^0 & \mathbf{B}_{n1}^0 \end{bmatrix},$$

$$\mathbf{W}_1 = BlockDiag[\ldots, \mathbf{W}, \mathbf{W}, \ldots].$$

Or, simply, (C.5) has the form (E.21), where

$$\mathbf{f} = \mathbf{f}_1, \quad \mathbf{f}^{\mathrm{d}} = \bar{\mathbf{f}}_1^{\mathrm{d}}, \quad \mathbf{f}^0 = \mathbf{f}_1^0, \quad \mathbf{w} = \mathbf{w}_1, \quad \mathbf{W}^0 = \mathbf{W}_1,$$

$$\mathbf{B}^{\mathrm{d}} = \mathbf{B}_3, \ \mathbf{B}^0 = \mathbf{B}_1.$$

The next set of necessary exceptions that apply to the model $(i, 1)$ arises in the form of the $\mathbf{R}$ matrix that relates the dependent tendons set to the generalized coordinates ($\mathbf{l}^{\mathrm{d}} = \mathbf{R}\mathbf{q}$). For any $(i, 1)$ case $\mathbf{R}$ takes the form following the same procedure as in (4.12) and (4.13):

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_0 \\ \tilde{\mathbf{R}}_{11} \end{bmatrix}, \tag{C.6}$$

where

$$\tilde{\mathbf{R}}_{11} = \begin{bmatrix} \mathbf{E}_3 & \tilde{\mathbf{E}}_4 & \mathbf{0} & \cdots & \cdots & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_3 & \tilde{\mathbf{E}}_4 & \ddots & & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 & \mathbf{E}_3 & \tilde{\mathbf{E}}_4 & \ddots & \vdots \\ \vdots & \ddots & \mathbf{E}_2 & \mathbf{E}_3 & \tilde{\mathbf{E}}_4 & \mathbf{0} \\ \mathbf{0} & & \ddots & \ddots & \ddots & \tilde{\mathbf{E}}_4 \\ \tilde{\mathbf{E}}_4 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{E}_2 & \mathbf{E}_3 \end{bmatrix},$$

$$\tilde{\mathbf{E}}_4 = \frac{1}{2}\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}.$$

The transformation matrix $\mathbf{T}$ that is applied to the control inputs takes the following form. The only exception to (5.17) is that there are no $\mathbf{I}_8$ blocks due to the fact that there are no stages between the boundary conditions at the base and the top of the structure. The second set of $\mathbf{I}_2$ blocks are also not needed since $m = 1$. Hence, the appropriate $\mathbf{T}$ matrix for $\tilde{\mathbf{u}} = \mathbf{T}\mathbf{u}$ is

$$\mathbf{T} = \left[\begin{array}{ccc|cc|c} 0 & \cdots & & & & 01 \\ \hline \mathbf{T}_1' & & & \mathbf{S} & & \\ & \ddots & & & \ddots & \\ & & \mathbf{T}_1' & & & \mathbf{S} \\ & & & \mathbf{I}_2 & & \\ & & & & \ddots & \\ & & & & & \mathbf{I}_2 \end{array}\right] \in \mathbb{R}^{(10n+1)\times 8n},$$

where $\mathbf{S}$ is defined by (5.18), and

$$\mathbf{T}_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{8\times 6}.$$

There are n $\mathbf{T}_1'$ blocks, n $\mathbf{S}$ blocks, n $\mathbf{I}_2$ blocks. The control inputs are now defined as

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1^{\mathrm{d}} \\ \mathbf{u}_1^0 \end{bmatrix}.$$

The $\mathbf{u}_1^{\mathrm{d}}$ block becomes

$$\mathbf{u}_1'^{\mathrm{d}} = \begin{bmatrix} \mathbf{u}_{11}'^{\mathrm{d}} \\ \mathbf{u}_{21}'^{\mathrm{d}} \\ \mathbf{u}_{31}'^{\mathrm{d}} \\ \vdots \\ \mathbf{u}_{n1}'^{\mathrm{d}} \end{bmatrix}, \quad \mathbf{u}_{i1}'^{\mathrm{d}} = \begin{bmatrix} u_{2n1} \\ u_{3n1} \\ u_{4n1} \\ u_{6n1} \\ u_{7n1} \\ u_{8n1} \end{bmatrix} \in \mathbb{R}^{6\times 1}, \quad i = 1, 2, 3, \ldots, n, \;\; j = m = 1.$$

The next section of the appendix will explicitly show all matrix forms for the specific example $(n, m) = (3, 1)$.

Appendix D. Example case $(n, m) = (3,1)$

Given the equation for the dynamics of the shell class of tensegrity structures:

$$\ddot{\mathbf{q}} + (\mathbf{K}_r(\dot{\mathbf{q}}) + \mathbf{K}_p(\mathbf{q}))\mathbf{q} = \mathbf{B}(\mathbf{q})\mathbf{u} + \mathbf{D}(\mathbf{q})\mathbf{w}.$$

We explicitly write out the matrices that define the problem:

$$\mathbf{q}=\mathbf{q}_1=\begin{bmatrix}\mathbf{q}_{11}\\ \mathbf{q}_{21}\\ \mathbf{q}_{31}\end{bmatrix}=\begin{bmatrix}\mathbf{q}_{111}\\ \mathbf{q}_{211}\\ \mathbf{q}_{311}\\ \mathbf{q}_{411}\\ \mathbf{q}_{121}\\ \hline \mathbf{q}_{221}\\ \mathbf{q}_{321}\\ \mathbf{q}_{421}\\ \hline \mathbf{q}_{131}\\ \mathbf{q}_{231}\\ \mathbf{q}_{331}\\ \mathbf{q}_{431}\end{bmatrix}=\mathbf{Q}_{11}\mathbf{l}_1=\begin{bmatrix}\mathbf{A}&\mathbf{0}&\mathbf{0}\\ \mathbf{D}&\mathbf{B}&\mathbf{0}\\ \mathbf{D}&\mathbf{E}&\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{l}_{11}\\ \mathbf{l}_{21}\\ \mathbf{l}_{31}\end{bmatrix}=\begin{bmatrix}\mathbf{A}&\mathbf{0}&\mathbf{0}\\ \mathbf{D}&\mathbf{B}&\mathbf{0}\\ \mathbf{D}&\mathbf{E}&\mathbf{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{\rho}_{11}\\ \mathbf{r}_{111}\\ \mathbf{r}_{211}\\ \mathbf{t}_{511}\\ \mathbf{t}_{111}\\ \hline \mathbf{r}_{121}\\ \mathbf{r}_{221}\\ \mathbf{t}_{521}\\ \hline \mathbf{t}_{121}\\ \mathbf{r}_{131}\\ \mathbf{r}_{231}\\ \mathbf{t}_{531}\end{bmatrix},$$

where $\mathbf{Q}_{11}$ is 36×36. Futhermore,

$$\mathbf{K}_r(\dot{\mathbf{q}})=\begin{bmatrix}\boldsymbol{\Omega}_{11}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\boldsymbol{\Omega}_{21}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\boldsymbol{\Omega}_{31}\end{bmatrix},$$

which is also a 36×36 matrix.

$$\mathbf{K}_p(\mathbf{q})=\mathbf{H}\tilde{\mathbf{K}}=\mathbf{H}(\mathbf{B}^{\mathrm{d}}\mathbf{K}^{\mathrm{d}}+\mathbf{B}^0\mathbf{K}^0)=\mathbf{H}_1(\mathbf{B}_3\bar{\mathbf{K}}_1^{\mathrm{d}}+\mathbf{B}_1^0\mathbf{K}_1^0),$$

$$\mathbf{K}_p=\begin{bmatrix}\mathbf{H}_{11}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{H}_{21}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{H}_{31}\end{bmatrix}\left(\begin{bmatrix}\mathbf{B}_{1n1}&\mathbf{B}_{11}^{\mathrm{d}}&\mathbf{B}_{21}^{\mathrm{d}}&\hat{\mathbf{B}}_{n1}^{\mathrm{d}}\\ \mathbf{0}&\hat{\mathbf{B}}_{01}^{\mathrm{d}}&\mathbf{B}_{11}^{\mathrm{d}}&\mathbf{B}_{21}^{\mathrm{d}}\\ \bar{\mathbf{B}}_{1n1}&\mathbf{B}_{21}^{\mathrm{d}}&\hat{\mathbf{B}}_{01}^{\mathrm{d}}&\mathbf{B}_{11}^{\mathrm{d}}\end{bmatrix}\begin{bmatrix}\mathbf{K}_{131}\\ \mathbf{K}_{11}^{\mathrm{d}}\\ \mathbf{K}_{21}^{\mathrm{d}}\\ \mathbf{K}_{31}^{\mathrm{d}}\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}\mathbf{B}_{11}^0&\mathbf{0}&\mathbf{0}\\ \mathbf{B}_{01}^0&\mathbf{B}_{11}^0&\mathbf{0}\\ \mathbf{0}&\mathbf{B}_{01}^0&\mathbf{B}_{n1}^0\end{bmatrix}\begin{bmatrix}\mathbf{K}_{11}^0\\ \mathbf{K}_{21}^0\\ \mathbf{K}_{31}^0\end{bmatrix}\right).$$

$$(36\times 36)=(36\times 36)[(36\times 75)*(75\times 36)+(36\times 18)*(18\times 36)]$$

In order to form $\bar{\mathbf{K}}_1^{\mathrm{d}}$, $\mathbf{R}$ is needed. In order to form $\mathbf{K}_1^0$, $\mathbf{Q}^{-1}$ is needed. Therefore, we obtain $\mathbf{R}$ as follows:

$$\mathbf{l}^{\mathrm{d}} = \mathbf{l}_1^{\mathrm{d}} = \begin{bmatrix} \mathbf{t}_{131}^{\mathrm{d}} \\ \mathbf{t}_1^{\mathrm{d}} \end{bmatrix} = \mathbf{R}\mathbf{q} = \begin{bmatrix} \mathbf{R}_0 \\ \tilde{\mathbf{R}}_{11} \end{bmatrix} [\mathbf{q}_1],$$

$$\left[\begin{array}{c} \mathbf{t}_{131}^{\mathrm{d}} \\ \hline \mathbf{t}_{11}^{\mathrm{d}} \\ \mathbf{t}_{21}^{\mathrm{d}} \\ \mathbf{t}_{31}^{\mathrm{d}} \end{array}\right] = \left[\begin{array}{ccc} \mathbf{E}_6 & \mathbf{0} & \mathbf{E}_7 \\ \hline \mathbf{E}_3 & \tilde{\mathbf{E}}_4 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_3 & \tilde{\mathbf{E}}_4 \\ \tilde{\mathbf{E}}_4 & \mathbf{E}_2 & \mathbf{E}_3 \end{array}\right] \begin{bmatrix} \mathbf{q}_{11} \\ \mathbf{q}_{21} \\ \mathbf{q}_{31} \end{bmatrix},$$

where the matrix dimension is 75×36.

$$\mathbf{B}(\mathbf{q}) = \mathbf{H}\tilde{\mathbf{B}}\mathbf{T} = \mathbf{H}[\mathbf{B}^{\mathrm{d}}\mathbf{P}^{\mathrm{d}}, \mathbf{B}^0\mathbf{P}^0]\mathbf{T}$$

$$[\mathbf{B}^{\mathrm{d}}\mathbf{P}^{\mathrm{d}}] = [\mathbf{B}_3\bar{\mathbf{P}}_1^{\mathrm{d}}] = \begin{bmatrix} \mathbf{B}_{1n1} & \mathbf{B}_{11}^{\mathrm{d}} & \mathbf{B}_{21}^{\mathrm{d}} & \hat{\mathbf{B}}_{n1}^{\mathrm{d}} \\ \mathbf{0} & \hat{\mathbf{B}}_{01}^{\mathrm{d}} & \mathbf{B}_{11}^{\mathrm{d}} & \mathbf{B}_{21}^{\mathrm{d}} \\ \bar{\mathbf{B}}_{1n1} & \mathbf{B}_{21}^{\mathrm{d}} & \hat{\mathbf{B}}_{01}^{\mathrm{d}} & \mathbf{B}_{11}^{\mathrm{d}} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{1n1}^{\mathrm{d}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{11}^{\mathrm{d}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{21}^{\mathrm{d}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_{31}^{\mathrm{d}} \end{bmatrix},$$

where the dimension are:

$$(36 \times 25) = (36 \times 75) * (75 \times 25)$$

$$[\mathbf{B}^0\mathbf{P}^0] = [\mathbf{B}_1\mathbf{P}_1^0] = \begin{bmatrix} \mathbf{B}_{11}^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{01}^0 & \mathbf{B}_{11}^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{01}^0 & \mathbf{B}_{n1}^0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_{11}^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{21}^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{31}^0 \end{bmatrix},$$

and the dimensions are:

$$(36 \times 6) = (36 \times 18) * (18 \times 6)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{31} \end{bmatrix} [\mathbf{B}_3\bar{\mathbf{P}}_1^{\mathrm{d}},\ \mathbf{B}_1\mathbf{P}_1^0] \left[\begin{array}{ccccc|c} 0 & \cdots & & \cdots & 0 & 01 \\ \hline \mathbf{T}_1' & \mathbf{0} & \mathbf{0} & \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_1' & \mathbf{0} & \mathbf{0} & \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_1' & \mathbf{0} & \mathbf{0} & \mathbf{S} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 \end{array}\right].$$

and the dimensions are:

$$(36 \times 24) = (36 \times 36) * (36 \times 31) * (31 \times 24)$$

The control inputs $u_{\alpha ij}$ are defined as

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}^{\mathrm{d}} \\ \mathbf{u}^{0} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1'^{\mathrm{d}} \\ \mathbf{u}_1^{0} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_{11}'^{\mathrm{d}} \\ \mathbf{u}_{21}'^{\mathrm{d}} \\ \mathbf{u}_{31}'^{\mathrm{d}} \\ \mathbf{u}_{11}^{0} \\ \mathbf{u}_{21}^{0} \\ \mathbf{u}_{31}^{0} \end{bmatrix} \quad (24 \times 1),$$

where

$$\mathbf{u}_{i1}'^{\mathrm{d}} = \begin{bmatrix} u_{2n1} \\ u_{3n1} \\ u_{4n1} \\ u_{6n1} \\ u_{7n1} \\ u_{8n1} \end{bmatrix} \in \mathbb{R}^{6\times 1}, \quad i = 1, 2, 3, \; j = 1.$$

$$\mathbf{Bu} \in \mathbb{R}^{(36\times 1)}.$$

The external forces applied to the nodes arise in the product $\mathbf{Dw}$, where

$$\mathbf{D}(\mathbf{q}) = \mathbf{HW}^0 = \mathbf{H}[\,\mathbf{W}_1\,] = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{31} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{W} \end{bmatrix},$$

with dimensions $(36 \times 36) = (36 \times 36) * (36 \times 36)$

$$\mathbf{w} = \mathbf{w}_1 = \begin{bmatrix} \mathbf{w}_{11} \\ \mathbf{w}_{21} \\ \mathbf{w}_{31} \end{bmatrix} (36 \times 1)$$

so

$$\mathbf{Dw} \in \mathbb{R}^{(36\times 1)}.$$

Appendix E. Nodal forces

At the base, right end of Fig. 5, where $(i = 1,\ j = 1)$:

$$\bar{\mathbf{f}}_{11} = \begin{bmatrix} \hat{\mathbf{f}}_{111} + \hat{\mathbf{f}}_{211} \\ \hat{\mathbf{f}}_{111} - \hat{\mathbf{f}}_{211} \\ \hat{\mathbf{f}}_{311} + \hat{\mathbf{f}}_{411} \\ \hat{\mathbf{f}}_{311} - \hat{\mathbf{f}}_{411} \end{bmatrix} = \begin{bmatrix} (-\mathbf{f}_{211} + \mathbf{f}_{311} + \mathbf{f}_{1n1} + \mathbf{f}_{2n1} + \mathbf{w}_{111}) + (\mathbf{f}_{511} - \mathbf{f}_{111} - \mathbf{f}_{411} - \mathbf{f}_{811} + \mathbf{w}_{211}) \\ (-\mathbf{f}_{211} + \mathbf{f}_{311} + \mathbf{f}_{1n1} + \mathbf{f}_{2n1} + \mathbf{w}_{111}) - (\mathbf{f}_{511} - \mathbf{f}_{111} - \mathbf{f}_{411} - \mathbf{f}_{811} + \mathbf{w}_{211}) \\ (\mathbf{f}_{811} - \mathbf{f}_{711} + \mathbf{f}_{1011} - \mathbf{f}_{321} + \mathbf{f}_{421} + \mathbf{w}_{311}) + (\mathbf{f}_{611} - \mathbf{f}_{511} + \mathbf{f}_{212} + \mathbf{f}_{112} + \mathbf{w}_{411}) \\ (\mathbf{f}_{811} - \mathbf{f}_{711} + \mathbf{f}_{1011} - \mathbf{f}_{321} + \mathbf{f}_{421} + \mathbf{w}_{311}) - (\mathbf{f}_{611} - \mathbf{f}_{511} + \mathbf{f}_{212} + \mathbf{f}_{112} + \mathbf{w}_{411}) \end{bmatrix}.$$

At the center of the base, where $(1<i<n,\ j = 1)$:

$$\bar{\mathbf{f}}_{i1} = \begin{bmatrix} \hat{\mathbf{f}}_{1i1} + \hat{\mathbf{f}}_{2i1} \\ \hat{\mathbf{f}}_{1i1} - \hat{\mathbf{f}}_{2i1} \\ \hat{\mathbf{f}}_{3i1} + \hat{\mathbf{f}}_{4i1} \\ \hat{\mathbf{f}}_{3i1} - \hat{\mathbf{f}}_{4i1} \end{bmatrix} = \begin{bmatrix} (\mathbf{f}_{1(i-1)1} + \mathbf{f}_{2(i-1)1} - \mathbf{f}_{2i1} + \mathbf{f}_{3i1} + \mathbf{w}_{1i1}) + (-\mathbf{f}_{1i1} - \mathbf{f}_{4i1} + \mathbf{f}_{5i1} - \mathbf{f}_{8i1}) + \mathbf{w}_{2i1}) \\ (\mathbf{f}_{1(i-1)1} + \mathbf{f}_{2(i-1)1} - \mathbf{f}_{2i1} + \mathbf{f}_{3i1} + \mathbf{w}_{1i1}) - (-\mathbf{f}_{1i1} - \mathbf{f}_{4i1} + \mathbf{f}_{5i1} - \mathbf{f}_{8i1}) + \mathbf{w}_{2i1}) \\ (\mathbf{f}_{8i1} - \mathbf{f}_{7i1} + \mathbf{f}_{10i1} - \mathbf{f}_{3(i+1)1} + \mathbf{f}_{4(i+1)1} + \mathbf{w}_{3i1}) + (-\mathbf{f}_{5i1} + \mathbf{f}_{6i1} - \mathbf{f}_{10(i-1)} + \mathbf{f}_{1i2} + \mathbf{f}_{2i2} + \mathbf{w}_{4i1}) \\ (\mathbf{f}_{8i1} - \mathbf{f}_{7i1} + \mathbf{f}_{10i1} - \mathbf{f}_{3(i+1)1} + \mathbf{f}_{4(i+1)1} + \mathbf{w}_{3i1}) - (-\mathbf{f}_{5i1} + \mathbf{f}_{6i1} - \mathbf{f}_{10(i-1)} + \mathbf{f}_{1i2} + \mathbf{f}_{2i2} + \mathbf{w}_{4i1}) \end{bmatrix}.$$

At the left end of the base in Fig. 5, where $(i = n,\ j = 1)$:

$$\bar{\mathbf{f}}_{n1} = \begin{bmatrix} \hat{\mathbf{f}}_{1n1} + \hat{\mathbf{f}}_{2n1} \\ \hat{\mathbf{f}}_{1n1} - \hat{\mathbf{f}}_{2n1} \\ \hat{\mathbf{f}}_{3n1} + \hat{\mathbf{f}}_{4n1} \\ \hat{\mathbf{f}}_{3n1} - \hat{\mathbf{f}}_{4n1} \end{bmatrix} = \begin{bmatrix} (\mathbf{f}_{1(n-1)1} + \mathbf{f}_{2(n-1)1} + \mathbf{f}_{3n1} - \mathbf{f}_{2n1} + \mathbf{w}_{1n1}) + (-\mathbf{f}_{1n1} - \mathbf{f}_{4n1} + \mathbf{f}_{5n1} - \mathbf{f}_{8n1} + \mathbf{w}_{2n1}) \\ (\mathbf{f}_{1(n-1)1} + \mathbf{f}_{2(n-1)1} + \mathbf{f}_{3n1} - \mathbf{f}_{2n1} + \mathbf{w}_{1n1}) - (-\mathbf{f}_{1n1} - \mathbf{f}_{4n1} + \mathbf{f}_{5n1} - \mathbf{f}_{8n1} + \mathbf{w}_{2n1}) \\ (-\mathbf{f}_{7n1} + \mathbf{f}_{8n1} + \mathbf{f}_{10n1} - \mathbf{f}_{311} + \mathbf{f}_{411} + \mathbf{w}_{3n1}) + (-\mathbf{f}_{10(n-1)1} - \mathbf{f}_{5n1} + \mathbf{f}_{6n1} + \mathbf{f}_{1n2} + \mathbf{f}_{2n2} + \mathbf{w}_{4n1}) \\ (-\mathbf{f}_{7n1} + \mathbf{f}_{8n1} + \mathbf{f}_{10n1} - \mathbf{f}_{311} + \mathbf{f}_{411} + \mathbf{w}_{3n1}) - (-\mathbf{f}_{10(n-1)1} - \mathbf{f}_{5n1} + \mathbf{f}_{6n1} + \mathbf{f}_{1n2} + \mathbf{f}_{2n2} + \mathbf{w}_{4n1}) \end{bmatrix}. \tag{E.1}$$

At the second stage, where ($i \leqslant i \leqslant n,\ j = 2$):

$$\bar{\mathbf{f}}_{12} = \begin{bmatrix} \hat{\mathbf{f}}_{112} + \hat{\mathbf{f}}_{212} \\ \hat{\mathbf{f}}_{112} - \hat{\mathbf{f}}_{212} \\ \hat{\mathbf{f}}_{312} + \hat{\mathbf{f}}_{412} \\ \hat{\mathbf{f}}_{312} - \hat{\mathbf{f}}_{412} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6n1} + \mathbf{f}_{7n1} - \mathbf{f}_{9n2} - \mathbf{f}_{212} + \mathbf{f}_{312} + \mathbf{w}_{112}) + (-\mathbf{f}_{112} - \mathbf{f}_{412} + \mathbf{f}_{512} - \mathbf{f}_{812} + \mathbf{f}_{912} + \mathbf{w}_{212}) \\ (-\mathbf{f}_{6n1} + \mathbf{f}_{7n1} - \mathbf{f}_{9n2} - \mathbf{f}_{212} + \mathbf{f}_{312} + \mathbf{w}_{112}) - (-\mathbf{f}_{112} - \mathbf{f}_{412} + \mathbf{f}_{512} - \mathbf{f}_{812} + \mathbf{f}_{912} + \mathbf{w}_{212}) \\ (-\mathbf{f}_{712} + \mathbf{f}_{812} + \mathbf{f}_{1012} - \mathbf{f}_{322} + \mathbf{f}_{422} + \mathbf{w}_{312}) + (-\mathbf{f}_{512} + \mathbf{f}_{612} - \mathbf{f}_{10n2} + \mathbf{f}_{113} + \mathbf{f}_{213} + \mathbf{w}_{412}) \\ (-\mathbf{f}_{712} + \mathbf{f}_{812} + \mathbf{f}_{1012} - \mathbf{f}_{322} + \mathbf{f}_{422} + \mathbf{w}_{312}) - (-\mathbf{f}_{512} + \mathbf{f}_{612} - \mathbf{f}_{10n2} + \mathbf{f}_{113} + \mathbf{f}_{213} + \mathbf{w}_{412}) \end{bmatrix},$$

$$\bar{\mathbf{f}}_{i2} = \begin{bmatrix} \hat{\mathbf{f}}_{1i2} + \hat{\mathbf{f}}_{2i2} \\ \hat{\mathbf{f}}_{1i2} - \hat{\mathbf{f}}_{2i2} \\ \hat{\mathbf{f}}_{3i2} + \hat{\mathbf{f}}_{4i2} \\ \hat{\mathbf{f}}_{3i2} - \hat{\mathbf{f}}_{4i2} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6(i-1)1} + \mathbf{f}_{7(i-1)1} - \mathbf{f}_{9(i-1)2} - \mathbf{f}_{2i2} + \mathbf{f}_{3i2} + \mathbf{w}_{1i2}) + (-\mathbf{f}_{1i2} - \mathbf{f}_{4i2} + \mathbf{f}_{5i2} - \mathbf{f}_{8i2} + \mathbf{f}_{9i2} + \mathbf{w}_{2i2}) \\ (-\mathbf{f}_{6(i-1)1} + \mathbf{f}_{7(i-1)1} - \mathbf{f}_{9(i-1)2} - \mathbf{f}_{2i2} + \mathbf{f}_{3i2} + \mathbf{w}_{1i2}) - (-\mathbf{f}_{1i2} - \mathbf{f}_{4i2} + \mathbf{f}_{5i2} - \mathbf{f}_{8i2} + \mathbf{f}_{9i2} + \mathbf{w}_{2i2}) \\ (-\mathbf{f}_{7i2} + \mathbf{f}_{8i2} + \mathbf{f}_{10i2} - \mathbf{f}_{3(i+1)2} + \mathbf{f}_{4(i+1)2} + \mathbf{w}_{3i2}) + (-\mathbf{f}_{5i2} + \mathbf{f}_{6i2} - \mathbf{f}_{10(i-1)2} + \mathbf{f}_{1i3} + \mathbf{f}_{2i3} + \mathbf{w}_{4i2}) \\ (-\mathbf{f}_{7i2} + \mathbf{f}_{8i2} + \mathbf{f}_{10i2} - \mathbf{f}_{3(i+1)2} + \mathbf{f}_{4(i+1)2} + \mathbf{w}_{3i2}) - (-\mathbf{f}_{5i2} + \mathbf{f}_{6i2} - \mathbf{f}_{10(i-1)2} + \mathbf{f}_{1i3} + \mathbf{f}_{2i3} + \mathbf{w}_{4i2}) \end{bmatrix},$$

$$\bar{\mathbf{f}}_{n2} = \begin{bmatrix} \hat{\mathbf{f}}_{1n2} + \hat{\mathbf{f}}_{2n2} \\ \hat{\mathbf{f}}_{1n2} - \hat{\mathbf{f}}_{2n2} \\ \hat{\mathbf{f}}_{3n2} + \hat{\mathbf{f}}_{4n2} \\ \hat{\mathbf{f}}_{3n2} - \hat{\mathbf{f}}_{4n2} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6(n-1)1} + \mathbf{f}_{7(n-1)1} - \mathbf{f}_{9(n-1)2} - \mathbf{f}_{2n2} + \mathbf{f}_{3n2} + \mathbf{w}_{1n2}) + (-\mathbf{f}_{1n2} - \mathbf{f}_{4n2} + \mathbf{f}_{5n2} - \mathbf{f}_{8n2} + \mathbf{f}_{9n2} + \mathbf{w}_{2n2}) \\ (-\mathbf{f}_{6(n-1)1} + \mathbf{f}_{7(n-1)1} - \mathbf{f}_{9(n-1)2} - \mathbf{f}_{2n2} + \mathbf{f}_{3n2} + \mathbf{w}_{1n2}) - (-\mathbf{f}_{1n2} - \mathbf{f}_{4n2} + \mathbf{f}_{5n2} - \mathbf{f}_{8n2} + \mathbf{f}_{9n2} + \mathbf{w}_{2n2}) \\ (-\mathbf{f}_{7n2} + \mathbf{f}_{8n2} + \mathbf{f}_{10n2} - \mathbf{f}_{312} + \mathbf{f}_{412} + \mathbf{w}_{3n2}) + (-\mathbf{f}_{5n2} + \mathbf{f}_{6n2} - \mathbf{f}_{10(n-1)2} + \mathbf{f}_{1n3} + \mathbf{f}_{2n3} + \mathbf{w}_{4n2}) \\ (-\mathbf{f}_{7n2} + \mathbf{f}_{8n2} + \mathbf{f}_{10n2} - \mathbf{f}_{312} + \mathbf{f}_{412} + \mathbf{w}_{3n2}) - (-\mathbf{f}_{5n2} + \mathbf{f}_{6n2} - \mathbf{f}_{10(n-1)2} + \mathbf{f}_{1n3} + \mathbf{f}_{2n3} + \mathbf{w}_{4n2}) \end{bmatrix}. \tag{E.2}$$

At the typical stage ($1 < j < m,\ 1 \leqslant i \leqslant n$):

$$\bar{\mathbf{f}}_{1j} = \begin{bmatrix} \hat{\mathbf{f}}_{11j} + \hat{\mathbf{f}}_{21j} \\ \hat{\mathbf{f}}_{11j} - \hat{\mathbf{f}}_{21j} \\ \hat{\mathbf{f}}_{31j} + \hat{\mathbf{f}}_{41j} \\ \hat{\mathbf{f}}_{31j} - \hat{\mathbf{f}}_{41j} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6n(j-1)} + \mathbf{f}_{7n(j-1)} - \mathbf{f}_{9nj} - \mathbf{f}_{21j} + \mathbf{f}_{31j} + \mathbf{w}_{11j}) + (-\mathbf{f}_{11j} - \mathbf{f}_{41j} + \mathbf{f}_{51j} - \mathbf{f}_{81j} + \mathbf{f}_{91j} + \mathbf{w}_{21j}) \\ (-\mathbf{f}_{6n(j-1)} + \mathbf{f}_{7n(j-1)} - \mathbf{f}_{9nj} - \mathbf{f}_{21j} + \mathbf{f}_{31j} + \mathbf{w}_{11j}) - (-\mathbf{f}_{11j} - \mathbf{f}_{41j} + \mathbf{f}_{51j} - \mathbf{f}_{81j} + \mathbf{f}_{91j} + \mathbf{w}_{21j}) \\ (-\mathbf{f}_{71j} + \mathbf{f}_{81j} + \mathbf{f}_{101j} - \mathbf{f}_{32j} + \mathbf{f}_{42j} + \mathbf{w}_{31j}) + (-\mathbf{f}_{51j} + \mathbf{f}_{61j} - \mathbf{f}_{10nj} + \mathbf{f}_{11(j+1)} + \mathbf{f}_{21(j+1)} + \mathbf{w}_{41j}) \\ (-\mathbf{f}_{71j} + \mathbf{f}_{81j} + \mathbf{f}_{101j} - \mathbf{f}_{32j} + \mathbf{f}_{42j} + \mathbf{w}_{31j}) - (-\mathbf{f}_{51j} + \mathbf{f}_{61j} - \mathbf{f}_{10nj} + \mathbf{f}_{11(j+1)} + \mathbf{f}_{21(j+1)} + \mathbf{w}_{41j}) \end{bmatrix},$$

$$\bar{\mathbf{f}}_{ij} = \begin{bmatrix} \hat{\mathbf{f}}_{1ij} + \hat{\mathbf{f}}_{2ij} \\ \hat{\mathbf{f}}_{1ij} - \hat{\mathbf{f}}_{2ij} \\ \hat{\mathbf{f}}_{3ij} + \hat{\mathbf{f}}_{4ij} \\ \hat{\mathbf{f}}_{3ij} - \hat{\mathbf{f}}_{4ij} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6(i-1)(j-1)} + \mathbf{f}_{7(i-1)(j-1)} - \mathbf{f}_{9(i-1)j} - \mathbf{f}_{2ij} + \mathbf{f}_{3ij} + \mathbf{w}_{1ij}) + (-\mathbf{f}_{1ij} - \mathbf{f}_{4ij} + \mathbf{f}_{5ij} - \mathbf{f}_{8ij} + \mathbf{f}_{9ij} + \mathbf{w}_{2ij}) \\ (-\mathbf{f}_{6(i-1)(j-1)} + \mathbf{f}_{7(i-1)(j-1)} - \mathbf{f}_{9(i-1)j} - \mathbf{f}_{2ij} + \mathbf{f}_{3ij} + \mathbf{w}_{1ij}) - (-\mathbf{f}_{1ij} - \mathbf{f}_{4ij} + \mathbf{f}_{5ij} - \mathbf{f}_{8ij} + \mathbf{f}_{9ij} + \mathbf{w}_{2ij}) \\ (-\mathbf{f}_{7ij} + \mathbf{f}_{8ij} + \mathbf{f}_{10ij} - \mathbf{f}_{3(i+1)j} + \mathbf{f}_{4(i+1)j} + \mathbf{w}_{3ij}) + (-\mathbf{f}_{5ij} + \mathbf{f}_{6ij} - \mathbf{f}_{10(i-1)j} + \mathbf{f}_{1i(j+1)} + \mathbf{f}_{2i(j+1)} + \mathbf{w}_{4ij}) \\ (-\mathbf{f}_{7ij} + \mathbf{f}_{8ij} + \mathbf{f}_{10ij} - \mathbf{f}_{3(i+1)j} + \mathbf{f}_{4(i+1)j} + \mathbf{w}_{3ij}) - (-\mathbf{f}_{5ij} + \mathbf{f}_{6ij} - \mathbf{f}_{10(i-1)j} + \mathbf{f}_{1i(j+1)} + \mathbf{f}_{2i(j+1)} + \mathbf{w}_{4ij}) \end{bmatrix},$$

$$\bar{\mathbf{f}}_{nj} = \begin{bmatrix} \hat{\mathbf{f}}_{1nj} + \hat{\mathbf{f}}_{2nj} \\ \hat{\mathbf{f}}_{1nj} - \hat{\mathbf{f}}_{2nj} \\ \hat{\mathbf{f}}_{3nj} + \hat{\mathbf{f}}_{4nj} \\ \hat{\mathbf{f}}_{3nj} - \hat{\mathbf{f}}_{4nj} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6(n-1)(j-1)} + \mathbf{f}_{7(n-1)(j-1)} - \mathbf{f}_{9(n-1)j} - \mathbf{f}_{2nj} + \mathbf{f}_{3nj} + \mathbf{w}_{1nj}) + (-\mathbf{f}_{1nj} - \mathbf{f}_{4nj} + \mathbf{f}_{5nj} - \mathbf{f}_{8nj} + \mathbf{f}_{9nj} + \mathbf{w}_{2nj}) \\ (-\mathbf{f}_{6(n-1)(j-1)} + \mathbf{f}_{7(n-1)(j-1)} - \mathbf{f}_{9(n-1)j} - \mathbf{f}_{2nj} + \mathbf{f}_{3nj} + \mathbf{w}_{1nj}) - (-\mathbf{f}_{1nj} - \mathbf{f}_{4nj} + \mathbf{f}_{5nj} - \mathbf{f}_{8nj} + \mathbf{f}_{9nj} + \mathbf{w}_{2nj}) \\ (-\mathbf{f}_{7nj} + \mathbf{f}_{8nj} + \mathbf{f}_{10nj} - \mathbf{f}_{31j} + \mathbf{f}_{41j} + \mathbf{w}_{3nj}) + (-\mathbf{f}_{5nj} + \mathbf{f}_{6nj} - \mathbf{f}_{10(n-1)j} + \mathbf{f}_{1n(j+1)} + \mathbf{f}_{2n(j+1)} + \mathbf{w}_{4nj}) \\ (-\mathbf{f}_{7nj} + \mathbf{f}_{8nj} + \mathbf{f}_{10nj} - \mathbf{f}_{31j} + \mathbf{f}_{41j} + \mathbf{w}_{3nj}) - (-\mathbf{f}_{5nj} + \mathbf{f}_{6nj} - \mathbf{f}_{10(n-1)j} + \mathbf{f}_{1n(j+1)} + \mathbf{f}_{2n(j+1)} + \mathbf{w}_{4nj}) \end{bmatrix}. \tag{E.3}$$

$(1 \leqslant i \leqslant n,\ j = m)$:

$$\bar{\mathbf{f}}_{1m} = \begin{bmatrix} \hat{\mathbf{f}}_{11m} + \hat{\mathbf{f}}_{21m} \\ \hat{\mathbf{f}}_{11m} - \hat{\mathbf{f}}_{21m} \\ \hat{\mathbf{f}}_{31m} + \hat{\mathbf{f}}_{41m} \\ \hat{\mathbf{f}}_{31m} - \hat{\mathbf{f}}_{41m} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6n(m-1)} + \mathbf{f}_{7n(m-1)} - \mathbf{f}_{21m} + \mathbf{f}_{31m} - \mathbf{f}_{9nm} + \mathbf{w}_{1nm}) + (-\mathbf{f}_{41m} + \mathbf{f}_{51m} - \mathbf{f}_{81m} - \mathbf{f}_{11m} + \mathbf{f}_{91m} + \mathbf{w}_{2nm}) \\ (-\mathbf{f}_{6n(m-1)} + \mathbf{f}_{7n(m-1)} - \mathbf{f}_{21m} + \mathbf{f}_{31m} - \mathbf{f}_{9nm} + \mathbf{w}_{1nm}) - (-\mathbf{f}_{41m} + \mathbf{f}_{51m} - \mathbf{f}_{81m} - \mathbf{f}_{11m} + \mathbf{f}_{91m} + \mathbf{w}_{2nm}) \\ (-\mathbf{f}_{71m} + \mathbf{f}_{81m} - \mathbf{f}_{32m} + \mathbf{f}_{42m} + \mathbf{w}_{31m}) + (-\mathbf{f}_{51m} + \mathbf{f}_{61m} - \mathbf{f}_{6nm} + \mathbf{f}_{7nm} + \mathbf{w}_{41m}) \\ (-\mathbf{f}_{71m} + \mathbf{f}_{81m} - \mathbf{f}_{32m} + \mathbf{f}_{42m} + \mathbf{w}_{31m}) - (-\mathbf{f}_{51m} + \mathbf{f}_{61m} - \mathbf{f}_{6nm} + \mathbf{f}_{7nm} + \mathbf{w}_{41m}) \end{bmatrix},$$

$$\bar{\mathbf{f}}_{im} = \begin{bmatrix} \hat{\mathbf{f}}_{1im} + \hat{\mathbf{f}}_{2im} \\ \hat{\mathbf{f}}_{1im} - \hat{\mathbf{f}}_{2im} \\ \hat{\mathbf{f}}_{3im} + \hat{\mathbf{f}}_{4im} \\ \hat{\mathbf{f}}_{3im} - \hat{\mathbf{f}}_{4im} \end{bmatrix} =$$

$$\begin{bmatrix} (-\mathbf{f}_{6(i-1)(m-1)}+\mathbf{f}_{7(i-1)(m-1)}-\mathbf{f}_{9(i-1)m}-\mathbf{f}_{2im}+\mathbf{f}_{3im}+\mathbf{w}_{1im})+(-\mathbf{f}_{1im}-\mathbf{f}_{4im}+\mathbf{f}_{5im}-\mathbf{f}_{8im}+\mathbf{f}_{9im}+\mathbf{w}_{2im}) \\ (-\mathbf{f}_{6(i-1)(m-1)}+\mathbf{f}_{7(i-1)(m-1)}-\mathbf{f}_{9(i-1)m}-\mathbf{f}_{2im}+\mathbf{f}_{3im}+\mathbf{w}_{1im})-(-\mathbf{f}_{1im}-\mathbf{f}_{4im}+\mathbf{f}_{5im}-\mathbf{f}_{8im}+\mathbf{f}_{9im}+\mathbf{w}_{2im}) \\ (-\mathbf{f}_{7im}+\mathbf{f}_{8im}-\mathbf{f}_{3(i+1)m}+\mathbf{f}_{4(i+1)m}+\mathbf{w}_{3im})+(-\mathbf{f}_{5im}-\mathbf{f}_{6(i-1)m}+\mathbf{f}_{6im}+\mathbf{f}_{7(i-1)m}+\mathbf{w}_{4im}) \\ (-\mathbf{f}_{7im}+\mathbf{f}_{8im}-\mathbf{f}_{3(i+1)m}+\mathbf{f}_{4(i+1)m}+\mathbf{w}_{3im})-(-\mathbf{f}_{5im}-\mathbf{f}_{6(i-1)m}+\mathbf{f}_{6im}+\mathbf{f}_{7(i-1)m}+\mathbf{w}_{4im}) \end{bmatrix},$$

$$\bar{\mathbf{f}}_{nm} = \begin{bmatrix} \hat{\mathbf{f}}_{1nm}+\hat{\mathbf{f}}_{2nm} \\ \hat{\mathbf{f}}_{1nm}-\hat{\mathbf{f}}_{2nm} \\ \hat{\mathbf{f}}_{3nm}+\hat{\mathbf{f}}_{4nm} \\ \hat{\mathbf{f}}_{3nm}-\hat{\mathbf{f}}_{4nm} \end{bmatrix}$$

$$= \begin{bmatrix} (-\mathbf{f}_{6(n-1)(m-1)}+\mathbf{f}_{7(n-1)(m-1)}-\mathbf{f}_{9(n-1)m}-\mathbf{f}_{2nm}+\mathbf{f}_{3nm}+\mathbf{w}_{1nm})+(-\mathbf{f}_{1nm}-\mathbf{f}_{4nm}+\mathbf{f}_{5nm}-\mathbf{f}_{8nm}+\mathbf{f}_{9nm}+\mathbf{w}_{2nm}) \\ (-\mathbf{f}_{6(n-1)(m-1)}+\mathbf{f}_{7(n-1)(m-1)}-\mathbf{f}_{9(n-1)m}-\mathbf{f}_{2nm}+\mathbf{f}_{3nm}+\mathbf{w}_{1nm})-(-\mathbf{f}_{1nm}-\mathbf{f}_{4nm}+\mathbf{f}_{5nm}-\mathbf{f}_{8nm}+\mathbf{f}_{9nm}+\mathbf{w}_{2nm}) \\ (-\mathbf{f}_{7nm}+\mathbf{f}_{8nm}-\mathbf{f}_{31m}+\mathbf{f}_{41m}+\mathbf{w}_{3nm})+(-\mathbf{f}_{5nm}+\mathbf{f}_{6nm}-\mathbf{f}_{6(n-1)m}+\mathbf{f}_{7(n-1)m}+\mathbf{w}_{4nm}) \\ (-\mathbf{f}_{7nm}+\mathbf{f}_{8nm}-\mathbf{f}_{31m}+\mathbf{f}_{41m}+\mathbf{w}_{3nm})-(-\mathbf{f}_{5nm}+\mathbf{f}_{6nm}-\mathbf{f}_{6(n-1)m}+\mathbf{f}_{7(n-1)m}+\mathbf{w}_{4nm}) \end{bmatrix}. \tag{E.4}$$

$$\mathbf{f}^0_{ij} = \begin{bmatrix} \mathbf{f}_5 \\ \mathbf{f}_1 \end{bmatrix}_{ij}, \quad \mathbf{f}^{\mathrm{d}}_{ij} = \begin{bmatrix} \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_6 \\ \mathbf{f}_7 \\ \mathbf{f}_8 \\ \mathbf{f}_9 \\ \mathbf{f}_{10} \end{bmatrix}_{ij}, \quad \mathbf{w}_{ij} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \mathbf{w}_4 \end{bmatrix}_{ij},$$

$$\bar{\mathbf{f}}_{11} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{n1} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{11}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{12} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{21}$$

$$+ \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^0_{11} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix} \mathbf{f}^0_{12} + \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{f}_{1n1} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{11}.$$

Or, in matrix notation, with the implied matrix definitions:

$$\bar{\mathbf{f}}_{11} = \mathbf{B}^{\rm d}_{n1}\mathbf{f}^{\rm d}_{n1} + \mathbf{B}^{\rm d}_{11}\mathbf{f}^{\rm d}_{11} + \mathbf{B}^{\rm d}_{12}\mathbf{f}^{\rm d}_{12} + \mathbf{B}^{\rm d}_{21}\mathbf{f}^{\rm d}_{21} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{11} + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{12} + \mathbf{B}_{1n1}\mathbf{f}_{1n1} + \mathbf{W}\mathbf{w}_{11}, \tag{E.5}$$

$$\bar{\mathbf{f}}_{i1} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\rm d}_{(i-1)1} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\rm d}_{i1}$$
$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\rm d}_{i2} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\rm d}_{(i+1)1}$$
$$+ \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{(i-1)1} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{i1} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix} \mathbf{f}^{0}_{i2} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{i1}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{i1} = \mathbf{B}^{\rm d}_{01}\mathbf{f}^{\rm d}_{(i-1)1} + \mathbf{B}^{\rm d}_{11}\mathbf{f}^{\rm d}_{i1} + \mathbf{B}^{\rm d}_{12}\mathbf{f}^{\rm d}_{i2} + \mathbf{B}^{\rm d}_{21}\mathbf{f}^{\rm d}_{(i+1)1} + \mathbf{B}^{0}_{01}\mathbf{f}^{0}_{(i-1)1} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{i1} + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{i2} + \mathbf{W}\mathbf{w}_{i1} \tag{E.6}$$

$$\bar{\mathbf{f}}_{n1} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\rm d}_{(n-1)1} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\rm d}_{n1}$$
$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\rm d}_{n2} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\rm d}_{11}$$
$$+ \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{(n-1)1} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{n1} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix} \mathbf{f}^{0}_{n2} + \begin{bmatrix} -\mathbf{I}_3 \\ \mathbf{I}_3 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{f}_{1n1}$$
$$+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{n1}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{n1} = \mathbf{B}^{\mathrm{d}}_{01}\mathbf{f}^{\mathrm{d}}_{(n-1)1} + \mathbf{B}^{\mathrm{d}}_{11}\mathbf{f}^{\mathrm{d}}_{n1} + \mathbf{B}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{n2} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{11} + \mathbf{B}^{0}_{01}\mathbf{f}^{0}_{(n-1)1} + \mathbf{B}^{0}_{n1}\mathbf{f}^{0}_{n1} + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{n2} + \bar{\mathbf{B}}_{1n1}\mathbf{f}_{1n1} + \mathbf{W}\mathbf{w}_{n1}, \tag{E.7}$$

$$\bar{\mathbf{f}}_{12} = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{12} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{22}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{13} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{n1}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{n2} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix}\mathbf{f}^{0}_{12} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}\mathbf{f}^{0}_{13}$$

$$+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix}\mathbf{w}_{12}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{12} = \bar{\mathbf{B}}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{12} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{22} + \mathbf{B}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{13} + \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{n1} + \mathbf{B}^{\mathrm{d}}_{n2}\mathbf{f}^{\mathrm{d}}_{n2} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{12} + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{13} + \mathbf{W}\mathbf{w}_{12}, \tag{E.8}$$

$$\bar{\mathbf{f}}_{i2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{(i-1)1} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{i2}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{i3} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{(i-1)2}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{(i+1)2} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix}\mathbf{f}^{0}_{i2}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}\mathbf{f}^{0}_{i3} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix}\mathbf{w}_{i2}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{i2} = \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{(i-1)1} + \bar{\mathbf{B}}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{i2} + \mathbf{B}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{i3} + \mathbf{B}^{\mathrm{d}}_{n2}\mathbf{f}^{\mathrm{d}}_{(i-1)2} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{(i+1)2} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{i2}$$

$$+ \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{i3} + \mathbf{W}\mathbf{w}_{i2}, \tag{E.9}$$

$$\bar{\mathbf{f}}_{n2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{12} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{(n-1)1}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{(n-1)2} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{n2}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{f}^{\mathrm{d}}_{n3} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}\mathbf{f}^{0}_{n3} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix}\mathbf{f}^{0}_{n2}$$

$$+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix}\mathbf{w}_{n2}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{n2} = \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{12} + \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{(n-1)1} + \mathbf{B}^{\mathrm{d}}_{n2}\mathbf{f}^{\mathrm{d}}_{(n-1)2} + \bar{\mathbf{B}}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{n2} + \mathbf{B}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{n3} + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{n3}$$

$$+ \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{n2} + \mathbf{W}\mathbf{w}_{n2}, \tag{E.10}$$

$$
\bar{\mathbf{f}}_{1j} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{n(j-1)} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{1j}
$$

$$
+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{nj} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{2j}
$$

$$
+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{1(j+1)} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{1j} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix} \mathbf{f}^{0}_{1(j+1)}
$$

$$
+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{1j}.
$$

Or, in matrix form,

$$
\begin{aligned} \bar{\mathbf{f}}_{1j} = {} & \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{n(j-1)} + \bar{\mathbf{B}}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{1j} + \mathbf{B}^{\mathrm{d}}_{n2}\mathbf{f}^{\mathrm{d}}_{nj} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{2j} + \mathbf{B}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{1(j+1)} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{1j} \\ & + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{1(j+1)} + \mathbf{W}\mathbf{w}_{1j}, \end{aligned} \tag{E.11}
$$

$$
\bar{\mathbf{f}}_{ij} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(i-1)(j-1)} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{ij}
$$

$$
+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(i-1)j} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(i+1)j}
$$

$$
+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{i(j+1)} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{ij} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix} \mathbf{f}^{0}_{i(j+1)}
$$

$$
+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{ij}.
$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{ij} = \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{(i-1)(j-1)} + \bar{\mathbf{B}}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{ij} + \mathbf{B}^{\mathrm{d}}_{n2}\mathbf{f}^{\mathrm{d}}_{(i-1)j} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{(i+1)j} + \mathbf{B}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{i(j+1)} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{ij} + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{i(j+1)} + \mathbf{W}\mathbf{w}_{ij}, \tag{E.12}$$

$$\begin{aligned}
\bar{\mathbf{f}}_{nj} = &\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(n-1)(j-1)} \\
&+ \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{nj} \\
&+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(n-1)j} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{1j} \\
&+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{n(j+1)} + \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{nj} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix} \mathbf{f}^{0}_{n(j+1)} \\
&+ \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{nj}.
\end{aligned}$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{nj} = \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{(n-1)(j-1)} + \bar{\mathbf{B}}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{nj} + \mathbf{B}^{\mathrm{d}}_{n2}\mathbf{f}^{\mathrm{d}}_{(n-1)j} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{1j} + \mathbf{B}^{\mathrm{d}}_{12}\mathbf{f}^{\mathrm{d}}_{n(j+1)} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{nj} + \mathbf{B}^{0}_{12}\mathbf{f}^{0}_{n(j+1)} + \mathbf{W}\mathbf{w}_{nj}, \tag{E.13}$$

$$\bar{\mathbf{f}}_{1m} = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{1m} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{2m}$$

$$+\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{nm} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{n(m-1)}$$

$$+\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{1m} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{1m}.$$

Or, in matrix form,

$$\bar{\mathbf{f}}_{1m} = \mathbf{B}^{\mathrm{d}}_{1m}\mathbf{f}^{\mathrm{d}}_{1m} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{2m} + \mathbf{B}^{\mathrm{d}}_{nm}\mathbf{f}^{\mathrm{d}}_{nm} + \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{n(m-1)} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{1m} + \mathbf{W}\mathbf{w}_{1m}, \tag{E.14}$$

$$\bar{\mathbf{f}}_{im} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(i-1)(m-1)} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(i-1)m}$$

$$+\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{im} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(i+1)m}$$

$$+\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{im} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{im}.$$

Or, in matrix notation,

$$\bar{\mathbf{f}}_{im} = \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{(i-1)(m-1)} + \mathbf{B}^{\mathrm{d}}_{nm}\mathbf{f}^{\mathrm{d}}_{(i-1)m} + \mathbf{B}^{\mathrm{d}}_{1m}\mathbf{f}^{\mathrm{d}}_{im} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{(i+1)m} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{im} + \mathbf{W}\mathbf{w}_{im}, \tag{E.15}$$

$$\bar{\mathbf{f}}_{nm} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(n-1)(m-1)} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{(n-1)m}$$

$$+\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{nm} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{f}^{\mathrm{d}}_{1m}$$

$$+\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \mathbf{f}^{0}_{nm} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \mathbf{w}_{nm}.$$

Or, in matrix notation,

$$\bar{\mathbf{f}}_{nm} = \bar{\mathbf{B}}^{\mathrm{d}}_{n1}\mathbf{f}^{\mathrm{d}}_{(n-1)(m-1)} + \mathbf{B}^{\mathrm{d}}_{nm}\mathbf{f}^{\mathrm{d}}_{(n-1)m} + \mathbf{B}^{\mathrm{d}}_{1m}\mathbf{f}^{\mathrm{d}}_{nm} + \mathbf{B}^{\mathrm{d}}_{21}\mathbf{f}^{\mathrm{d}}_{1m} + \mathbf{B}^{0}_{11}\mathbf{f}^{0}_{nm} + \mathbf{W}\mathbf{w}_{nm}. \tag{E.16}$$

Now assemble (E.5)–(E.16) into the form

$$\mathbf{f}_1 = \mathbf{B}_3\bar{\mathbf{f}}^{\mathrm{d}}_1 + \mathbf{B}_4\mathbf{f}^{\mathrm{d}}_2 + \mathbf{B}_1\mathbf{f}^{0}_1 + \mathbf{B}_2\mathbf{f}^{0}_2 + \mathbf{W}_1\mathbf{w}_1, \tag{E.17}$$

where

$$\bar{\mathbf{f}}^{\mathrm{d}}_1 = \begin{bmatrix} \mathbf{f}_{1n1} \\ \mathbf{f}^{\mathrm{d}}_1 \end{bmatrix}, \quad \mathbf{f}^{\mathrm{d}}_j = \begin{bmatrix} \mathbf{f}^{\mathrm{d}}_{1j} \\ \mathbf{f}^{\mathrm{d}}_{2j} \\ \vdots \\ \mathbf{f}^{\mathrm{d}}_{nj} \end{bmatrix}, \quad \mathbf{f}^{0}_j = \begin{bmatrix} \mathbf{f}^{0}_{1j} \\ \mathbf{f}^{0}_{2j} \\ \vdots \\ \mathbf{f}^{0}_{nj} \end{bmatrix}, \quad \mathbf{w}_j = \begin{bmatrix} \mathbf{w}_{1j} \\ \mathbf{w}_{2j} \\ \vdots \\ \mathbf{w}_{nj} \end{bmatrix}, \quad \mathbf{f}_j = \begin{bmatrix} \bar{\mathbf{f}}_{1j} \\ \bar{\mathbf{f}}_{2j} \\ \vdots \\ \bar{\mathbf{f}}_{nj} \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} \mathbf{B}_{1n1} & \mathbf{B}^{\mathrm{d}}_{11} & \mathbf{B}^{\mathrm{d}}_{21} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{\mathrm{d}}_{n1} \\ \mathbf{0} & \mathbf{B}^{\mathrm{d}}_{01} & \ddots & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & & \ddots & \ddots & \ddots & \mathbf{B}^{\mathrm{d}}_{21} \\ \bar{\mathbf{B}}_{1n1} & \mathbf{B}^{\mathrm{d}}_{21} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{\mathrm{d}}_{01} & \mathbf{B}^{\mathrm{d}}_{11} \end{bmatrix},$$

$$\mathbf{B}_4 = \mathit{BlockDiag}[\cdots, \mathbf{B}^{\mathrm{d}}_{12}, \mathbf{B}^{\mathrm{d}}_{12}, \cdots],$$

$$\mathbf{B}_2 = \mathit{BlockDiag}[\cdots, \mathbf{B}^{0}_{12}, \mathbf{B}^{0}_{12}, \cdots],$$

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{B}^{0}_{11} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{B}^{0}_{01} & \ddots & \ddots & & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{B}^{0}_{11} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{0}_{01} & \mathbf{B}^{0}_{n1} \end{bmatrix},$$

$$\mathbf{W}_1 = \mathit{BlockDiag}[\cdots, \mathbf{W}, \mathbf{W}, \cdots].$$

Also from (E.5)–(E.16)

$$\mathbf{f}_2 = \mathbf{B}_5\bar{\mathbf{f}}^{\mathrm{d}}_1 + \mathbf{B}_6\mathbf{f}^{\mathrm{d}}_2 + \mathbf{B}_4\mathbf{f}^{\mathrm{d}}_3 + \mathbf{B}_7\mathbf{f}^{0}_2 + \mathbf{B}_2\mathbf{f}^{0}_3 + \mathbf{W}_1\mathbf{w}_2, \tag{E.18}$$

where

$$\mathbf{B}_5 = [\mathbf{0}, \bar{\mathbf{B}}_5], \quad \mathbf{B}_7 = \mathit{BlockDiag}[\cdots, \mathbf{B}^{0}_{11}, \mathbf{B}^{0}_{11}, \cdots],$$

$$\bar{\mathbf{B}}_5 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \bar{\mathbf{B}}^{\mathrm{d}}_{n1} \\ \bar{\mathbf{B}}^{\mathrm{d}}_{n1} & \ddots & \ddots & & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \bar{\mathbf{B}}^{\mathrm{d}}_{n1} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_6 = \begin{bmatrix} \bar{\mathbf{B}}^{\mathrm{d}}_{12} & \mathbf{B}^{\mathrm{d}}_{21} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{\mathrm{d}}_{n2} \\ \mathbf{B}^{\mathrm{d}}_{n2} & \ddots & \ddots & \ddots & & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & & \ddots & \ddots & \ddots & \mathbf{B}^{\mathrm{d}}_{21} \\ \mathbf{B}^{\mathrm{d}}_{21} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{\mathrm{d}}_{n2} & \bar{\mathbf{B}}^{\mathrm{d}}_{12} \end{bmatrix}.$$

Also from (E.5)–(E.16)

$$\mathbf{f}_3 = \bar{\mathbf{B}}_5\mathbf{f}^{\mathrm{d}}_2 + \mathbf{B}_6\mathbf{f}^{\mathrm{d}}_3 + \mathbf{B}_4\mathbf{f}^{\mathrm{d}}_4 + \mathbf{B}_7\mathbf{f}^0_3 + \mathbf{B}_2\mathbf{f}^0_4 + \mathbf{W}_1\mathbf{w}_3, \tag{E.19}$$

$$\mathbf{f}_m = \bar{\mathbf{B}}_5\mathbf{f}^{\mathrm{d}}_{(m-1)} + \mathbf{B}_8\mathbf{f}^{\mathrm{d}}_m + \mathbf{B}_7\mathbf{f}^0_m + \mathbf{W}_1\mathbf{w}_m, \tag{E.20}$$

$$\mathbf{B}_8 = \begin{bmatrix} \mathbf{B}^{\mathrm{d}}_{1m} & \mathbf{B}^{\mathrm{d}}_{21} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{\mathrm{d}}_{nm} \\ \mathbf{B}^{\mathrm{d}}_{nm} & \ddots & \ddots & \ddots & & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & & \ddots & \ddots & \ddots & \mathbf{B}^{\mathrm{d}}_{21} \\ \mathbf{B}^{\mathrm{d}}_{21} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{\mathrm{d}}_{nm} & \mathbf{B}^{\mathrm{d}}_{1m} \end{bmatrix}.$$

Or, simply, the vector form of (E.17)–(E.19) is

$$\mathbf{f} = \mathbf{B}^{\mathrm{d}}\mathbf{f}^{\mathrm{d}} + \mathbf{B}^0\mathbf{f}^0 + \mathbf{W}^0\mathbf{w}, \tag{E.21}$$

where

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_m \end{bmatrix}, \quad \mathbf{f}^{\mathrm{d}} = \begin{bmatrix} \bar{\mathbf{f}}^{\mathrm{d}}_1 \\ \mathbf{f}^{\mathrm{d}}_2 \\ \vdots \\ \mathbf{f}^{\mathrm{d}}_m \end{bmatrix}, \quad \mathbf{f}^0 = \begin{bmatrix} \mathbf{f}^0_1 \\ \vdots \\ \mathbf{f}^0_m \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_m \end{bmatrix},$$

$$\mathbf{W}^0 = BlockDiag[\cdots, \mathbf{W}_1, \mathbf{W}_1, \cdots],$$

$$\mathbf{B}^{\mathrm{d}} = \begin{bmatrix} \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_5 & \mathbf{B}_6 & \ddots & \ddots & \vdots \\ \mathbf{0} & \bar{\mathbf{B}}_5 & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \mathbf{B}_6 & \mathbf{B}_4 \\ \mathbf{0} & \cdots & \mathbf{0} & \bar{\mathbf{B}}_5 & \mathbf{B}_8 \end{bmatrix}, \quad \mathbf{B}^0 = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_7 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \mathbf{B}_2 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{B}_7 \end{bmatrix}.$$

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