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Equilibrium Conditions of a Tensegrity Structure

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ABSTRACT

The class I tensegrity structures are reduced to linear algebra problems, after first characterizing the problem in a vector space where direction cosines are not needed. That is, we describe the components of all member vectors as opposed to the usual practice of characterizing the statics problem in terms of the magnitude of tension vectors. While our approach enlarges(by a factor of 3) the vector space required to describe the problem, the advantage is that enlarging the vector space makes the mathematical structure of the problem amenable to linear algebra treatment.

Kew Words tensegrity, statics, equilibrium condition, linear algebra

1 INTRODUCTION

Tensegrity structures pose a wonderful blend of geometry and mechanics. In addition, they have engineering appeal in problems requiring large changes is shape of the structure. Most existing *smart structure* methods are limited to small displacements. Since class I tensegrity structures [8] have no bar-to-bar connections, the control of tendons allows very large shape changes.

This paper characterizes the static equilibria of class I tensegrity structures. Furthermore, we use vectors to describe each element(bars and tendons), eliminating the need to use direction cosines and the subsequent transcendental functions that follow their use.

By enlarging the vector space in which we characterize the problem, the mathematical structure of the equations admit treatment by linear algebra methods, for the most part.

Our results characterize the equilibria conditions of class I tensegrity structures in terms of a very small number of variables since the necessary and sufficient conditions of the linear algebra treatment has eliminated several of the original variables.

Notation

We let $\mathbf{I}_n$ define the $n \times n$ identity matrix, and $\mathbf{0}$ define an $n \times m$ matrix of zeros. (The dimensions of $\mathbf{0}$ will be clear from the context.) We also let $\rho(\mathbf{A})$ define the rank of the matrix $\mathbf{A}$.

2 NETWORK REPRESENTATION OF STRUCTURES

In this paper, we choose to represent a tensegrity structure as an *oriented graph* in real three dimensional space $\mathcal{R}^3$ defined in terms of n_p nodes and $n_s + n_b$ directed branches which are all represented as vectors in $\mathcal{R}^3$. A *loop* is any closed path in the graph. As we shall see, the advantage of this approach is that the both the magnitude and the direction cosines of the forces are contained in vectors which can be solved using linear algebra. Thus linear algebra plays a larger role in this approach compared to the usual approach in mechanics and finite element methods using direction cosines.

In this oriented graph, the *nodes* consist of the ends of the bars as represented by the n_p nodes (or vectors) $\{\mathbf{p}_k\}$. Hence if there are n_b bars, then there are $n_p = 2n_b$ nodes. We choose to identify two different types of directed branches; the n_s string branches (or vectors) $\{\mathbf{s}_n\}$ and the n_b bar branches (or vectors) $\{\mathbf{b}_m\}$.

Geometric Connectivity

Each directed branch can undergo a displacement in reaching its equilibrium state. String vectors can change both their length and orientation while bar vectors can only change their orientation. Node vectors can change both their length and orientation but subject to a *Law of Geometric Connectivity* which we state as follows:

$$\textit{The vector sum of all branch vectors in any loop is zero.} \tag{1.1}$$

These equations are in the form of a set of linear algebraic equations in the branch vectors.

Force Equilibrium

In our study of tensegrity structures, we are concerned with structures in which bars sustain compressive forces. We therefore choose to distinguish between the string (or tensile) forces $\{\mathbf{t}_n\}$ and the bar (or compressive) forces $\{\mathbf{f}_m\}$ which are defined in terms of the string and bar vectors respectively as follows.

Definition: Given the tensile force $\mathbf{t}_n$ in the string characterized by the string vector $\mathbf{s}_n$ and the compressive force $\mathbf{f}_n$ in the bar characterized by the bar vector $\mathbf{b}_n$, the *tensile force coefficient* $\gamma_m \geq 0$ and the *compressive force coefficient* $\lambda_n \geq 0$ are defined by

$$\mathbf{t}_n = \gamma_n \mathbf{s}_n \ ; \quad \mathbf{f}_m = \lambda_m \mathbf{b}_m \tag{1.2}$$

The *force* of the tensegrity structure is defined by the external force vector $\mathbf{w} \in \mathcal{R}^{3n_p}$, the tensegrity compression vector $\mathbf{f} \in \mathcal{R}^{3n_b}$, and the tensegrity tension vector $\mathbf{t} \in \mathcal{R}^{3n_s}$ where

$$\mathbf{w}^T = [\mathbf{w}_1^T, \mathbf{w}_2^T, \ \dots\ , \mathbf{w}_{n_p}^T] \ ; \ \mathbf{f}^T = [\mathbf{f}_1^T, \mathbf{f}_2^T, \ \dots\ , \mathbf{f}_{n_b}^T] \ ; \ \mathbf{t}^T = [\mathbf{t}_1^T, \mathbf{t}_2^T, \ \dots\ , \mathbf{t}_{n_s}^T] \tag{1.3}$$

Force Convention:

Suppose each node $\mathbf{p}_k$ is subjected to compressive vector forces $\{\mathbf{f}_{mk}\}$, tensile vector forces $\{\mathbf{t}_{nk}\}$ and external force $\mathbf{w}_k$. Then the *Law for Static Equilibrium* may be stated as follows:

$$\sum_n \mathbf{t}_{nk} - \sum_m \mathbf{f}_{mk} - \mathbf{w}_k = \mathbf{0} \tag{1.4}$$

where a positive sign is assigned to a (tensile, compressive, external) force *leaving* a node, and a negative sign is assigned to a (tensile, compressive, external) force *entering* a node. The negative sign in (1.4) is a consequence of the fact that we choose to define nonnegative compressive force coefficients λ_n.

Consider a class 1 tensegrity structure consisting of n_p nodes, n_b bars and n_s strings. Suppose the positions of the positions of the nodes are described by the n_p vectors $\{\mathbf{p}_1, \mathbf{p}_2, \ \dots\ , \mathbf{p}_{n_p}\}$, the positions of the bars are described by the n_b vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ \dots\ , \mathbf{b}_{n_b}\}$, and the positions of the strings are described by the n_s vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ \dots\ , \mathbf{s}_{n_s}\}$.

Definition: The *geometry* of the tensegrity structure is defined by the tensegrity node vector $\mathbf{p} \in \mathcal{R}^{3n_p}$, the tensegrity bar vector $\mathbf{b} \in \mathcal{R}^{3n_b}$, and the tensegrity string vector $\mathbf{s} \in \mathcal{R}^{3n_s}$ where

$$\mathbf{p}^T = [\mathbf{p}_1^T, \mathbf{p}_2^T, \ \dots\ , \mathbf{p}_{n_p}^T] \ ; \ \mathbf{b}^T = [\mathbf{b}_1^T, \mathbf{b}_2^T, \ \dots\ , \mathbf{b}_{n_b}^T] \ ; \ \mathbf{s}^T = [\mathbf{s}_1^T, \mathbf{s}_2^T, \ \dots\ , \mathbf{s}_{n_s}^T] \tag{1.5}$$

From network analysis and the law for static equilibrium, we have the following result.

Lemma 2.1 *There exists an $3n_p \times 3n_s$ matrix $\mathbf{S}$ and an $3n_p \times 3n_b$ matrix $\mathbf{B}$ such that*

$$\mathbf{A} \begin{bmatrix} \mathbf{t} \\ -\mathbf{f} \end{bmatrix} = \mathbf{w} \ ; \ \mathbf{A} \stackrel{\Delta}{=} [\mathbf{S}, \mathbf{B}] \tag{1.6}$$

or equivalently

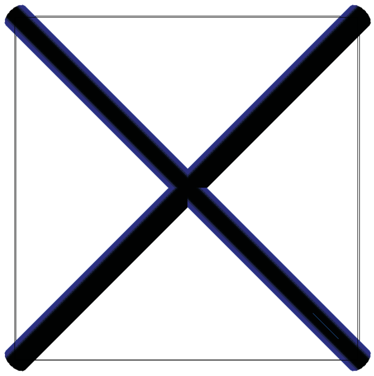

Figure 1 2-Bar 4-String Class 1 Tensegrity

$$\mathbf{St} = \mathbf{Bf} + \mathbf{w}. \tag{1.7}$$

In particular, if we consecutively number the n_s+n_b *branches* $\{\mathbf{s}_1, \mathbf{s}_2, \ \dots, \mathbf{s}_{n_s}, \mathbf{b}_1, \mathbf{b}_2, \ \dots, \mathbf{b}_{n_b}\}$ *as* $\{1, 2, \ \dots., n_s, n_s+1, \ \dots, n_s+n_b\}$*, then the* $3n_p \times (3n_s + 3n_b)$ *matrix* $\mathbf{A} = [\mathbf{A}_{ij}]$ *is defined by*

$$\mathbf{A}_{ij} = \begin{cases} \mathbf{I}_3 & \text{if tension (compression) } j \textit{ leaves } (\textit{enters}) \text{ node } i \\ -\mathbf{I}_3 & \text{if tension (compression) } j \textit{ enters } (\textit{leaves}) \text{ node } i \\ \mathbf{0} & \text{if tension (compression) } j \textit{ is not incident with } \text{node } i \end{cases} \tag{1.8}$$

Also: (i) each column of $\mathbf{A}$ *has exactly one block* $\mathbf{I}_3$ *and one block* $-\mathbf{I}_3$ *with all other column blocks* $\mathbf{0}$*, and (ii) for any row* i *there exists a column* j *such that* $\mathbf{A}_{ij} = \pm\mathbf{I}_3$*.*

In network analysis, the matrix $\mathbf{A}$ is known as the *incidence matrix.* This matrix is not the *reduced* incidence matrix since we have included the datum node which means that one block row of equations in (1.6) are dependent on the other rows. This fact does not cause any difficulties in subsequent developments. On the contrary, some symmetry is preserved in the algebraic equations. Network analysis also gives us the following result.

Lemma 2.2 *The string vector* $\mathbf{t}$ *and the bar vector* $\mathbf{b}$ *are related to the node vector* $\mathbf{p}$ *by the equation*

$$\mathbf{A}^T\mathbf{p} = \begin{bmatrix} \mathbf{s} \\ \mathbf{b} \end{bmatrix} \quad \text{or} \quad \begin{array}{c} \mathbf{S}^T\mathbf{p} = \mathbf{s} \\ \mathbf{B}^T\mathbf{p} = \mathbf{b} \end{array} \tag{1.9}$$

The proof is constructive. Firstly, from the network, it follows that components of the string vector $\mathbf{s}$ and the bar vector $\mathbf{b}$ can be written as a linear combination of components of the node vector $\mathbf{p}$. Also if branch k is a bar which leaves node i and enters node j, then $\mathbf{b}_k = \mathbf{p}_j - \mathbf{p}_i$, whereas if branch k is a string which leaves node i

and enters node j, then $\mathbf{s}_k = \mathbf{p}_j - \mathbf{p}_i$. Hence we have $\mathbf{Cp} = [\mathbf{s}^T, \mathbf{b}^T]$ where the matrix $\mathbf{C}$ consists only of block matrices of the form $\{\mathbf{0}, \pm\mathbf{I}_3\}$. A closer examination then reveals that $\mathbf{C} = \mathbf{A}^T$.

Example 2.3 *Consider the 2-bar 4-string class 1 tensegrity structure illustrated in Fig. 1 with tensile force vectors* $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\}$ *and compressive force vectors* $\{\mathbf{f}_1, \mathbf{f}_2\}$*. The Geometric Connectivity conditions are:*

$$\begin{aligned} -\mathbf{p}_1 + \mathbf{p}_3 &= \mathbf{s}_1 \ ; \ \ \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{s}_2 \\ -\mathbf{p}_2 + \mathbf{p}_4 &= \mathbf{s}_3 \ ; \ \ \mathbf{p}_1 - \mathbf{p}_4 = \mathbf{s}_4 \end{aligned} \tag{1.10}$$

Also, in terms of the stated force convention, the conditions for Static Equilibrium at nodes 1-4 are:

$$\begin{aligned} \mathbf{t}_1 - \mathbf{t}_4 - \mathbf{f}_1 - \mathbf{w}_1 &= \mathbf{0} \ ; \ \ -\mathbf{t}_2 + \mathbf{t}_3 + \mathbf{f}_1 - \mathbf{w}_2 = \mathbf{0} \\ -\mathbf{t}_1 + \mathbf{t}_2 - \mathbf{f}_2 - \mathbf{w}_3 &= \mathbf{0} \ ; \ \ -\mathbf{t}_3 + \mathbf{t}_4 + \mathbf{f}_2 - \mathbf{w}_4 = \mathbf{0} \end{aligned} \tag{1.11}$$

The static equilibrium conditions and the geometric conditions can be written in the form (1.6), (1.8), (1.9) where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}$$

with

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \ ; \ \mathbf{B} = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix}$$

We now derive a *canonical network representation* in which the bar vectors are a subset of the transformed network nodes.

Lemma 2.4 *Given the tensegrity node vector* $\mathbf{p}$*, define the nodal transformation*

$$\mathbf{q} = \mathbf{P}^{-1}\mathbf{p} \tag{1.12}$$

for some nonsingular matrix $\mathbf{P} \in \mathcal{R}^{3n_p \times 3n_p}$ *where*

$$\mathbf{P} = [\mathbf{P}_1^T, \mathbf{P}_2^T] \ ; \ \mathbf{P}_1, \mathbf{P}_2 \in \mathcal{R}^{3n_b \times 3n_p} \tag{1.13}$$

Then:

(i) In terms of the transformed tensegrity node $\mathbf{q}$*, the tensegrity geometry is given by*

$$\mathbf{S}_q^T\mathbf{q} = \mathbf{s} \ ; \ \mathbf{B}_q^T\mathbf{q} = \mathbf{b} \tag{1.14}$$

where

$$\mathbf{S}_q^T = [\mathbf{S}_1^T, \mathbf{S}_2^T] \ ; \ \mathbf{B}_q^T = [\mathbf{B}_1^T, \mathbf{B}_2^T] \tag{1.15}$$

with

$$\mathbf{S}_1 = \mathbf{P}_1\mathbf{S} \ ; \ \mathbf{S}_2 = \mathbf{P}_2\mathbf{S} \ ; \ \mathbf{B}_1 = \mathbf{P}_1\mathbf{B} \ ; \ \mathbf{B}_2 = \mathbf{P}_2\mathbf{B} \tag{1.16}$$

(ii) The tensegrity force equilibrium is given by

$$\mathbf{S}_1\mathbf{t} = \mathbf{B}_1\mathbf{f} + \mathbf{P}_1\mathbf{w} \ ; \ \mathbf{S}_2\mathbf{t} = \mathbf{B}_2\mathbf{f} + \mathbf{P}_2\mathbf{w} \tag{1.17}$$

Proof: Part (i) follows directly from the definition of $\mathbf{P}$. Part (ii) follows from the expansion of the equilibrium force equation $\mathbf{S}_q\mathbf{t} = \mathbf{B}_q\mathbf{f} + \mathbf{P}^T\mathbf{w}$.

Lemma 2.5 *Given the tensegrity node vector* $\mathbf{p}$*, bar vector* $\mathbf{b}$ *and string vector* $\mathbf{s}$*, the geometry of the class 1 tensegrity structure can be described by the algebraic equations*

$$\mathbf{B}^T\mathbf{p} = \mathbf{b} \quad ; \quad \mathbf{S}^T\mathbf{p} = \mathbf{s} \tag{1.18}$$

where the $3n_p \times 3n_b$ *matrix* $\mathbf{B}$ *is given by*

$$\mathbf{B}^T = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & . & . & \mathbf{0}_3 \\ . & . & . & . & . & . & . & . \\ \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \tag{1.19}$$

for some $3n_p \times 3n_s$ *matrix* $\mathbf{S}$ *which consist only of block matrices* $\{\mathbf{I}_3, -\mathbf{I}_3, \mathbf{0}_3\}$ *with*

$$n_p = 2n_b \tag{1.20}$$

In particular:

(i) $\mathbf{P}_1, \mathbf{P}_2 \in \mathcal{R}^{3n_b \times 3n_p}$ *in (1.13) are given by*

$$\begin{aligned} -\mathbf{P}_1 &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & . & . & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & . & \mathbf{0}_3 \\ . & . & . & . & . & . & . & . \\ \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix} \\ \mathbf{P}_2 &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & . & . & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3 & . & \mathbf{0}_3 \\ . & . & . & . & . & . & . & . \\ \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \end{aligned} \tag{1.21}$$

(ii) The inverse transformation

$$\mathbf{P}^{-1} = [\mathbf{P}_1^T, \mathbf{P}_2^T]^{-1} = \begin{bmatrix} \mathbf{B}^T \\ \mathbf{J}^T \end{bmatrix} \tag{1.22}$$

where $\mathbf{B}$ *is given by (1.19) and* $\mathbf{J} \in \mathcal{R}^{3n_p \times 3n_b}$ *is given by*

$$\mathbf{J}^T = \begin{bmatrix} \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & . & . & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & . & \mathbf{0}_3 \\ . & . & . & . & . & . & . & . \\ \mathbf{0}_3 & \mathbf{0}_3 & . & . & . & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{1.23}$$

(iii) The transformed tensegrity node $\mathbf{q}$ *is given by*

$$\mathbf{q}^T = [\mathbf{q}_b^T, \mathbf{q}_e^T] \tag{1.24}$$

where $\mathbf{q}_b, \mathbf{q}_e \in \mathcal{R}^{3n_b}$ *are given by*

$$\mathbf{q}_b = \mathbf{b} \ ; \ \mathbf{q}_e^T = [\mathbf{q}_2^T, \mathbf{q}_4^T, \ \dots \ , \mathbf{q}_{n_p}^T] \tag{1.25}$$

with

$$\mathbf{S}_q^T = [\mathbf{S}_1^T, \mathbf{S}_2^T] \ ; \ \mathbf{B}_q^T = [\mathbf{I}_{n_b}, \mathbf{0}_{n_b}] \tag{1.26}$$

where $\mathbf{S}_1 \in \mathcal{R}^{3n_b \times 3n_s}$; $\mathbf{S}_2 \in \mathcal{R}^{3n_b \times 3n_s}$ *are given by (1.16).*

(iv) The transformed force equilibrium conditions (1.17) are given by

$$\mathbf{S}_1\mathbf{t} = \mathbf{f} + \mathbf{P}_1\mathbf{w} \ ; \ \mathbf{S}_2\mathbf{t} = \mathbf{P}_2\mathbf{w} \tag{1.27}$$

Proof: Since each bar has two end points, $n_p = 2n_b$. Without any loss of generality, we can assume in a class 1 tensegrity that

$$\mathbf{b}_m = -\mathbf{p}_{2m-1} + \mathbf{p}_{2m} \ , \ m = 1, 2, \ \dots \ , n_b$$

which then gives $\mathbf{B}^T\mathbf{p} = \mathbf{b}$ with $\mathbf{B}$ as in (1.19). The block structure of the $\mathbf{S}$ matrix comes from the vector equations which relate the $\{\mathbf{s}_n, \mathbf{p}_r\}$. It follows from (1.19) and (1.21) that

$$\mathbf{P}^T\mathbf{B} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \mathbf{B} = \begin{bmatrix} \mathbf{I}_{3n_b} \\ \mathbf{0}_{3n_b} \end{bmatrix} \ ; \ \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \mathbf{S} = \mathbf{P}^T\mathbf{S}$$

which then gives (1.27) and (1.16).

Example 2.6 *The equations (1.10), (1.11) of the 2-bar 4-string tensegrity introduced in Example 2.3 can be written in the form (1.6) and (1.9) where*

$$\mathbf{B}^T = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \ ; \ \mathbf{S}^T = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}$$

Then from Lemma 2.4, we have (1.27) where

$$\begin{aligned}\mathbf{S}_1 &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix} ; \ \mathbf{S}_2 = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \\ \mathbf{P}_1 &= \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \end{bmatrix} ; \ \mathbf{P}_2 = \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \end{aligned} \tag{1.28}$$

3 ALGEBRAIC CONDITIONS FOR EQUILIBRIUM

Definition 3.1 *A tensegrity structure with tensile force coefficients $\{\gamma_n \geq 0\}$, compressive force coefficients $\{\lambda_n \geq 0\}$, node vector $\mathbf{p}$, string vector $\mathbf{s}$ and bar vector $\mathbf{b}$ is said to be in equilibrium if the element relationships (1.2), the force equations (1.7) and the geometric equations (1.9) are all satisfied.*

From lemma 2.1 and lemma 2.2, we have the that $\mathbf{t}^T\mathbf{S}^T\mathbf{p} - \mathbf{f}^T\mathbf{B}^T\mathbf{p} = \mathbf{w}^T\mathbf{p}$, or equivalently, $\mathbf{t}^T\mathbf{s} - \mathbf{f}^T\mathbf{b} = \mathbf{w}^T\mathbf{p}$. Hence from (1.2), we deduce the mechanical equivalent of Tellegen's Theorem for electrical networks.

Corollary 3.2 *For a tensegrity structure in equilibrium*

$$\sum_n \gamma_n ||\mathbf{s}_n||^2 - \sum_m \lambda_m ||\mathbf{b}_m||^2 = \sum_k \mathbf{w}_k^T \mathbf{p}_k \tag{1.29}$$

Connelly [1] defined a tensegrity framework $G(\mathbf{p})$ as a graph on $\mathbf{p}$ where each edge is either a cable or a bar. A *stress state* $\{\gamma_n, \lambda_m\}$ for $G(\mathbf{p})$ is said to be a *self-stress* if in the absence of an external load (i.e. $\mathbf{w} = \mathbf{0}$), force equilibrium is satisfied at each node.

The force $||\mathbf{t}_n||$ in a linear spring of length $||\mathbf{s}_n||$ having spring constant k_n and rest length ℓ_{n0} is given by

$$||\mathbf{t}_n|| = k_n(||\mathbf{s}_n|| - \ell_{n0})$$

Hence in terms of the tensile force coefficient γ_n in (1.2), we have

$$\gamma_n = k_n\left(1 - \frac{\ell_{n0}}{||\mathbf{s}_n||}\right) \tag{1.30}$$

The *potential energy PE_n in the spring* is also given by

$$PE_n = \frac{1}{2} k_n (||\mathbf{s}_n|| - \ell_{n0})^2 \tag{1.31}$$

so that when the cable has *zero rest length*:

$$\gamma_n = k_n \text{ is constant } ; \ PE_n = \frac{1}{2}\gamma_n ||\mathbf{s}_n||^2 \tag{1.32}$$

In analogy with total potential energy, Connelly [1] defined the energy form associated with the stress state $\{\gamma_n, \lambda_m\}$ as:

$$E(\mathbf{p}) \triangleq \frac{1}{2}\{\sum_n \gamma_n ||\mathbf{s}_n||^2 - \sum_m \lambda_m ||\mathbf{b}_m||^2\} \tag{1.33}$$

where all members are assumed to behave as elastic strings in which the cables have zero rest length and the bars have infinite rest length. The idea is that when the end points of a bar are displaced, the energy builds up as the square of the extension. The function $E(\mathbf{p})$ when $\mathbf{w} = \mathbf{0}$ is said to have an absolute minimum corresponding to the rest length of the element [2].

In this context, the more general statement in Corollary 3.2 then says then in the presence of external forces $\{\mathbf{w}_k\}$, the energy associated with the stressed state must be balanced by the sum of the external work done $\mathbf{w}_k^T \mathbf{p}_k$ at each node.

Requirements for Equilibrium: Given an external force vector $\mathbf{w}$, the problem of determining the geometric and force configuration of a tensegrity structure consisting of n_s strings and n_b bars in equilibrium is therefore equivalent to finding a solution $\mathbf{b}, \mathbf{q}_e \in \mathcal{R}^{3n_b}$ of the equations:

$$\mathbf{s} = \mathbf{S}_1^T \mathbf{b} + \mathbf{S}_2^T \mathbf{q}_e \tag{1.34}$$

$$\mathbf{t} = \mathbf{\Gamma s} \; ; \; \mathbf{\Gamma} \triangleq \text{diag}\{\gamma_1 \mathbf{I}_3, \gamma_2 \mathbf{I}_3, \; \dots \; , \gamma_{n_s} \mathbf{I}_3\} \tag{1.35}$$

$$\mathbf{S}_2 \mathbf{t} = \mathbf{P}_2 \mathbf{w} \tag{1.36}$$

$$\mathbf{f} = \mathbf{S}_1 \mathbf{t} - \mathbf{P}_1 \mathbf{w} \tag{1.37}$$

$$\mathbf{f} = \mathbf{\Lambda b} \; ; \; \mathbf{\Lambda} \triangleq \text{diag}\{\lambda_1 \mathbf{I}_3, \lambda_2 \mathbf{I}_3, \; \dots \; , \lambda_{n_b} \mathbf{I}_3\} \tag{1.38}$$

for given matrices

$$\mathbf{S}_1, \mathbf{S}_2 \in \mathcal{R}^{3n_b \times 3n_s} \; ; \; \mathbf{P}_1, \mathbf{P}_2 \in \mathcal{R}^{3n_b \times 3n_p} \tag{1.39}$$

where

$$n_p = 2n_b \; ; \; n_s > n_b \tag{1.40}$$

The diagonal matrices $\{\mathbf{\Gamma}, \mathbf{\Lambda}\}$ shall be referred to as the *tensile force matrix* and *compressive force matrix* respectively.

Materials Properties: Conditions for equilibrium will depend on both the tensile force coefficients of the strings and the compressive force coefficients of the bars. A tensile force coefficient γ_n of a (linear) string may be designed by selection of the spring constant k_n and the ratio $\ell_{m0}/||\mathbf{s}_n||$ of the rest to stretched length of the string subject only to the requirement that the *yield force* (i.e. the force that causes the string to reach its elastic limit) is not exceeded. In the (mathematically ideal) case of zero rest length, $k_n = \gamma_n$.

On the other hand, the compressive force coefficient λ_n of a bar cannot be designed by an adjustment of the materials properties. However the bar force must not exceed the *buckling force* where an ideal bar of radius r_0 and length L_0 buckles at a compressive force F_0 given by

$$F_0 = \frac{E_0 \pi r_0^4}{L_0^2} \ ; \ E_0 = \text{Young}'\text{s Modulus of bar}$$

Using an earlier derivation, the next result provides an expression for both the tensile and compressive force coefficients in an equilibrium structure.

Lemma 3.3 *Suppose the transformed tensegrity structure is in equilibrium. Then:*
(i) Assuming a linear force relationship for the strings, the tensile force coefficient $\{\gamma_n\}$ *in the string* $\mathbf{s}_n$ *is given by (1.30) where* $k_n > 0$ *is the spring constant,* $\ell_{n0} > 0$ *is the rest length of the* n*th string, and* $||\mathbf{s}_n||$ *is the stretched length of the* n*th string*
(ii) The compressive force coefficient $\{\lambda_m\}$ *in the bar* $\mathbf{b}_m$ *is given by*

$$\lambda_m = \frac{||\mathbf{f}_m||}{L_m} \ ; \ L_m \triangleq ||\mathbf{b}_m|| \tag{1.41}$$

where L_m *is the (constant) length of the bar, and*

$$\begin{aligned} \mathbf{f}_m &= \mathbf{S}_{m1}\mathbf{\Gamma}\mathbf{s} - \mathbf{P}_{m1}\mathbf{w} \ ; \ \mathbf{s} = \mathbf{S}_q^T\mathbf{q} \\ \mathbf{\Gamma} &= \text{diag}\{\gamma_1\mathbf{I}_3, \gamma_2\mathbf{I}_3, \ .. \ , \gamma_{n_s}\mathbf{I}_3\} \end{aligned} \tag{1.42}$$

where $\mathbf{S}_{m1}, \mathbf{P}_{m1} \in \mathcal{R}^{3\times 3n_p}$ *for* $1 \leq m \leq n_b$ *are the* m*th blocks of* $\{\mathbf{S}_1, \mathbf{P}_1\}$ *respectively in (1.21), (1.16) where*

$$\mathbf{S}_1 = \begin{bmatrix} \mathbf{S}_{11} \\ \mathbf{S}_{21} \\ \cdot \\ \cdot \\ \mathbf{S}_{n_b,1} \end{bmatrix} \ ; \ \mathbf{P}_1 = \begin{bmatrix} \mathbf{P}_{11} \\ \mathbf{P}_{21} \\ \cdot \\ \cdot \\ \mathbf{P}_{n_b,1} \end{bmatrix} \tag{1.43}$$

Proof: From (1.2), (1.27) and (1.43), we have $\lambda_m \mathbf{b}_m = \mathbf{f}_m$. Then since $\lambda_m \geq 0$, we get (1.41).

3.1 Prestressed Equilibrium Structure

We now proceed to derive necessary and sufficient conditions for the existence of a structure in equilibrium that is prestressed in the absence of any external load (i.e. $\mathbf{w} = \mathbf{0}$). The derivation relies heavily on the *singular value decomposition* $\text{svd}(\mathbf{A}) = \{\mathbf{U}_A, \mathbf{\Sigma}_A, \mathbf{V}_A\}$ of a matrix $\mathbf{A}$.

Lemma 3.4 *(i) Suppose an* $n \times m$ *matrix* $\mathbf{A}$ *has rank* r_A*, then there exists an* $n \times n$ *unitary matrix* $\mathbf{U}_A$*, an* $m \times m$ *unitary matrix* $\mathbf{V}_A$ *and a positive definite* $r_A \times r_A$ *diagonal matrix* $\mathbf{\Sigma}_{1A}$ *such that*

$$\mathbf{A} = \mathbf{U}_A \mathbf{\Sigma}_A \mathbf{V}_A^T \ ; \ \mathbf{\Sigma}_A = \begin{bmatrix} \mathbf{\Sigma}_{1A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{1.44}$$

(ii) If $\{\mathbf{U}_A, \mathbf{V}_A\}$ *are partitioned such that*

$$\mathbf{U}_A = [\mathbf{U}_{1A}, \mathbf{U}_{2A}] \ ; \ \mathbf{V} = [\mathbf{V}_{1A}, \mathbf{V}_{2A}] \tag{1.45}$$

with

$$\mathbf{U}_{1A} \in \mathcal{R}^{n\times r_A} \ ; \ \mathbf{U}_{2A} \in \mathcal{R}^{n\times(n-r_A)} \ ; \ \mathbf{V}_{1A} \in \mathcal{R}^{m\times r_A} \ ; \ \mathbf{V}_{2A} \in \mathcal{R}^{m\times(m-r_A)} \tag{1.46}$$

then

$$\begin{aligned} \mathbf{U}_{1A}^T.\mathbf{U}_{1A} &= \mathbf{I}_{r_A} \ ; \ \mathbf{U}_{1A}^T.\mathbf{U}_{2A} = \mathbf{0} \ ; \ \mathbf{U}_{2A}^T.\mathbf{U}_{2A} = \mathbf{I}_{n-r_A} \\ \mathbf{V}_{1A}^T.\mathbf{V}_{1A} &= \mathbf{I}_{r_A} \ ; \ \mathbf{V}_{1A}^T.\mathbf{V}_{2A} = \mathbf{0} \ ; \ \mathbf{V}_{2A}^T.\mathbf{V}_{2A} = \mathbf{I}_{m-r_A} \\ \mathbf{U}_{2A}^T\mathbf{A} &= \mathbf{0} \ ; \ \mathbf{A}\mathbf{V}_{2A} = \mathbf{0} \end{aligned} \tag{1.47}$$

(iii) The algebraic equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has a solution if and only if $\mathbf{U}_{2A}^T\mathbf{b} = \mathbf{0}$. *When this condition is satisfied, then all solutions* $\mathbf{x}$ *are of the form*

$$\mathbf{x} = \mathbf{V}_{1A}\mathbf{\Sigma}_{1A}^{-1}\mathbf{U}_{1A}^T\mathbf{b} + \mathbf{V}_{2A}\mathbf{z}_x \ ; \ \mathbf{U}_{2A}^T\mathbf{b} = \mathbf{0}$$

where $\mathbf{z}_x \in \mathcal{R}^{n-r_A}$ *is arbitrary.*

We now establish necessary and sufficient conditions for a solution of equations (1.34)-(1.40) in the absence of external forces (i.e. $\mathbf{w} = \mathbf{0}$) by examining each of these equations in turn beginning with the solution of (1.34), (1.35). The next result follows from lemma 3.4.

Lemma 3.5 *Suppose*

$$r \triangleq \rho(\mathbf{S}_2) \leq \min\{3n_p, 3n_s\} \tag{1.48}$$

and let $\mathbf{S}_2$ *have the singular value decomposition* $\{\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}\}$ *given by*

$$\mathbf{S}_2 = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \in \mathcal{R}^{3n_b\times 3n_s} \tag{1.49}$$

where

$$\begin{aligned} \mathbf{U} &= [\mathbf{U}_1, \mathbf{U}_2] \ ; \ \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ ; \ \mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2] \\ \mathbf{U}_1, \mathbf{V}_1 &\in \mathcal{R}^{3n_b\times r} \ ; \ \mathbf{U}_2, \mathbf{V}_2 \in \mathcal{R}^{3n_b\times(3n_b-r)} \end{aligned} \tag{1.50}$$

Then a necessary and sufficient condition for (1.34) to have a solution $\mathbf{q}_e \in \mathcal{R}^{3n_b}$ *is given by*

$$\mathbf{V}_2^T(\mathbf{s} - \mathbf{S}_1^T\mathbf{b}) = \mathbf{0} \tag{1.51}$$

Furthermore, when (1.51) is satisfied, all solutions $\mathbf{q}_e$ *are of the form*

$$\mathbf{q}_e = \mathbf{U}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{V}_1^T(\mathbf{s} - \mathbf{S}_1^T\mathbf{b}) + \mathbf{U}_2\mathbf{z}_e \tag{1.52}$$

where $\mathbf{z}_e \in \mathcal{R}^{3n_b - r}$ *is arbitrary.*

We now consider the solution of (1.36) when $\mathbf{w} = \mathbf{0}$.

Lemma 3.6 *When* $\mathbf{w} = \mathbf{0}$*, all solutions* $\mathbf{t}$ *of (1.36) which guarantee (1.51) are of the form*

$$\mathbf{t} = \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b}\ ;\ \mathbf{M} \stackrel{\Delta}{=} \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \tag{1.53}$$

Proof: From (1.49), (1.50) and lemma 3.4, we have $\mathbf{t} = \mathbf{V}_2\mathbf{z}_t$ where $\mathbf{z}_t$ is the free solution of (1.36). Then from (1.35)

$$\mathbf{V}_2^T(\mathbf{s} - \mathbf{S}_1^T\mathbf{b}) = \mathbf{V}_2^T(\mathbf{\Gamma}^{-1}\mathbf{V}_2\mathbf{z}_t - \mathbf{S}_1^T\mathbf{b})$$

Since $\mathbf{V}_2$ has full column rank, the matrix $\mathbf{M} = \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2$ is invertible. Hence (1.51) is satisfied when $\mathbf{z}_t = \mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b}$, and this gives (1.53).

We now consider the solution of (1.37), (1.38) when $\mathbf{w} = \mathbf{0}$.

Lemma 3.7 *When* $\mathbf{w} = \mathbf{0}$*, a necessary and sufficient condition for (1.37), (1.38) to have a solution* $\mathbf{b} \in \mathcal{R}^{3n_b}$ *is given by*

$$(\mathbf{X} - \mathbf{\Lambda})\mathbf{b} = \mathbf{0} \tag{1.54}$$

where

$$\mathbf{X} \stackrel{\Delta}{=} (\mathbf{S}_1\mathbf{V}_2)\mathbf{M}^{-1}(\mathbf{S}_1\mathbf{V}_2)^T\ ;\ \mathbf{M} \stackrel{\Delta}{=} \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \tag{1.55}$$

In particular, define

$$r_b \stackrel{\Delta}{=} \rho(\mathbf{\Lambda} - \mathbf{X}) \tag{1.56}$$

Then:

(i) When $r_b = 3n_b$*, then* $\mathbf{b} = \mathbf{0}$ *is the only solution of (1.54),*
(ii) When $r_b = 0$*, any* $\mathbf{b} \in \mathcal{R}^{3n_b}$ *is a solution of (1.54), and*
(iii) When $0 < r_b < 3n_b$*, all solutions* $\mathbf{b}$ *satisfy the equation*

$$[\mathbf{I} - \mathbf{V}_{X1}\mathbf{\Sigma}_{X11}^{-1}\mathbf{U}_{X1}^T\mathbf{\Lambda}]\mathbf{b} = \mathbf{U}_{X2}\mathbf{z}_b \tag{1.57}$$

where $\mathbf{z}_b \in \mathcal{R}^{3n_b - r_b}$ *is free, and where* $\{\mathbf{U}_X, \mathbf{\Sigma}_X, \mathbf{V}_X\}$ *is the singular value decomposition of the matrix* $\mathbf{X} \in \mathcal{R}^{3n_b \times 3n_b}$*; that is*

$$\mathbf{U}_X = [\mathbf{U}_{X1}, \mathbf{U}_{X2}] \ ; \ \mathbf{\Sigma}_b = \begin{bmatrix} \mathbf{\Sigma}_{X11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ ; \ \mathbf{V}_X = [\mathbf{V}_{X1}, \mathbf{V}_{X2}] \ ; \ r_X = rank(\mathbf{X})$$

with $\mathbf{U}_{X1} \in \mathcal{R}^{3n_b \times r_X}$, $\mathbf{V}_{X2} \in \mathcal{R}^{3n_b \times (3n_b - r_X)}$.

Proof: From (1.53), (1.38) and (1.36)

$$\mathbf{S}_1\mathbf{t} - \mathbf{f} = \mathbf{S}_1\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b} - \mathbf{\Lambda}\mathbf{b} = (\mathbf{X} - \mathbf{\Lambda})\mathbf{b}$$

The result then follows from the singular value decomposition of $\mathbf{X}$ after writing (1.54) in the form $\mathbf{X}\mathbf{b} = \mathbf{f}, \ \mathbf{f} = \mathbf{\Lambda}\mathbf{b}$.

Now $\mathbf{\Gamma}$ positive definite implies $\mathbf{M}$ in (1.55) is positive definite. Then $\mathbf{M}$ positive definite implies $\mathbf{X}$ in (1.55) is positive semidefinite with at least one non-zero eigenvalue. Since (1.54) has a nontrivial solution $\mathbf{b}$ if and only if $\det(\mathbf{X} - \mathbf{\Gamma}) = 0$, the following lemma is important.

Lemma 3.8 *Given a symmetric positive semidefinite matrix* $\mathbf{X}$. *Then there exists a positive definite diagonal matrix* $\mathbf{\Lambda}$ *such that*

$$\det(\mathbf{X} - \mathbf{\Lambda}) = 0$$

if and only if $\mathbf{X} \neq 0$.

Proof: Since $\mathbf{X} = \mathbf{X}^T \geq 0$, $\mathbf{X}$ cannot have negative eigenvalues. Suppose λ_0 is a positive eigenvalue of $\mathbf{X}$. Then $\mathbf{\Lambda} = \lambda_0\mathbf{I}$ implies $\det(\mathbf{\Lambda} - \mathbf{X}) = \det(\lambda_0\mathbf{I} - \mathbf{X}) = 0$.

Remarks: For $0 < r_b < 3n_b$, $(\mathbf{\Lambda} - \mathbf{X})\mathbf{b} = \mathbf{0}$ always has at least one solution $\mathbf{b} \neq \mathbf{0}$ of the form (1.57) where the freedom in the choice of $\mathbf{b}$ is available in the choice of a free vector $\mathbf{z}_b$. However for $n_b \geq 3$, an *additional necessary condition* for the existence of a nontrivial tensegrity structure is that the n_b bar vectors $\{\mathbf{b}_m\}$ span $\mathcal{R}^3$. This condition places additional requirements on the component matrices of the singular value decomposition of the matrix $\mathbf{X}$. In the 2-bar 4-string planar tensegrity and the $(3;9;3)$ 3-bar 9-string single stage shell tensegrity structure, no nontrivial tensegrity structures have been found in the case when $r_b \neq 0$; only in the case when $r_b = 0$ do equilibrium solutions exist. We therefore specialize our results to this case.

Theorem 3.9 *Consider a class 1 tensegrity structure as defined by the geometry and force equations in the absence of external load as described by the geometric conditions*

$$\mathbf{B}^T\mathbf{p} = \mathbf{b} \ ; \ \mathbf{S}^T\mathbf{p} = \mathbf{s} \ ; \ \mathbf{p} \in \mathcal{R}^{3n_p}, \ \mathbf{b} \in \mathcal{R}^{3n_b}, \ \mathbf{s} \in \mathcal{R}^{3n_s}$$

with $n_p = 2n_b$, *and the equilibrium force equations*

$$\mathbf{S}\mathbf{t} = \mathbf{B}\mathbf{f} \ ; \ \mathbf{t} = \mathbf{\Gamma}\mathbf{s}, \ \mathbf{f} = \mathbf{\Lambda}\mathbf{b}$$

where for $\{\gamma_m > 0, \ \lambda_n > 0\}$,

$$\mathbf{\Gamma} = \text{diag}\{\gamma_1\mathbf{I}_3, \gamma_2\mathbf{I}_3, \ \ldots \ , \gamma_{n_s}\mathbf{I}_3\} \ ; \ \mathbf{\Lambda} = \text{diag}\{\lambda_1\mathbf{I}_3, \lambda_2\mathbf{I}_3, \ \ldots \ , \lambda_{n_b}\mathbf{I}_3\}$$

Then given any tensile force coefficients $\{\gamma_m > 0,\ 1 \leq m \leq n_s\}$*, there exits compressive force coefficients* $\{\lambda_n > 0,\ 1 \leq m \leq n_b\}$ *which define an equilibrium structure if* $\{\mathbf{\Lambda}, \mathbf{\Gamma}\}$ *satisfy the condition:*

$$\mathbf{\Lambda} = \mathbf{X} \ ; \ \mathbf{X} \triangleq (\mathbf{S}_1\mathbf{V}_2)\mathbf{M}^{-1}(\mathbf{S}_1\mathbf{V}_2)^T, \ \mathbf{M} \triangleq \mathbf{V}_2\mathbf{\Gamma}^{-1}\mathbf{V}_2^T \tag{1.58}$$

Moreover, given condition (1.58), any vector $\mathbf{p}$ *of the form*

$$\mathbf{p} = \mathbf{P}\mathbf{Q}[\mathbf{b}^T, \mathbf{z}_e^T] \tag{1.59}$$

where

$$\mathbf{Q} \triangleq \begin{bmatrix} \mathbf{I}_{3n_b} & \mathbf{0} \\ \mathbf{L} & \mathbf{U}_2 \end{bmatrix} \ ; \ \mathbf{L} \triangleq \mathbf{U}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{V}_1^T[\mathbf{\Gamma}^{-1}\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T - \mathbf{I}_{3n_s}]\mathbf{S}_1^T \tag{1.60}$$

for arbitrary vectors $\{\mathbf{b} \in \mathcal{R}^{3n_b},\ \mathbf{z}_e \in \mathcal{R}^{3n_b - r}\}$ *defines a corresponding tensegrity node. Equivalently, the tensegrity node is constrained according to the equation*

$$\mathbf{U}_{Q2}^T\mathbf{P}^{-1}\mathbf{p} = \mathbf{0} \tag{1.61}$$

where $\mathbf{U}_Q\mathbf{\Sigma}_Q\mathbf{V}_Q^T$ *is the singular value decomposition of* $\mathbf{Q}$ *in (1.60) with*

$$\mathbf{U}_Q = [\mathbf{U}_{Q1}, \mathbf{U}_{Q2}] \ ; \ \mathbf{U}_{Q2} \in \mathcal{R}^{3n_p \times (3n_p - r_Q)}, \ r_Q = \rho(\mathbf{Q}) \tag{1.62}$$

The corresponding tensegrity tension vector $\mathbf{t}$*, string vector* $\mathbf{s}$*, and compression vector* $\mathbf{f}$ *are given in terms of the bar vector* $\mathbf{b}$ *by*

$$\mathbf{t} = \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b} \ ; \ \mathbf{s} = \mathbf{\Gamma}^{-1}\mathbf{t} \ ; \ \mathbf{f} = \mathbf{\Lambda}\mathbf{b} \tag{1.63}$$

Construction Procedure:

One procedure for construction of a class 1 tensegrity structure in equilibrium is provided as follows:

1. Given the matrices $\mathbf{S}$ and $\mathbf{B}$ from the network topology, find a nonsingular matrix $\mathbf{P} = [\mathbf{P}_1^T, \mathbf{P}_2^T]$ such that $\mathbf{B}_q^T = \mathbf{B}^T\mathbf{P} = [\mathbf{I}_{n_b}, \mathbf{0}_{n_b}]$, and calculate $\{\mathbf{S}_1 = \mathbf{P}_1\mathbf{S}, \mathbf{S}_2 = \mathbf{P}_2\mathbf{S}\}$. (Note that if $\mathbf{B}$ is defined as in (1.19), then $\{\mathbf{P}_1, \mathbf{P}_2\}$ are given by (1.21).)

2. Choose $\{\gamma_m > 0\}$ such that $\mathbf{X} \geq 0$ in (1.55) is diagonal, and then select $\{\lambda_n > 0\}$ such that $\mathbf{\Lambda} = \mathbf{X}$.

3. Select any bar vector $\mathbf{b}$ such that $\{\mathbf{b}_1, \mathbf{b}_2, \ \dots \ , \mathbf{b}_{n_b}\}$ span $\mathcal{R}^3$, and calculate $\{\mathbf{t}, \mathbf{s}, \mathbf{f}\}$ from (1.63).

4. Choose the free vector $\mathbf{z}_e$ in (1.59), (1.60) to give a suitable node vector $\mathbf{p}$.

In the design of tensegrity structures, an alternative approach for the selection of the node vector $\mathbf{p}$ and bar vector $\mathbf{b}$ may be useful. That is, we can first select the node vector $\mathbf{p}$ as in (1.59), (1.60) (which satisfies the constraint (1.61)) subject (if necessary) to other design constraints. The corresponding bar vector $\mathbf{b}$ is then determined from the first $3n_b$ components of the vector $\mathbf{P}^{-1}\mathbf{p}$.

3.2 Computation of Equilibrium Configurations

An inspection of all matrices in all algebraic expressions reveals that each matrix $\mathbf{A} = [a_{ij}]$ is a tensor product of the form $\mathbf{A} = \tilde{\mathbf{A}} \bigotimes \mathbf{I} = [a_{ij}\mathbf{I}]$ where $\tilde{\mathbf{A}}$ is $m \times n$ and the identity matrix $\mathbf{I}$ is of either dimension $3n \times 3m$, or $2n \times 2m$ for planar tensegrity structures. Hence all matrix manipulations with the exception of the final determination of the vectors $\{\mathbf{p}, \mathbf{b}, \mathbf{s}, \mathbf{t}, \mathbf{f}\}$ can be carried out in lower dimension thereby reducing the computational burden.

In particular, the sufficient condition (1.58) for the existence of a prestress equilibrium can be written in the form

$$\tilde{\mathbf{\Lambda}} = \tilde{\mathbf{X}} \; ; \; \tilde{\lambda} = \mathbf{\Lambda} \bigotimes \mathbf{I}, \;\; \tilde{\mathbf{X}} = \mathbf{X} \bigotimes \mathbf{I}$$

where

$$\tilde{\mathbf{X}} = (\tilde{\mathbf{S}}_1\tilde{\mathbf{V}}_2)\tilde{\mathbf{M}}^{-1}(\tilde{\mathbf{S}}_1\tilde{\mathbf{V}}_2)^T \; ; \; \tilde{\mathbf{M}} = \tilde{\mathbf{V}}_2^T\tilde{\mathbf{\Lambda}}^{-1}\tilde{\mathbf{V}}_2$$

We now illustrate the construction procedure

Example 3.10 *For the 2-bar 4-string planar tensegrity structure, we can replace* $\{\mathbf{S}, \mathbf{B}, \mathbf{S}_1, \mathbf{S}_2, \mathbf{P}_1, \mathbf{P}_2\}$ *in examples 2.3 and 2.6 by* $\{\tilde{\mathbf{S}}, \tilde{\mathbf{B}}, \tilde{\mathbf{S}}_1, \tilde{\mathbf{S}}_2, \tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2\}$ *where*

$$\begin{aligned}
\tilde{\mathbf{S}} &= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \; ; \; \tilde{\mathbf{B}} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \\
\tilde{\mathbf{S}}_1 &= \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \; ; \; \tilde{\mathbf{S}}_2 = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \\
\tilde{\mathbf{P}}_1 &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \; ; \; \tilde{\mathbf{P}}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Then the singular value decomposition $\{\tilde{\mathbf{U}}, \tilde{\mathbf{\Sigma}}, \tilde{\mathbf{V}}\}$ *of the (reduced dimension) matrix* $\tilde{\mathbf{S}}_2$ *of (reduced) rank* $\tilde{r} = \rho(\tilde{\mathbf{S}}_2) = 1$ *is given by* $\tilde{\mathbf{\Sigma}}_{11} = 2\sqrt{2}$ *and*

$$\tilde{\mathbf{U}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \; ; \; \tilde{\mathbf{U}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; ; \; \tilde{\mathbf{V}}_1 = 0.5 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

and

$$\tilde{\mathbf{V}}_2 = \begin{bmatrix} -0.8660 & -0.0000 & 0 \\ -0.2887 & 0.5774 & -0.5774 \\ 0.2887 & 0.7887 & 0.2113 \\ -0.2887 & 0.2113 & 0.7887 \end{bmatrix}$$

$$\tilde{\mathbf{S}}_1\tilde{\mathbf{V}}_2 = \begin{bmatrix} -0.5774 & -0.2113 & -0.7887 \\ 0.5774 & 0.5774 & -0.5774 \end{bmatrix}$$

We also have that $(\tilde{\mathbf{S}}_1\tilde{\mathbf{V}}_2)(\tilde{\mathbf{S}}_1\tilde{\mathbf{V}}_2)^T = \mathbf{I}_2$.

(i) Suppose $\{\gamma_k = 1; k = 1,2,3,4\}$ *so that the (reduced) matrix* $\tilde{\mathbf{\Gamma}} = \mathbf{I}_4$. *It then follows that the (reduced) matrix* $\tilde{\mathbf{X}} = \mathbf{I}_4$ *which satisfies condition (1.58) in Theorem 3.9. Hence when there is no external load (i.e.* $\mathbf{w} = \mathbf{0}$*), (1.54) has a (nonzero) solution* $\mathbf{b}^T = [\mathbf{b}_1^T, \mathbf{b}_2^T]$ *when* $\lambda_1 = \lambda_2 = 1$. *For example, when* $\mathbf{b}_1 = [4,0], \mathbf{b}_2 = [0,4]$, *the nodes* $\{\mathbf{p}_1 = [-3,-1], \mathbf{p}_2 = [1,-1], \mathbf{p}_3 = [-1,3], \mathbf{p}_4 = [-1,-1]\}$ *define an equilibrium solution.*

(ii) We hypothesize that there is no equilibrium solution unless: for some $\overline{\gamma}$, $\{\gamma_n = \overline{\gamma}\}$ *for* $1 \le n \le 4$ *and* $\{\lambda_m = \overline{\gamma}\}$ *for* $1 \le m \le 2$. *Such tensegrity structures may be said to have force symmetry (but not necessarily geometric symmetry).*

No equilibrium solutions have been found for other choices of $\{\gamma_n, \lambda_n\}$. *An analytical investigation was undertaken for the case* $\{\gamma_1 = \gamma_3 = 1,\ \gamma_2 = \gamma_4 = a\}$. *In this case, it may be shown that*

$$\tilde{\mathbf{X}} = 0.5 \begin{bmatrix} a+1 & a-1 \\ a-1 & a+1 \end{bmatrix}$$

with singular values (and eigenvalues) given by $\{1, a\}$. *In all choices for* $\{\lambda_n\}$ *that led to the rank of* $\tilde{\mathbf{\Lambda}} - \tilde{\mathbf{X}}$ *having rank 2, the* 4×2 *matrix* $\tilde{\mathbf{V}}_{b2}$ *is of the form* $\tilde{\mathbf{V}}_{b2}^T = [\tilde{\mathbf{A}}_1^T, \pm\tilde{\mathbf{A}}_1^T]$; *that is, in (1.57), the two bar vectors* $\{\mathbf{b}_1, \mathbf{b}_2\}$ *are always parallel, so the equilibrium structure is trivial with* $\{\mathbf{p}_1 = \mathbf{p}_3, \mathbf{p}_2 = \mathbf{p}_4\}$.

Example 3.11 *A* $(3;9;3)$ *class 1 tensegrity [9] consisting of one stage of* $n_b = 3$ *bars (so* $n_p = 6$ *nodes) with* $n_s = 9$ *strings has topology as illustrated in Fig. 2*

The corresponding (reduced) geometric matrices $\{\tilde{\mathbf{S}}, \tilde{\mathbf{B}}\}$ *are given by*

$$\tilde{\mathbf{S}} = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} ; \tilde{\mathbf{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

A symmetrical force configuration was investigated with equal 'base' string force coefficients $\{\gamma_1 = \gamma_2 = \gamma_1 \triangleq \gamma_{base}\}$, *equal 'top' string coefficients* $\{\gamma_5 = \gamma_6 = \gamma_7 \triangleq \gamma_{top}\}$, *and equal vertical string coefficients* $\{\gamma_4 = \gamma_8 = \gamma_9 \triangleq \gamma_{vert}\}$. *The following conclusions were reached:*

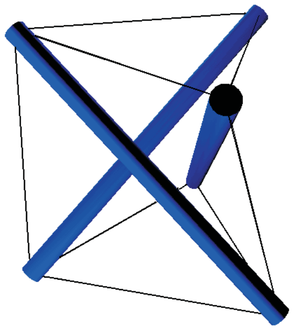

Figure 2 3-Bar 9-String $(3;9;3)$ Class 1 Tensegrity

(i) No equilibrium solution is possible when $\gamma_{base} = \gamma_{top} = \gamma_{vert}$. In this case $\tilde{\mathbf{X}}$ is not diagonal, and once again, all choices of $\tilde{\mathbf{\Lambda}}$ such that $\rho(\tilde{\mathbf{\Lambda}} - \tilde{\mathbf{X}}) \neq 0$ led to parallel bar vectors.

(ii) For all choices of $\{\gamma_{base}, \gamma_{top}\}$, an equilibrium solution with $\tilde{\mathbf{\Lambda}} = \tilde{\mathbf{X}}$ is provided by $\lambda_1 = \lambda_2 = \lambda_3 = \overline{\lambda}$ where $\gamma_{vert} = \overline{\lambda}$. For example, when $\{\gamma_{base} = 1, \gamma_{top} = 1\}$, then $\gamma_{vert} = \overline{\lambda} = 1.7321$, and when $\{\gamma_{base} = 1, \gamma_{top} = 5\}$, then $\gamma_{vert} = \overline{\lambda} = 3.8730$. In all cases, $\tilde{\mathbf{X}}$ is a scaled identity matrix.

Recall that once the condition $\mathbf{\Lambda} = \mathbf{X}$ is satisfied, the choice for the bar vectors was completely arbitrary subject only to $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ spanning $\mathcal{R}^3$. Alternatively, as required in (1.61), the construction may be achieved subject to the node vector $\mathbf{p}$ being in the null space of the matrix $\mathbf{U}_{Q2}^T\mathbf{P}^{-1}$. It then follows that parameters

$$\alpha_1 = \frac{\gamma_{base}}{\gamma_{top}} \; ; \; \alpha_2 = \frac{\overline{\lambda}}{\gamma_{top}}$$

may be used to optimize the node positions with respect to some performance measure (such as being restricted to lie on a given surface).

4 CONCLUSION

This paper reduces the study of the tensegrity equilibria to a series of linear algebra problems. Of course the existence conditions for the linear algebra problems are nonlinear in the design variables. Our formulation offers insight and identifies the free parameters that may be used to achieve desired structureal shapes.

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