Open–Loop Control of Class–2 Tensegrity Towers

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ABSTRACT

This paper concerns open-loop control laws for reconfiguration of tensegrity towers. By postulating the control strategy as an equilibrium tracking control, very little control energy is required. Several different reconfiguration scenarios are possible for different string connectivity schemes. This includes unit radius control, twist angle control and truncation parameter control. All these control laws allow a nonuniform distribution of the control parameters among units. By defining a wave–like reference signal and injecting it in the open–loop control law, we demonstrate the concept of self–propelled tensegrity structure that are capable of locomotion.

Keywords: Tensegrity, Equilibrium, Reconfiguration, Open-loop control, Locomotion

1. INTRODUCTION

Tensegrity structures are prestressable truss-like systems that involve string elements as tensile members. Introduced a half century ago, the dominant themes of early publications were characterization of static equilibria and rigidity properties, *e.g.* [1, 2], [3], [4], [5]. These two research areas naturally established themselves as the main focus of the research attention because of the presence of the strings in a tensegrity. Tensegrities are believed to be a promising technology for shape controllable structures, because large shape changes can be accomplished by controlling rest lengths of the strings. Available materials to manufacture strings are lighter and stronger than those for compressive elements, so that utilizing them in a structure can reduce mass and enable lightweight designs.

The strings are elements with a nonlinear character of the force-strain relation. This complicates any form of syntheses of a tensegrity structure, either static or dynamic. This partially explains why the success of dynamic modelling of tensegrity structures, e.g. [6], has not been followed with an equivalent success in control design. Several results on control of tensegrity structures are available, [7–12]. The control algorithm for reconfiguration of tensegrity plates proposed in [13] and generalized for the whole class of modular tensegrity structures in [14] provides a solution for the tracking control problem associated with the tensegrity reconfiguration. Moreover, this result is based on slowly varying nonlinear control concept for tracking equilibrium manifold, and enabled solutions for the whole class of modular tensegrity structures independently of their size and shape. Here we investigate further possibilities of this concept and its application on the class of tensegrity towers in which bar elements touch each other (class–2 tensegrity).

The rest of the paper is outlined as follows. In the Section 2. we characterize equilibrium of tensegrity structures. Section 3. concerns specifics of the equilibrium analysis of modular tensegrity structures with a detailed equilibrium analysis of the module used to built the tensegrity towers. A solution for the reconfiguration control is given in Section 4. followed with examples in Section 5.

2. TENSEGRITY EQUILIBRIUM CONDITIONS AND CONSTITUTIVE EQUATION

Definition 2.1. *The* **nodes** $\nu_k,\ k = 1, \dots, n_n$ *of a tensegrity structure, are the points where bars and strings of the structure connect. A nodal vector* $\mathbf{p}_k \in \mathbb{R}^3$ *represents the position of the node* ν_k*. The sets of all nodes of a tensegrity structure and associated nodal vectors are denoted* $\mathbb{N}$ *and* $\mathbb{P}$ *respectively.*

Definition 2.2. *An* **element** $e_i = \{[\nu_k, \nu_j], z_i\},\ k \neq j,\ i = 1, \dots, n_e$*, of a tensegrity structure is either a bar or a string that connects the two nodes* ν_k *and* ν_j *of the tensegrity. The pair* $[\nu_k, \nu_j]$ *is an ordered pair, and* z_i *identifies the element type. For the tensegrity structure with the element set* $\mathbb{E}$*,* z_i *is defined as follows,*

$$z_i = \begin{cases} 1, & e_i \in \mathbb{E}_s, \\ -1, & e_i \in \mathbb{E}_b, \end{cases} \tag{1}$$

where $\mathbb{E}_s \in \mathbb{E}$ *and* $\mathbb{E}_b \in \mathbb{E}$ *are the subsets of the string and bar elements of the tensegrity structure.*

DEFINITION 2.3. *An* **element vector** $\mathbf{g}_i \in \mathbb{R}^3$ *is a vector along the length of an element* $e_i = \{[\nu_k, \nu_j], z_i\}$*. It emanates from the first node* ν_k *and terminates at the second node* ν_j *of the element, i.e.,*

$$\mathbf{g}_i = \mathbf{p}_j - \mathbf{p}_k.$$

It is obvious that magnitude of an element vector $\mathbf{g}_i$ is equal to its length $\|\mathbf{g}_i\|$, which is denoted by l_i.

DEFINITION 2.4. *The element force vector* $\mathbf{f}_{ji} \in \mathbb{R}^3$ *represents the contribution of the internal force of the element* e_i*, to the balance of the forces at the node* ν_j *and it can be written as,*

$$\mathbf{f}_{ji} = c_{ji}\lambda_i \mathbf{g}_i, \quad f_i = \lambda_i \|\mathbf{g}_i\| = \lambda_i l_i, \tag{2}$$

where **element force coefficient** λ_i *is a scalar.*

Obviously scalars c_{ji} in (2) that are defined as typical elements of the matrix $C(\mathbb{E}) \in \mathbb{R}^{n_n \times n_e}$, have one of the three possible values, $c_{ji} = \pm 1$ or $c_{ji} = 0$.

Let $\mathbb{R}^n_m$ denote the vector space of vectors $\mathbf{x}$ that have the following structure:

$$\mathbf{x} \in \mathbb{R}^n_m \Rightarrow \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdot \\ \mathbf{x}_n \end{bmatrix}, \quad \mathbf{x}_i \in \mathbb{R}^m, \quad \mathbb{R}^m = \mathbb{R}^m_1 \tag{3}$$

Vector of nodal vectors $\mathbf{p} \in \mathbb{R}^{n_n}_3$, vector of element vectors $\mathbf{g}(\mathbb{E}, \mathbb{P}) \in \mathbb{R}^{n_e}_3$, vector of force densities $\boldsymbol{\lambda} \in \mathbb{R}^{n_e}$ and vector $\mathbf{z} \in \mathbb{R}^{n_e}$ are formed by collecting all node vectors $\mathbf{p}_i$, element vectors $\mathbf{g}_i$, force densities λ_i and all individual element type identifiers,

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \cdot \\ \mathbf{p}_{n_n} \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \cdot \\ \mathbf{g}_{n_e} \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \lambda_{n_e} \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ z_{n_e} \end{bmatrix}. \tag{4}$$

Let the member-node incidence matrix of the oriented graph associated with $\mathbb{E}$ be denoted $M(\mathbb{E}) \in \mathbb{R}^{n_e \times n_n}$ and let $\mathbf{M} \in \mathbb{R}^{3n_e \times 3n_n}$ be defined as $\mathbf{M} = M \otimes I_3$. The typical element m_{ij} of the matrix M is $m_{ij} = 1$ or $m_{ij} = -1$ if the element e_i terminates at or emanates from the node ν_j, otherwise $m_{ij} = 0$. Let the n_s string elements in $\mathbb{E}_s$ be numbered first. Then, vector $\mathbf{g}$ and matrix $\mathbf{M}$ can be partitioned as follows,

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_s \\ \mathbf{g}_b \end{bmatrix} = \mathbf{M}\mathbf{p}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{S}^T \\ \mathbf{B}^T \end{bmatrix}, \quad \mathbf{S} \in \mathbb{R}^{3n_n \times 3n_s}.$$

One can show that equilibrium conditions for the prestressed structure with properly loaded strings, and matrix $\mathbf{C} = C \otimes I_3$ can be defined and written as,

$$\mathbf{C}\tilde{\mathbf{g}}\boldsymbol{\lambda} = 0, \quad \|\boldsymbol{\lambda}\| > 0, \quad \underline{\lambda}_i \geq 0, \quad e_i \in \mathbb{E}_s, \tag{5}$$

$$\mathbf{C} = \begin{bmatrix} -\mathbf{S} & \mathbf{B} \end{bmatrix}, \quad \mathbf{g} = \mathbf{M}\mathbf{p} = \begin{bmatrix} \mathbf{S}^T \\ \mathbf{B}^T \end{bmatrix} \mathbf{p}, \tag{6}$$

if the linear operator $\tilde{\cdot}$ acting on the vector $\mathbf{x} \in \mathbb{R}^n_m$ is defined as follows,

$$\tilde{\mathbf{x}} := blockdiag\{\mathbf{x}_1 \ldots, \mathbf{x}_i, \ldots \mathbf{x}_n\} \in \mathbb{R}^{mn \times n}, \quad \mathbf{x}_i \in \mathbb{R}^m.$$

Let the tensegrity structure Γ defined by the triple $\Gamma = \{\mathbb{E}, \mathbb{P}, \boldsymbol{\lambda}\}$ admit element and nodal symmetry $I(\mathbf{x})$ as defined in [15], so that all its elements and nodes can be grouped respectively in n_{ec} and n_{nc} *element* and *node equivalence classes*. Assume that elements in the same equivalency class are constrained to share common force densities. Then, the full vector of force densities $\boldsymbol{\lambda}$ can be expressed as a linear mapping from the reduced set of the independent force density variables $\underline{\boldsymbol{\lambda}} \in R^{n_{ec}}$, by defining matrix $Q \in R^{n_e \times n_{ec}}$. As it is shown in [15], this assumption does not additionally restrict the domain of the feasible geometry variables defining an equilibrium structure. The size of the problem can be reduced further by keeping only the set of independent equations in (5). This can be accomplished by multiplying the equality in (5) from the left with a sparse matrix $\mathbf{D} \in \mathbb{R}^{3n_{nc} \times 3n_n}$. The structure of the matrices Q and $\mathbf{D}$ corresponding to the symmetric problem is given in [15]. If a change of geometry variables is defined so that $\mathbf{p} = \mathbf{p}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots)$, and the shape constrains in the form $\boldsymbol{\varphi}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots) = 0$ included in the problem, the symmetric tensegrity form-finding problem becomes,

$$
\begin{aligned}
&\mathbf{D}\mathbf{C}\tilde{\mathbf{g}}Q\underline{\boldsymbol{\lambda}} = 0, \quad \mathbf{C} = \left[\begin{array}{cc} -\mathbf{S} & \mathbf{B} \end{array}\right], \\
&\underline{\mathbf{p}} = \underline{\mathbf{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, \ldots), \quad \mathbf{p} = \mathcal{R}\underline{\mathbf{p}}, \quad \mathbf{g} = \mathbf{M}\mathbf{p} = \left[\begin{array}{c} \mathbf{S}^T \\ \mathbf{B}^T \end{array}\right]\mathbf{p}, \\
&\boldsymbol{\varphi}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots) = 0, \\
&\|\boldsymbol{\lambda}\| > 0, \quad \underline{\lambda}_i \geq 0, \quad e_i \in \mathbb{E}_s.
\end{aligned}
\tag{7}
$$

A more detailed explanation of tensegrity constitutive equations (5-6) and their form for symmetric structures (7) is given in [15].

3. CLASS–2 TENSEGRITY TOWERS AS COMPOSITION OF TENSEGRITY STRUCTURES

As shown in [14] the solution of the tensegrity constitutive equations for modular tensegrity structures can be greatly simplified by exploiting their particular structured form that arises from the specific way that the elements of a modular structure are connected and from the associated nodal symmetry. Since the class–2 tensegrity towers belong to the category of modular tensegrity structures, the composition rules defined in [14] will be implemented to simplify their equilibrium analysis. The main result associated with modular tensegrity structures is given in the next section in Theorem 3.4. For the proof of the Theorem consult [14].

Definition 3.1. **Node** ν_r *of the structure* Γ *is said to be* **attached to element** $e_i = \{[\nu_j, \nu_k], z_i\}$ *if element* e_i *is replaced in the structure definition with elements* $e_q = \{[\nu_j, \nu_r], z_i\}$ *and* $e_s = \{[\nu_r, \nu_k], z_i\}$. *This will formally be written as* $[e_q, e_s] = \nu_r @ e_i$.

Definition 3.2. **Node** ν_r *of structure* Γ *is said to be* **attached to node** ν_j *if node* ν_r *is replaced by node* ν_j *in the definition of all elements incident with node* ν_r. *This will formally be written as* $\nu_r \leftarrow \nu_j$.

Node attachment operation $\nu_r \leftarrow \nu_j$ should not be confused with the node placement $\mathbf{p}_r = \mathbf{p}_j$. While the former operation removes node ν_r from the set $\mathbb{N}$, and consequently $\mathbf{p}_r$ from the set $\mathbb{P}$, the later operation only place node ν_r at the position of the node ν_j so that both these overlapping nodes continue to exist.

Definition 3.3. **Superposition of** *two overlapping* **elements** $e_i = \{[\nu_j, \nu_k], z_i\}$ *and* $e_q = \{[\nu_j, \nu_k], z_i\}$ *or* $e_q = \{[\nu_k, \nu_j], z_i\}$ *of structure* Γ *is the operation in which element* e_q *is deleted from the set* $\mathbb{E}$. *This will formally be written as* $e_q \leftarrow e_i$.

The following theorem concerns the composition of equilibrium structures and the main property associated with the result of this composition.

Theorem 3.4. *Let the tensegrity structure* $\Gamma = \{\mathbb{E}, \mathbb{P}, \boldsymbol{\lambda}\}$ *be defined from the two equilibrium tensegrity structures* $\Gamma_1 = \{\mathbb{E}^1, \mathbb{P}^1, \boldsymbol{\lambda}^1\}$ *and* $\Gamma_2 = \{\mathbb{E}^2, \mathbb{P}^2, \boldsymbol{\lambda}^2\}$ *by attaching some nodes of structure* Γ_1 *to elements or nodes of structure* Γ_2, *and by attaching some nodes of structure* Γ_2 *to elements or nodes of structure* Γ_1, *so that all of the following conditions are satisfied:*

(i) If node ν_r *is attached to node* ν_j *then the nodal vectors satisfy* $\mathbf{p}_r = \mathbf{p}_j$.

(ii) If node ν_r *is attached to element* $e_i = \{[\nu_j, \nu_k], z_i\}$, *so that* $[e_q, e_s] = \nu_r @ e_i$, *then the nodal vector* $\mathbf{p}_r$ *satisfies*

$$\mathbf{p}_r = \mathbf{p}_j + a(\mathbf{p}_k - \mathbf{p}_j), \quad \textit{for} \quad 0 < a < 1 \tag{8}$$

and force densities λ_p *and* λ_q *of elements* e_p *and* e_q *satisfy,*

$$\lambda_h = \frac{\lambda_i \|\mathbf{g}_i\|_2}{\|\mathbf{g}_h\|_2} = \frac{f_i}{l_h}, \quad h = q, s. \tag{9}$$

(iv) If overlapping elements e_i *with force density* λ_i *and* e_j *with force density* λ_j *are generated, and replaced by their superposition* $e_j \leftarrow e_i$*, the force density of the remaining element is* $\lambda_j + \lambda_i$*, i.e.,*

$$\lambda_i \leftarrow \lambda_j + \lambda_i.$$

Then, structure Γ *is an equilibrium structure and it is said to be the* **composition** *of the two* **component structures** Γ_1 *and* Γ_2*.*

An n_m stage class–2 tensegrity tower can be regarded as a composition of n_m component structures. The case where one-stage shell–class structures defined in [6] serve as component structures will be analyzed here. The only constraint for n_m of these equilibrium tensegrity structure to be compatible for composition by stacking them up to build a tower is that the radii of the top and bottom of the two adjacent structures are the same, so that the top nodes of one structure can be attached to the bottom nodes of the adjacent structure. See Figure 1. Then, for the given collection of n_m compatible equilibrium tensegrity modules, Theorem 3.4 guaranties that the series of $n_m - 1$ compositions of these n_m component structures, yields an class–2 tensegrity tower in equilibrium. In the view of Theorem 3.4 we start equilibrium analysis of the tower by analyzing the equilibrium of its components.

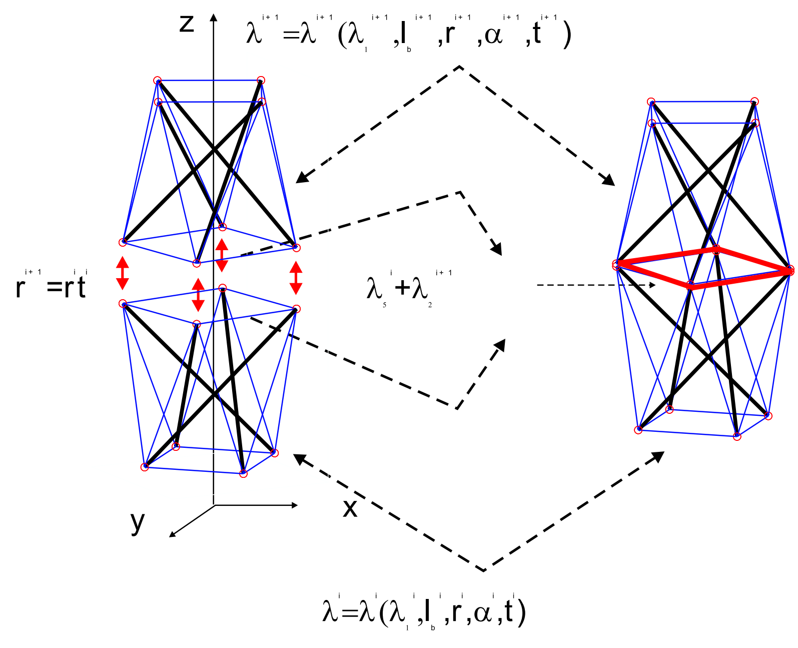

Figure 1. Composition of tower modules

3.1. Geometry and equilibrium of one–stage shell–class tensegrity module

For the n-bar shell–class tensegrity in the configuration that admits n-fold rotational symmetry C_n about z-axis as the nodal symmetry, nodal positions can be expressed in terms of geometry parameters l_b, r, α, t that are defined in [15] and depicted in Figure 2., and the constant matrix $\mathcal{R}$ that reflects symmetry of the structure. As suggested in [15], nodal vector of the structure, $\mathbf{p} \in \mathbb{R}_3^{2n}$, can be related to these parameters in the following compact form,

$$\mathbf{p} = \mathcal{R}\underline{\mathbf{p}}, \quad \underline{\mathbf{p}} = \bar{\mathbf{p}}(n, l_b, r, \alpha, t) = \begin{bmatrix} \mathbf{p}_1(n, l_b, r, \boldsymbol{\alpha}, t) \\ \mathbf{p}_{2n}(n, l_b, r, \alpha, t) \end{bmatrix}. \tag{10}$$

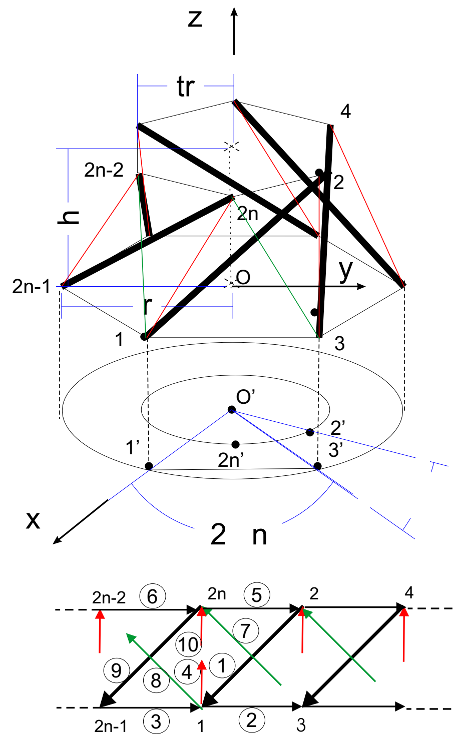

Figure 2. One–stage shell-class module geometry and connectivity

Define matrix R of the rotation about z axis,

$$R(x) = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

Then, nodal parametrization (10) takes the following form,

$$\begin{array}{lll} \mathbf{p}_1 = \begin{bmatrix} r & 0 & 0 \end{bmatrix}^T, & \mathbf{p}_3 = R(\frac{2\pi}{n})\mathbf{p}_1, & \mathbf{p}_{2n-1} = R^{-1}(\frac{2\pi}{n})\mathbf{p}_1, \\ \mathbf{p}_{2n} = R(\alpha) \begin{bmatrix} tr & 0 & h \end{bmatrix}^T, & \mathbf{p}_2 = R(\frac{2\pi}{n})\mathbf{p}_{2n}, & \mathbf{p}_{2n-2} = R^{-1}(\frac{2\pi}{n})\mathbf{p}_{2n}, \end{array} \tag{12}$$

where, the height h of the structure is computed as,

$$h = \sqrt{l_b^2 - r^2 - r^2t^2 + 2r^2t\cos(\frac{2\pi}{n} + \alpha)}. \tag{13}$$

The requirement that the height h of the structure be a positive real number yields the following definition of the feasible set for the geometry variables,

$$0 < r < \frac{l_b}{1 + t^2 - 2t\cos(\frac{2\pi}{n} + \alpha)}, \quad 0 < t, \quad 0 < l_b. \tag{14}$$

3.1.1. Equilibrium of a one–stage shell–class tensegrity module

Note that (12) parameterizes all configurations that admit the given nodal symmetry, whether or not (7) is satisfied. One must solve (7) to find the subset of these symmetric configurations that yield equilibrium tensegrity structures. Due to the element symmetry, corresponding constitutive equations (7) reduce to the balance of the forces equations at only two representative nodes ν_1 and ν_{2n} if elements in the same equivalency class share the same force densities. Element vectors of the elements appearing in the force balance equations for nodes ν_1 and ν_{2n} are computed as,

$$\mathbf{g}_1 = \mathbf{p}_1 - \mathbf{p}_2, \quad \mathbf{g}_2 = \mathbf{p}_3 - \mathbf{p}_1, \quad \mathbf{g}_3 = \mathbf{p}_1 - \mathbf{p}_{2n-1}, \quad \mathbf{g}_4 = R(-m\frac{2\pi}{n})\mathbf{p}_{2n} - \mathbf{p}_1,$$
$$\mathbf{g}_5 = \mathbf{p}_2 - \mathbf{p}_{2n}, \quad \mathbf{g}_6 = \mathbf{p}_{2n} - \mathbf{p}_{2n-2}, \quad \mathbf{g}_7 = \mathbf{p}_{2n} - R(q\frac{2\pi}{n})\mathbf{p}_3,$$
$$\mathbf{g}_8 = R(-q\frac{2\pi}{n})\mathbf{p}_{2n-2} - \mathbf{p}_1, \quad \mathbf{g}_9 = \mathbf{p}_{2n-1} - \mathbf{p}_{2n}, \quad \mathbf{g}_{10} = \mathbf{p}_{2n} - R(m\frac{2\pi}{n})\mathbf{p}_1.$$

The set of equations defining equilibrium configuration of the module becomes,

$$\begin{aligned} \mathbf{g}_1\lambda_1 + \mathbf{g}_2\lambda_2 - \mathbf{g}_3\lambda_3 + \mathbf{g}_4\lambda_4 + \mathbf{g}_8\lambda_8 = 0, \\ -\mathbf{g}_4\lambda_4 + \mathbf{g}_5\lambda_5 - \mathbf{g}_6\lambda_6 - \mathbf{g}_7\lambda_7 - \mathbf{g}_9\lambda_9 - \mathbf{g}_{10}\lambda_{10} = 0, \\ \lambda_i \geq 0. \end{aligned}$$

Element symmetry of the structure allows further reduction of the number of force density variables,

$$\lambda_9 = \lambda_1, \quad \lambda_3 = \lambda_2, \quad \lambda_6 = \lambda_5, \quad \lambda_8 = \lambda_7, \quad \lambda_{10} = \lambda_4, \tag{15}$$

so that the equilibrium conditions reduce to,

$$\begin{aligned} \mathbf{g}_1\lambda_1 + \mathbf{g}_2\lambda_2 - \mathbf{g}_3\lambda_2 + \mathbf{g}_4\lambda_4 + \mathbf{g}_8\lambda_7 = 0, \\ -\mathbf{g}_4\lambda_4 + \mathbf{g}_5\lambda_4 - \mathbf{g}_6\lambda_5 - \mathbf{g}_7\lambda_7 - \mathbf{g}_9\lambda_1 - \mathbf{g}_{10}\lambda_4 = 0, \\ \lambda_i \geq 0. \end{aligned}$$

This can be written in the more compact form,

$$\mathbf{DC}\tilde{\mathbf{g}}Q\underline{\lambda} = 0, \tag{16}$$
$$\underline{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 & \lambda_7 \end{bmatrix}^T, \tag{17}$$
$$\lambda_i \geq 0, \tag{18}$$

if one defines matrices $\mathbf{D}, \mathbf{C}, \tilde{\mathbf{g}}, Q$ corresponding to the problem and cast the problem in the standard form (7). Then, $\underline{\lambda}$ that solves (16) is computed as a vector in the null space of $\mathbf{DC}\tilde{\mathbf{g}}Q$. Since the basis $\underline{\Lambda}$ of the null space of $DC\tilde{\mathbf{g}}Q$ is one dimensional, solution $\underline{\lambda}$ is unique up to the scaling with a positive constant, and is given by:

$$\lambda_1 \geq 0, \tag{19}$$
$$\lambda_2 = \lambda_1 \frac{t\csc^2(\frac{\pi}{n})\sin(\frac{(m+1)\pi}{n})\sin(\frac{(q+2)\pi}{n})}{2\cos(\frac{(m+q+1)\pi}{n} - \alpha)}, \tag{20}$$
$$\lambda_5 = \lambda_1 \frac{\csc^2(\frac{\pi}{n})\sin(\frac{(m+1)\pi}{n})\sin(\frac{(q+2)\pi}{n})}{2t\cos(\frac{(m+q+1)\pi}{n} - \alpha)}, \tag{21}$$
$$m - q \neq 1, \tag{22}$$
$$\lambda_4 = \lambda_1 \frac{\cos(\frac{q\pi}{n} - \alpha)\sin(\frac{(q+2)\pi}{n})}{\cos(\frac{(m+q+1)\pi}{n} - \alpha)\sin(\frac{(q-m+1)\pi}{n})}, \tag{23}$$
$$\lambda_7 = -\lambda_1 \frac{\cos(\frac{(1-m)\pi}{n} + \alpha)\sin(\frac{(1+m)\pi}{n})}{\sin(\frac{(q-m+1)\pi}{n})\cos(\frac{(m+q+1)\pi}{n} - \alpha)} \tag{24}$$
$$m - q = 1, \tag{25}$$
$$\lambda_4 = \lambda_1, \tag{26}$$
$$\lambda_7 = 0, \tag{27}$$

which can be written in the corresponding more compact form,

$$\underline{\lambda} = \underline{\Lambda}\lambda_1, \quad \underline{\Lambda} = \underline{\Lambda}(n, \alpha, t, m, q). \tag{28}$$

The connectivity for which $m - q = 1$ corresponds to the case where the strings, e_4 and e_{10}, overlap, so that they can be substituted with a single string.

For the given number of bars n, and the string connectivity parameters m and q, permissible α can be computed from the condition that,

$$\lambda_i \geq 0. \tag{29}$$

Solving,

$$\lambda_7 = 0, \tag{30}$$

for α, gives the values of α where λ_7 changes the sign,

$$\alpha = \frac{\pi}{2} + (m-1)\frac{\pi}{n}. \tag{31}$$

Solving,

$$\lambda_4 = 0, \tag{32}$$

for α, the values of α where λ_4 switches the sign are obtained,

$$\alpha = \frac{\pi}{2} + \frac{q\pi}{n}. \tag{33}$$

Finally, admissible α is defined as,

$$\underline{\alpha} = \min\{\tfrac{\pi}{2} + \tfrac{q\pi}{n}, \tfrac{\pi}{2} + (m-1)\tfrac{\pi}{n}\}, \tag{34}$$
$$\overline{\alpha} = \max\{\tfrac{\pi}{2} + \tfrac{q\pi}{n}, \tfrac{\pi}{2} + (m-1)\tfrac{\pi}{n}\}, \tag{35}$$
$$\alpha \in [\underline{\alpha}, \overline{\alpha}]\,. \tag{36}$$

The results obtained from the equilibrium analysis of the module will be summarized in the following theorem.

THEOREM 3.5. *The equilibrium force densities given by (19-27), and the set of admissible geometry parameters defined by (14) and (34-36), represent the complete parameterization of all equilibrium configurations of the symmetric one-stage tensegrity module with symmetric element forces.*

Note that, if the string e_4, and the string, e_7, are both present in the structure, equilibrium geometry is not unique. That is, (34-36) defines the range of α that yields an equilibrium tensegrity geometry. In the case where either of these two different groups of strings is not present in the structure, the equilibrium geometry becomes unique and it is defined by the corresponding limits of α. It is important to point out that this analysis concerns only the prestress modes of the structure where elements in the same equivalency class have common force coefficients. It can be shown that for configurations where twist angle α does not lie on the bounds of the feasible set (34-36), there exist more than one prestress mode of the structure. For example, it can be shown that the three–bar structure has three prestress modes, and the four–bar structure has two prestress modes that are all characterized with asymmetric distribution of force density variables.

4. SLOWLY VARYING NONLINEAR SYSTEMS AND OPEN LOOP–CONTROL

4.1. Parametrization of tensegrity structure nonlinear dynamic model

Several different nonlinear models of tensegrity structure have been devised, e.g. [6]. What is common for all of them is that rest lengths of elastic elements are parameters of the model. The open loop control strategy that is postulated to control reconfiguration of equilibrium tensegrity structures is a result of a well known result from nonlinear control theory.

PROPOSITION 1. *Let a parameterized model of a nonlinear system be given in the state space form*

$$\frac{d\mathbf{x}}{d\tau} = f(\mathbf{x}, \delta), \quad \mathbf{x} \in R^n, \tag{37}$$

where δ represents the set of parameters defining the model of the system. Let $\mathbf{g}(\delta)$ satisfying

$$0 = f(\mathbf{g}(\delta), \delta), \tag{38}$$

be an exponentially stable equilibrium manifold of the system. If a sufficiently slowly varying function $\delta = \delta(\tau)$ is defined, then trajectory $\mathbf{x}(\tau)$ of the system $\dot{\mathbf{x}} = f(\mathbf{x}, \delta(\tau))$ tracks the equilibrium manifold $\mathbf{g}(\delta(\tau))$.

For more detailed analysis related to this topic consult [16].

4.2. Equilibrium rest lengths parametrization

For a given equilibrium tensegrity structure $\Gamma = \{\mathbb{E}, \mathbb{P}, \Lambda\}$, depending on the material model used to build its elastic elements, rest lengths l_{0_i} of the elastic elements e_i are computed as,

$$l_{0_i} = l_{0_i}(\lambda_i, l_i, z_i, y_i, a_i, \ldots), \tag{39}$$

where the meaning of the parameters y_i, $a_i, \ldots$ appearing in (39) depends on the material strain–stress relationship. In particular, for the linear elastic material model of elastic elements with the cross section area a_i and Young's modulus y_i, the force–strain relationship given by the Hooke's law,

$$f_i = \lambda_i l_i = z_i \frac{y_i a_i}{l_{0_i}}(l_i - l_{0_i}),$$

results that (39) has the following form,

$$l_{0_i} = \frac{z_i l_i y_i a_i}{l_i \lambda_i + z_i y_i a_i}, \quad \text{and, if} \quad y_i \to \infty, \Rightarrow l_{0_i} \to l_i. \tag{40}$$

Note that the lengths l_i of all elements of the structure Γ depend only on the structure geometry $\mathbb{P}$ and connectivity $\mathbb{E}$. In the equilibrium that is defined by the set Ω of feasible geometry and force parameters ,

$$\begin{gathered} \mathbf{p} = \mathbf{p}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots), \\ \boldsymbol{\lambda} = \Lambda \underline{\underline{\boldsymbol{\lambda}}}, \quad \Lambda = \Lambda(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots), \\ \underline{\underline{\boldsymbol{\lambda}}}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots \in \Omega, \end{gathered} \tag{41}$$

the corresponding equilibrium rest lengths are defined as,

$$\mathbf{l}_0 = \mathbf{l}_0(\underline{\underline{\boldsymbol{\lambda}}}, \mathbb{E}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots, \mathbf{z}, \mathbf{y}, \mathbf{a}). \tag{42}$$

Once the material and cross sections have been assigned to all elements of the structure Γ, (42) reduces to,

$$\mathbf{l}_0 = \mathbf{l}_0(\underline{\underline{\boldsymbol{\lambda}}}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots). \tag{43}$$

Recall that the rest lengths l_0 of the elastic members serve as the parameters δ of the nonlinear dynamic model of the system (37), so that using (43), (37) yields the parameterized structure model,

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{l}_0(\underline{\underline{\boldsymbol{\lambda}}}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots)) = f(\mathbf{x}, \underline{\underline{\boldsymbol{\lambda}}}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots). \tag{44}$$

Proposition 1 suggests that the system (44) tracks the equilibrium configuration,

$$\mathbf{p}(\tau) = \mathbf{p}(\boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\gamma}(\tau), \ldots)$$

if one defines the sufficiently slowly varying functions,

$$\underline{\underline{\boldsymbol{\lambda}}}(\tau), \boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\gamma}(\tau), \ldots \in \Omega,$$

that define the desired configuration $\mathbf{p}(\tau)$, and the force densities $\boldsymbol{\lambda}(\tau)$ at every time instance τ. For the tensegrity structures whose equilibrium has been analyzed in this text, (43) has the following general form,

$$\mathbf{l}_0 = \mathbf{l}_0(\underline{\underline{\boldsymbol{\lambda}}}, n, \mathbf{l}_b, \mathbf{r}, \boldsymbol{\alpha}, \mathbf{t}), \quad r_{i+1} = t_i r_i \tag{45}$$

Assume that all bar elements are rigid with fixed lengths $\mathbf{l}_b$ as it is postulated in the model [6]. Then the only parameters of the desired geometry of the system that can be time dependant are, $\underline{\underline{\boldsymbol{\lambda}}}(\tau), \mathbf{r}(\tau), \boldsymbol{\alpha}(\tau)$ and $\mathbf{t}(\tau)$, so that the string rest length open-loop control becomes,

$$l_{0_j}(\tau) = l_{0_j}(\underline{\underline{\boldsymbol{\lambda}}}(\tau), \mathbf{r}(\tau), \boldsymbol{\alpha}(\tau), \mathbf{t}(\tau)), \quad r_{i+1} = t_i r_i, \quad e_j \in \mathbb{E}_s. \tag{46}$$

5. EXAMPLES

5.1. Tower deployment with twist angle and truncation control

The control objective in this example is to deploy the class–2 tensegrity tower composed of four one–stage four–bar modules with bar lengths $l_b = 6$. To define the deployment control law (46), a monotonically decreasing function $r(\tau)$ on the interval $(0, T)$ is defined. The desired geometry parameter functions in (46) are defined as follows,

$$
\begin{aligned}
&r(\tau) = 3.2 - 0.05\tau, && \alpha(\tau) = \frac{\pi}{4}, && t = 1, \quad 0 < \tau \leq 10,\\
&r(\tau) = 3.2 - 0.05\tau, && \alpha(\tau) = \frac{\pi}{4} + 0.03(10 - \tau)\frac{\pi}{4}, && t = 1, \quad 10 < \tau \leq 25,\\
&r(\tau) = 1.95, && \alpha(\tau) = 1.45\frac{\pi}{4}, && t = 1, \quad 25 < \tau \leq 30.
\end{aligned}
$$

Unlike the geometry parameters that are common for all the modules, force densities $\lambda_1^i(\tau)$ are not. They are set to be constant throughout the deployment, so that $\underline{\underline{\boldsymbol{\lambda}}}(\tau) = \left[\begin{array}{cccc} \lambda_1^1(\tau) & \lambda_1^2(\tau) & \lambda_3^2(\tau) & \lambda_4^2(\tau) \end{array} \right]^T$, and

$$\underline{\underline{\boldsymbol{\lambda}}}(\tau) = \left[\begin{array}{cccc} 1 & 2 & 2 & 1 \end{array} \right], \quad 0 < \tau \leq 30.$$

For the closed form of the string rest length control consult [17]. Simulation results are depicted in Figure 3. The ratio between the initial and final height of the tower is approximately 4.5.

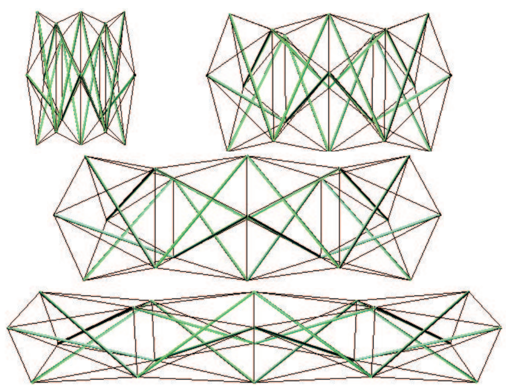

Figure 3. Deployment of tensegrity tower

5.2. Self–propelled tensegrity structures – a tensegrity worm

If the tensegrity structure changes its shape in a periodic wave-like mode as a result of an internal shape control, an interaction of the structure with its surrounding can induce a locomotion. The following is an example of a self-propelled tensegrity tower, where the envelope of the shape of the structure $r(\tau, x)$ represents the longitudinal wave that propagates with the velocity v_w along the length of the structure.

$$r(\tau, x) = r_0 + \sum_{k=1}^{n_k} A_k \cos(2\pi \frac{x - x_w(\tau)}{w_\lambda} k), \tag{47}$$

$$x_w(\tau) = v_w \tau_{rel}, \quad \tau_{rel} = mod(\tau, T), \quad v_w = \frac{w_\lambda}{T}. \tag{48}$$

The shape of the wave and its frequency content can be defined by selecting the magnitudes A_k of its n_k harmonics and the wave length w_λ. Assume that all modules of the tower have the same height h so that the location x_i of the nodes of the modules and the wave length w_λ are approximated as,

$$x_i = (i - 1)h, \quad w_\lambda = mh, \quad m \in \mathbb{Z}. \tag{49}$$

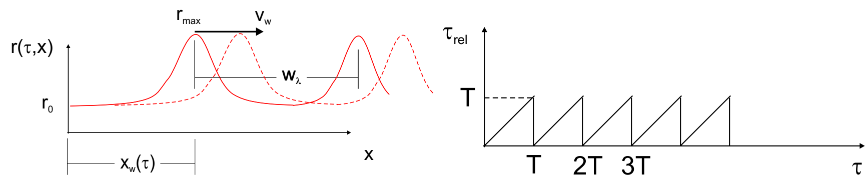

Figure 4. Internal transversal wave function

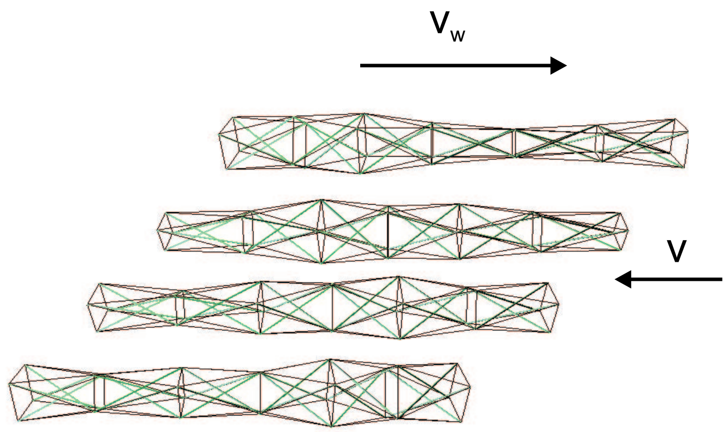

Figure 5. Self-propelled tensegrity actuated by applying transversal wave shape control

Then, the continuous function $r_i(\tau) \in C^1$ of the radii of the modules in the tensegrity tower whose shape envelope represents a transversal wave, and the corresponding feasible truncation ratio $t_i(\tau)$ of the modules can be written as,

$$r_i(\tau) = r_0 + \sum_{k=1}^{n_k} A_k \cos(2\pi \frac{(i-1) - \frac{m}{T}\tau_{rel}}{m} k),$$
$$t_i(\tau) = \frac{r_{i+1}(\tau)}{r_i(\tau)}.$$

Note that the requirement that $r_i(\tau)$ is a continuous function guarantees that the string rest length control function is also continuous and represents a physically realizable control, that does not require infinite power to achieve.

The locomotion of the tower, generated by the interaction of the tower with its environment, whether it is a fluid drag or friction from a contact surface, has the opposite direction from the direction of the wave propagation. The simulation results that are shown in Figure 5. represent the application of this shape control strategy on a class–2 tensegrity tower that is made by composition of six one-stage tensegrity modules.

6. CONCLUSIONS

This paper demonstrates application of an open-loop control strategy for reconfiguration of class–2 tensegrity towers. By exploiting modularity of the structure that enables a simplified equilibrium analysis, we provided a solution for its equilibrium geometry independently of the size of the structure. Based on this result, a string rest length open–loop control law is defined. This control drives the structure through a sequence of configurations that remain in close proximity of the equilibrium manifold. Non–uniqueness of the deployment trajectories opens the possibility for further optimization in the study of optimal structural and control performance. The results presented here define the optimization domain for the structure.

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