Volume 112, Advances in the Astronautical Sciences
Spaceflight Mechanics 2002

AAS 02-177

EQUILIBRIUM CONDITIONS OF A CLASS I TENSEGRITY STRUCTURE

Robert E. Skelton [1] , **Darrell Williamson** [2] , **JeongHeon Han** [3]

ABSTRACT

Static models of tensegrity structures are reduced to linear algebra problems, after first characterizing the problem in a vector space where direction cosines are not needed. That is, we describe the components of all member vectors as opposed to the usual practice of characterizing the statics problem in terms of the magnitude of tension vectors. While our approach enlarges (by a factor of 3) the vector space required to describe the problem, the computational space is not increased. The advantage of enlarging the vector space makes the mathematical structure of the problem amenable to linear algebra treatment. Using the linear algebraic techniques, many variables are eliminated from the final existence equations. This paper characterizes the existence conditions for all tensegrity equilibria.

Key Words: tensegrity, structure, statics, equilibrium conditions, linear algebra.

INTRODUCTION

The Tensegrity structures introduced by Kenneth Snelson pose a wonderful blend of geometry and mechanics. In addition, they have engineering appeal in problems requiring large changes is structural shape. We define "class I" tensegrity as a stable connection of bars and strings with only one bar connected to any given node. Nodes connecting more than one bar or other constraints form "class II" tensegrity. Thus, class I structures are unconstrained, whereas, class II may have boundary constraints or bar to bar connections. Most existing *smart structure* methods are limited to small displacements. Since class I tensegrity structures [8] have no bar-to-bar connections, the control of tendons allows very large shape changes. Therefore, an efficient set of analytical tools could be the enabler to a hoist of new engineering concepts for deployable and shape controllable structures.

This paper characterizes the static equilibria of class I tensegrity structures. Furthermore, we use vectors to describe each element(bars and tendons), eliminating the need to use direction cosines and the subsequent transcendental functions that follow their use.

It is well known in a variety of mathematical problems that enlarging the domain in which the problem is posed can often simplify the mathematical treatment. For example, nonlinear Riccati equations are known to be solvable by linear algebra in a space that is twice the size of the original problem statement. Many nonlinear problems admit solutions by linear techniques by enlarging the domain of the problem. The purpose of this paper is to show that by enlarging the vector space in

[1] University of California, San Diego, Dept. of Mechanical and Aerospace Engineering, La Jolla, CA, USA. email: bobskelton@ucsd.edu
[2] University of Wollongong, Faculty of Informatics, Wollongong, NSW 2522 Australia email: darrell_williamson@uow.edu.au
[3] University of California, San Diego, Dept. of Mechanical and Aerospace Engineering, La Jolla, CA, USA. email: jehan@mae.ucsd.edu

which we characterize the tensegrity statics problem, the mathematical structure of the equations admit treatment by linear algebra methods, for the most part.

Our results characterize the equilibria conditions of both Class I tensegrity structures in terms of a very small number of variables, since the necessary and sufficient conditions of the linear algebra treatment allows the elimination of several of the original variables. In the future, these results will be programmed into object-oriented software to design and simulate a large class of tensegrity structures. This paper provides the enabling technology for efficient algorithms to design tensegrity structures, which have been around for fifty years without efficient design procedures.

The paper is laid out as follows. Section 2 introduces the network representations of tensegrity structures as an oriented graph in real three dimensional space. Geometric connectivity, equilibrium, and a coordinate transformation will be introduced. Section 3 introduces the algebraic equilibrium conditions for a class I tensegrity structure. After we derive necessary and sufficient conditions for the existence of an unloaded tensegrity structure in equilibrium, we write the necessary and sufficient conditions for the externally loaded structure in equilibrium. A couple of examples will show how to construct a tensegrity structure that concludes the paper.

Notation

We let $\mathbf{I}_n$ define the $n \times n$ identity matrix, and $\mathbf{0}$ define an $n \times m$ matrix of zeros. (The dimensions of $\mathbf{0}$ will be clear from the context.) We also let $\rho(\mathbf{A})$ define the rank of the matrix $\mathbf{A}$. Let $\mathbf{A} \in \mathcal{R}^{m \times n}$ and $\mathbf{B} \in \mathcal{R}^{p \times q}$, then the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \bigotimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \in \mathcal{R}^{mp \times nq}$$

NETWORK REPRESENTATION OF STRUCTURES

In this paper, we choose to represent a tensegrity structure as an *oriented graph* in real three dimensional space $\mathcal{R}^3$ defined in terms of n_p nodes and $n_s + n_b$ directed branches which are all represented as vectors in $\mathcal{R}^3$. A *loop* is any closed path in the graph. As we shall see, the advantage of this approach is that both the magnitude and the direction of the forces are contained in vectors which can be solved using linear algebra. Thus linear algebra plays a larger role in this approach compared to the usual approach in mechanics and finite element methods using direction cosines.

In the oriented graph of a class I tensegrity structure, the *nodes* consist of the ends of the bars as represented by the n_p nodes (or vectors) $\{\mathbf{p}_k\}$. Hence if there are n_b bars, then there are $n_p = 2n_b$ nodes. We choose to identify two different types of directed branches; the n_s string branches (or vectors) $\{\mathbf{s}_n\}$ and the n_b bar branches (or vectors) $\{\mathbf{b}_m\}$.

Geometric Connectivity

Each directed branch can undergo a displacement in reaching its equilibrium state. String vectors can change both their length and orientation while rigid bar vectors can only change their orientation. Node vectors can change both their length and orientation but subject to a *Law of Geometric Connectivity* which we state as follows:

$$\textit{The vector sum of all branch vectors in any loop is zero.} \tag{1}$$

These equations are in the form of a set of linear algebraic equations in the branch vectors.

Force Equilibrium

In our study of tensegrity structures, we are concerned with structures in which bars sustain compressive forces. We therefore choose to distinguish between the string (or tensile) forces $\{\mathbf{t}_n\}$

and the bar (or compressive) forces $\{\mathbf{f}_m\}$ which are defined in terms of the string and bar vectors respectively as follows.

Definition 1 *Given the tensile force $\mathbf{t}_n$ in the string characterized by the string vector $\mathbf{s}_n$ and the compressive force $\mathbf{f}_n$ in the bar characterized by the bar vector $\mathbf{b}_n$, the tensile force coefficient $\gamma_m \geq 0$ and the compressive force coefficient $\lambda_n \geq 0$ are defined by*

$$\mathbf{t}_n = \gamma_n \mathbf{s}_n \ ; \quad \mathbf{f}_m = \lambda_m \mathbf{b}_m \tag{2}$$

The forces of the tensegrity structure are defined by the external force vector $\mathbf{w} \in \mathcal{R}^{3n_p}$, the compression vector $\mathbf{f} \in \mathcal{R}^{3n_b}$, and the tension vector $\mathbf{t} \in \mathcal{R}^{3n_s}$ where

$$\mathbf{w}^T = [\mathbf{w}_1^T, \mathbf{w}_2^T, \ \dots \ , \mathbf{w}_{n_p}^T] \ ; \ \mathbf{f}^T = [\mathbf{f}_1^T, \mathbf{f}_2^T, \ \dots \ , \mathbf{f}_{n_b}^T] \ ; \ \mathbf{t}^T = [\mathbf{t}_1^T, \mathbf{t}_2^T, \ \dots \ , \mathbf{t}_{n_s}^T] \tag{3}$$

Force Convention:

Suppose each node $\mathbf{p}_k$ is subjected to compressive vector forces $\{\mathbf{f}_{mk}\}$, tensile vector forces $\{\mathbf{t}_{nk}\}$ and external force $\mathbf{w}_k$. Then the *Law for Static Equilibrium* may be stated as follows:

$$\sum_n \mathbf{t}_{nk} - \sum_m \mathbf{f}_{mk} - \mathbf{w}_k = \mathbf{0} \tag{4}$$

where a positive sign is assigned to a (tensile, compressive, external) force *leaving* a node, and a negative sign is assigned to a (tensile, compressive, external) force *entering* a node. The negative sign in (4) is a consequence of the fact that we choose to define nonnegative compressive force coefficients λ_n.

Consider a class I tensegrity structure consisting of n_p nodes, n_b bars and n_s strings. Suppose the positions of the nodes are described by the n_p vectors $\{\mathbf{p}_1, \mathbf{p}_2, \ \dots \ , \mathbf{p}_{n_p}\}$, the positions of the bars are described by the n_b vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ \dots \ , \mathbf{b}_{n_b}\}$, and the positions of the strings are described by the n_s vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ \dots \ , \mathbf{s}_{n_s}\}$.

Definition 2 *The geometry of the tensegrity structure is defined by the tensegrity node vector $\mathbf{p} \in \mathcal{R}^{3n_p}$, the tensegrity bar vector $\mathbf{b} \in \mathcal{R}^{3n_b}$, and the tensegrity string vector $\mathbf{s} \in \mathcal{R}^{3n_s}$ where*

$$\mathbf{p}^T = [\mathbf{p}_1^T, \mathbf{p}_2^T, \ \dots \ , \mathbf{p}_{n_p}^T] \ ; \ \mathbf{b}^T = [\mathbf{b}_1^T, \mathbf{b}_2^T, \ \dots \ , \mathbf{b}_{n_b}^T] \ ; \ \mathbf{s}^T = [\mathbf{s}_1^T, \mathbf{s}_2^T, \ \dots \ , \mathbf{s}_{n_s}^T] \tag{5}$$

From the network, it follows that components of the string vector $\mathbf{s}$ and the bar vector $\mathbf{b}$ can be written as a linear combination of components of the node vector $\mathbf{p}$. Also if branch k is a bar which leaves node i and enters node j, then $\mathbf{b}_k = \mathbf{p}_j - \mathbf{p}_i$, whereas if branch k is a string which leaves node i and enters node j, then $\mathbf{s}_k = \mathbf{p}_j - \mathbf{p}_i$. Hence we have $\mathbf{A}^T\mathbf{p} = [\mathbf{s}^T, \mathbf{b}^T]^T$ where the matrix $\mathbf{A}$ consists only of block matrices of the form $\{\mathbf{0}, \pm\mathbf{I}_3\}$.

In particular, if we consecutively number the $n_s + n_b$ branches $\{\mathbf{s}_1, \mathbf{s}_2, \ \dots \ , \mathbf{s}_{n_s}, \mathbf{b}_1, \mathbf{b}_2, \ \dots \ , \mathbf{b}_{n_b}\}$ as $\{1, 2, \ \dots. \ , n_s, n_s + 1, \ \dots \ , n_s + n_b\}$, then the $3n_p \times (3n_s + 3n_b)$ matrix $\mathbf{A} = [\mathbf{A}_{ij}]$ is defined by

$$\mathbf{A}_{ij} = \begin{cases} \mathbf{I}_3 & \text{if tension (compression) } j \text{ } \textit{leaves (enters)} \text{ node } i \\ -\mathbf{I}_3 & \text{if tension (compression) } j \text{ } \textit{enters (leaves)} \text{ node } i \\ \mathbf{0} & \text{if tension (compression) } j \text{ } \textit{is not incident with} \text{ node } i \end{cases} \tag{6}$$

Also: (i) each column of $\mathbf{A}$ has exactly one block $\mathbf{I}_3$ and one block $-\mathbf{I}_3$ with all other column blocks $\mathbf{0}$, and (ii) for any row i there exists a column j such that $\mathbf{A}_{ij} = \pm\mathbf{I}_3$. Specifically, the "bar connectivity" matrix $\mathbf{B}$ and the "string connectivity" matrix $\mathbf{S}$ form the matrix $\mathbf{A}$ as follows.

$$\begin{bmatrix} \mathbf{s} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{S}^T \\ \mathbf{B}^T \end{bmatrix} \mathbf{p} = \mathbf{A}^T\mathbf{p} \tag{7}$$

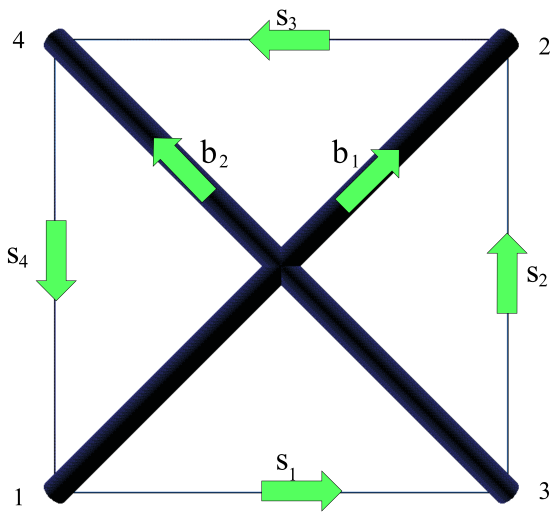


FIG. 1. 2-Bar 4-String Class I Tensegrity

In network analysis, the matrix $\mathbf{A}$ is known as the *incidence matrix*. This matrix is not the *reduced* incidence matrix since we have included the datum node which means that one block row of equations in (8) are dependent on the other rows. This fact does not cause any difficulties in subsequent developments. On the contrary, some symmetry is preserved in the algebraic equations. From Network analysis and the law for static equilibrium, we have the following result.

Lemma 1 *Consider a tensegrity structure as described by the geometric conditions given by (7). Then the equilibrium conditions for a class I tensegrity structure under the external load* $\mathbf{w}$ *are*

$$\mathbf{A}\begin{bmatrix} \mathbf{t} \\ -\mathbf{f} \end{bmatrix} = \mathbf{w}\ ;\ \ \mathbf{A} \triangleq [\mathbf{S}, \mathbf{B}] \tag{8}$$

or equivalently

$$\mathbf{St} = \mathbf{Bf} + \mathbf{w}. \tag{9}$$

Example 1 *Consider the 2-bar 4-string class I tensegrity structure illustrated in Figure 1 with tensile force vectors* $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\}$ *and compressive force vectors* $\{\mathbf{f}_1, \mathbf{f}_2\}$*. The Geometric Connectivity conditions are:*

$$\begin{aligned} -\mathbf{p}_1 + \mathbf{p}_3 &= \mathbf{s}_1\ ;\ \ \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{s}_2 \\ -\mathbf{p}_2 + \mathbf{p}_4 &= \mathbf{s}_3\ ;\ \ \mathbf{p}_1 - \mathbf{p}_4 = \mathbf{s}_4 \end{aligned} \tag{10}$$

Also, in terms of the stated force convention (4), the conditions for Static Equilibrium at nodes 1-4 are:

$$\begin{aligned} \mathbf{t}_1 - \mathbf{t}_4 - \mathbf{f}_1 - \mathbf{w}_1 &= \mathbf{0}\ ;\ \ -\mathbf{t}_2 + \mathbf{t}_3 + \mathbf{f}_1 - \mathbf{w}_2 = \mathbf{0} \\ -\mathbf{t}_1 + \mathbf{t}_2 - \mathbf{f}_2 - \mathbf{w}_3 &= \mathbf{0}\ ;\ \ -\mathbf{t}_3 + \mathbf{t}_4 + \mathbf{f}_2 - \mathbf{w}_4 = \mathbf{0} \end{aligned} \tag{11}$$

The static equilibrium conditions and the geometric conditions can be written in the form (6), (7), (8), where

$$\mathbf{S} = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \; ; \; \mathbf{B} = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \tag{12}$$

We now derive a *canonical network representation* in which the bar vectors are a subset of the transformed network nodes.

Lemma 2 *Given the tensegrity node vector* $\mathbf{p}$*, define the coordinate transformation*

$$\mathbf{p} = \mathbf{P}\mathbf{q} \tag{13}$$

for some nonsingular matrix $\mathbf{P} \in \mathcal{R}^{3n_p \times 3n_p}$ *where*

$$\mathbf{P} = [\mathbf{P}_1^T, \mathbf{P}_2^T] \; ; \; \mathbf{P}_1, \mathbf{P}_2 \in \mathcal{R}^{3n_b \times 3n_p} \tag{14}$$

Then:

(i) In terms of the transformed tensegrity node $\mathbf{q}$*, the tensegrity geometry is given by*

$$\mathbf{S}_q^T\mathbf{q} = \mathbf{s} \; ; \; \mathbf{B}_q^T\mathbf{q} = \mathbf{b} \tag{15}$$

where

$$\mathbf{S}_q^T = [\mathbf{S}_1^T, \mathbf{S}_2^T] \; ; \; \mathbf{B}_q^T = [\mathbf{B}_1^T, \mathbf{B}_2^T] \tag{16}$$

with

$$\mathbf{S}_1 = \mathbf{P}_1\mathbf{S} \; ; \; \mathbf{S}_2 = \mathbf{P}_2\mathbf{S} \; ; \; \mathbf{B}_1 = \mathbf{P}_1\mathbf{B} \; ; \; \mathbf{B}_2 = \mathbf{P}_2\mathbf{B} \tag{17}$$

(ii) The tensegrity force equilibrium is given by

$$\mathbf{S}_1\mathbf{t} = \mathbf{B}_1\mathbf{f} + \mathbf{P}_1\mathbf{w} \; ; \; \mathbf{S}_2\mathbf{t} = \mathbf{B}_2\mathbf{f} + \mathbf{P}_2\mathbf{w} \tag{18}$$

Proof : Part (i) follows directly from the definition of $\mathbf{P}$. Part (ii) follows from the expansion of the equilibrium condition $\mathbf{S}_q\mathbf{t} = \mathbf{B}_q\mathbf{f} + \mathbf{P}^T\mathbf{w}$. ∎

Since each bar has two end points, $n_p = 2n_b$. Without any loss of generality, we label the nodes of the bar $\mathbf{b}_m$ to be $\mathbf{p}_{2m}$ and $\mathbf{p}_{2m-1}$ for class I tensegrity structures, hence

$$\mathbf{b}_m = -\mathbf{p}_{2m-1} + \mathbf{p}_{2m} \; , \; m = 1, 2, \; ... \; , n_b.$$
$$\mathbf{B}^T = \text{blockdiag}\{\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}, \cdots, \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}\} \tag{19}$$

In particular, the following Lemma gives three special choices for coordinate transformations based on the special bar connectivity matrix given by (19).

Lemma 3 *The transformed equilibrium conditions (18) for all three coordinate transformations are given by*

$$\mathbf{S}_q\mathbf{t} - \mathbf{B}_q\mathbf{f} = \mathbf{P}^T\mathbf{w} \; ; \; \mathbf{S}_q^T = [\mathbf{S}_1^T, \mathbf{S}_2^T] \; ; \; \mathbf{B}_q^T = [\mathbf{I}_{n_b}, \mathbf{0}_{n_b}] \tag{20}$$

where $\mathbf{S}_1 \in \mathcal{R}^{3n_b \times 3n_s}$; $\mathbf{S}_2 \in \mathcal{R}^{3n_b \times 3n_s}$ *are given by (17).*

$$\mathbf{S}_1\mathbf{t} = \mathbf{f} + \mathbf{P}_1\mathbf{w} \; ; \; \mathbf{S}_2\mathbf{t} = \mathbf{P}_2\mathbf{w} \tag{21}$$

The transformed coordinate $\mathbf{q}$ *is given by*

$$\mathbf{q}^T = [\mathbf{b}^T, \mathbf{q}_e^T] \tag{22}$$

where $\mathbf{b}$ *is the bar vector and* $\mathbf{q}_e$ *is given in (26), (29), and (32).*

(i) Let the coordinate transformation matrix $\mathbf{P} \in \mathcal{R}^{3n_p \times 3n_p}$ *in (14) be given by*

$$\mathbf{P} = \left[\begin{array}{cc} \mathbf{P}_1^T & \mathbf{P}_2^T \end{array} \right] = \frac{1}{2} \left[\begin{array}{cc} \mathbf{B} & (\mathbf{I}_e - \mathbf{I}_o)\mathbf{B} \end{array} \right] \tag{23}$$

where $\mathbf{B}$ *is given by (19) and odd and even node selection matrix* $\mathbf{I}_o, \mathbf{I}_e \in \mathcal{R}^{3n_p \times 3n_p}$ *are defined by*

$$\mathbf{I}_o = \text{blockdiag}\{\mathbf{I}_3, \mathbf{0_3}, \cdots, \mathbf{I_3}, \mathbf{0_3}\} \ ; \quad \mathbf{I_e} = \text{blockdiag}\{\mathbf{0_3}, \mathbf{I_3}, \cdots, \mathbf{0_3}, \mathbf{I_3}\}. \tag{24}$$

Then the inverse transformation is

$$\mathbf{P}^{-1} = [\mathbf{P}_1^T, \mathbf{P}_2^T]^{-1} = 2 \left[\begin{array}{c} \mathbf{P}_1 \\ \mathbf{P}_2 \end{array} \right] = \left[\begin{array}{c} \mathbf{B}^T \\ \mathbf{B}^T(\mathbf{I}_e - \mathbf{I}_o) \end{array} \right] \tag{25}$$

$\mathbf{q}_e \in \mathcal{R}^{3n_b}$ *is the vector of the mass center of each of the bars given by*

$$\mathbf{q}_e^T = [\mathbf{p}_{c_1}^T, \mathbf{p}_{c_2}^T, \ \dots \ , \mathbf{p}_{c_{n_b}}^T] \tag{26}$$

where $\mathbf{p}_{c_j} \stackrel{\triangle}{=} \frac{1}{2}(\mathbf{p}_{2j} + \mathbf{p}_{2j-1})$.

(ii) Let the coordinate matrix $\mathbf{P} \in \mathcal{R}^{3n_p \times 3n_p}$ *in (14) be given by*

$$\mathbf{P} = \left[\begin{array}{cc} \mathbf{I}_o\mathbf{B} & (\mathbf{I}_e - \mathbf{I}_o)\,\mathbf{B} \end{array} \right] \tag{27}$$

where $\mathbf{B}$ *is given by (19) and odd and even node selection matrix* $\mathbf{I}_o, \mathbf{I}_e \in \mathcal{R}^{3n_p \times 3n_p}$ *are given by (24). Then the inverse transformation is*

$$\mathbf{P}^{-1} = [\mathbf{P}_1^T, \mathbf{P}_2^T]^{-1} = \left[\begin{array}{c} \mathbf{B}^T \\ \mathbf{B}^T\mathbf{I}_e \end{array} \right] \tag{28}$$

$\mathbf{q}_e \in \mathcal{R}^{3n_b}$ *is a vector of the even nodes given by*

$$\mathbf{q}_e^T = [\mathbf{p}_2^T, \mathbf{p}_4^T, \ \dots \ , \mathbf{p}_{2n_b}^T] \tag{29}$$

(iii) Let the coordinate matrix $\mathbf{P} \in \mathcal{R}^{3n_p \times 3n_p}$ *in (14) be given by*

$$\mathbf{P} = \left[\begin{array}{cc} \mathbf{I}_e\mathbf{B} & (\mathbf{I}_o - \mathbf{I}_e)\,\mathbf{B} \end{array} \right] \tag{30}$$

where $\mathbf{B}$ *is given by (19) and odd and even node selection matrix* $\mathbf{I}_o, \mathbf{I}_e \in \mathcal{R}^{3n_p \times 3n_p}$ *are given by (24). Then the inverse transformation is*

$$\mathbf{P}^{-1} = [\mathbf{P}_1^T, \mathbf{P}_2^T]^{-1} = \left[\begin{array}{c} \mathbf{B}^T \\ \mathbf{B}^T\mathbf{I}_o \end{array} \right] \tag{31}$$

$\mathbf{q}_e \in \mathcal{R}^{3n_b}$ *is a vector of the odd nodes given by*

$$\mathbf{q}_e^T = [\mathbf{p}_1^T, \mathbf{p}_3^T, \ \dots \ , \mathbf{p}_{2n_b-1}^T] \tag{32}$$

Proof : Proof of $\mathbf{P}\mathbf{P}^{-1} = \mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$ follows directly from identities

$$\mathbf{B}^T\mathbf{I}_e\mathbf{B} = \mathbf{B}^T\mathbf{I}_o\mathbf{B} = \mathbf{I} \ ; \quad \mathbf{I}_e\mathbf{I}_o = \mathbf{0} \ ; \quad \mathbf{I_e^2} = \mathbf{I_e} \ ; \quad \mathbf{I_o^2} = \mathbf{I_o}.$$

∎

Example 2 *The equations (10), (11) of the 2-bar 4-string tensegrity introduced in Example 1 can be written in the form (8) and (7) where*

$$\mathbf{B}^T = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \; ; \; \mathbf{S}^T = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}$$

Then from part (i) in Lemma 3, we have

$$\begin{aligned} \mathbf{P}_1 &= \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} \end{bmatrix} \; ; \; \mathbf{P}_2 = \begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \\ \mathbf{S}_1 &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix} \; ; \; \mathbf{S}_2 = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \end{aligned} \tag{33}$$

Or from part (ii) in Lemma 3, we have

$$\begin{aligned} \mathbf{P}_1 &= \frac{1}{2}\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \; ; \; \mathbf{P}_2 = \frac{1}{2}\begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \\ \mathbf{S}_1 &= \frac{1}{2}\begin{bmatrix} \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \; ; \; \mathbf{S}_2 = \frac{1}{2}\begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \end{aligned}$$

ANALYSIS OF THE TRANSFORMED EQUILIBRIUM CONDITIONS FOR A CLASS I TENSEGRITY STRUCTURE

Definition 3 *A tensegrity structure with tensile force coefficients $\{\gamma_n \geq 0\}$, compressive force coefficients $\{\lambda_n \geq 0\}$, node vector $\mathbf{p}$, string vector $\mathbf{s}$ and bar vector $\mathbf{b}$ is said to be in equilibrium if the element relationships (2), the force equations (9) and the geometric equations (7) are all satisfied.*

For the remainder of this paper, we choose to use the coordinate transformation derived in part (i) Lemma 3.

Requirements for Equilibrium: Given an external force vector $\mathbf{w}$, the problem of determining the geometric and force configuration of a tensegrity structure consisting of n_s strings and n_b bars in equilibrium is therefore equivalent to finding a solution $\mathbf{b}, \mathbf{q}_e \in \mathcal{R}^{3n_b}$ of the equations:

$$\begin{aligned} \mathbf{s} &= \mathbf{S}_1^T\mathbf{b} + \mathbf{S}_2^T\mathbf{q}_e && (34) \\ \mathbf{t} &= \mathbf{\Gamma}\mathbf{s} \; ; \; \mathbf{\Gamma} \stackrel{\Delta}{=} \mathrm{diag}\{\gamma_1\mathbf{I}_3, \gamma_2\mathbf{I}_3, \; ... \; , \gamma_{n_s}\mathbf{I}_3\} && (35) \\ \mathbf{S}_2\mathbf{t} &= \mathbf{P}_2\mathbf{w} && (36) \\ \mathbf{f} &= \mathbf{S}_1\mathbf{t} - \mathbf{P}_1\mathbf{w} && (37) \\ \mathbf{f} &= \mathbf{\Lambda}\mathbf{b} \; ; \; \mathbf{\Lambda} \stackrel{\Delta}{=} \mathrm{diag}\{\lambda_1\mathbf{I}_3, \lambda_2\mathbf{I}_3, \; ... \; , \lambda_{n_b}\mathbf{I}_3\} && (38) \end{aligned}$$

for given matrices

$$\mathbf{S}_1, \mathbf{S}_2 \in \mathcal{R}^{3n_b \times 3n_s} \; ; \; \mathbf{P}_1, \mathbf{P}_2 \in \mathcal{R}^{3n_b \times 3n_p} \tag{39}$$

where

$$n_p = 2n_b \; ; \; n_s > n_b \tag{40}$$

Beyond equilibrium requirements, one might require shape constraints by requiring $\mathbf{p}$ to take on a specific set of values $\mathbf{p} = \bar{\mathbf{p}}$, where $\mathbf{p} = \mathbf{P}\mathbf{q} = \mathbf{P}_1^T\mathbf{b} + \mathbf{P}_2^T\mathbf{q}_e$. Our focus in this paper is characterizing equilibria, hence the freedom in choosing the nodal vector $\mathbf{p}$ will appear as free variables in the vector $\mathbf{q}_e$, as the sequel shows. The diagonal matrices $\{\mathbf{\Gamma}, \mathbf{\Lambda}\}$ shall be referred to as the *tensile force matrix* and *compressive force matrix* respectively.

Prestressed Equilibrium(Unloaded Structure)

We now proceed to derive necessary and sufficient conditions for the existence of a structure in equilibrium that is prestressed in the absence of any external load (i.e. $\mathbf{w} = \mathbf{0}$) in (34) - (38). Our strategy for the examination of the conditions (34) - (38) is as follows. The solution of the linear algebra problem (36) yields nonunique $\mathbf{t}$ which lies in the right null space of $\mathbf{S}_2$. The existence condition of $\mathbf{q}_e$ for linear algebra problem (34) yields a condition on the left null space of $\mathbf{S}_2$. Hence (34) and (36) can be combined to obtain a unique expression for $\mathbf{t}$ in terms of $\mathbf{b}$. This is key to the main results of this paper.
We now establish necessary and sufficient conditions for a solution of equations (34)-(40) in the absence of external forces (i.e. $\mathbf{w} = \mathbf{0}$) by examining each of these equations in turn beginning with the solution of (34), (36).

Lemma 4 *Suppose*

$$r \triangleq \rho(\mathbf{S}_2) \leq \min\{3n_p, 3n_s\} \tag{41}$$

and let $\mathbf{S}_2$ *have the singular value decomposition* $\{\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}\}$ *given by*

$$\mathbf{S}_2 = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \in \mathcal{R}^{3n_b \times 3n_s} \tag{42}$$

where

$$\mathbf{U} \;=\; [\mathbf{U}_1, \mathbf{U}_2] \;;\; \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \;;\; \mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2]$$

$$\mathbf{U}_1 \in \mathcal{R}^{3n_b \times r} \;;\; \mathbf{U}_2 \in \mathcal{R}^{3n_b \times (3n_b - r)} \;;\; \mathbf{V}_1 \in \mathcal{R}^{3n_s \times r} \;;\; \mathbf{V}_2 \in \mathcal{R}^{3n_s \times (3n_s - r)}$$

Then a necessary and sufficient condition for (34) to have a solution $\mathbf{q}_e \in \mathcal{R}^{3n_b}$ *is given by*

$$\mathbf{V}_2^T(\mathbf{s} - \mathbf{S}_1^T\mathbf{b}) = \mathbf{0} \tag{43}$$

Furthermore, when (43) is satisfied, all solutions $\mathbf{q}_e$ *are of the form*

$$\mathbf{q}_e = \mathbf{U}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{V}_1^T(\mathbf{s} - \mathbf{S}_1^T\mathbf{b}) + \mathbf{U}_2\mathbf{z}_e \tag{44}$$

where $\mathbf{z}_e \in \mathcal{R}^{3n_b - r}$ *is arbitrary.*

We now consider the solution of (36) when $\mathbf{w} = \mathbf{0}$.

Lemma 5 *When* $\mathbf{w} = \mathbf{0}$*, all solutions* $\mathbf{t}$ *of (36) which guarantee (43) are of the form*

$$\mathbf{t} = \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b} \;;\; \mathbf{M} \triangleq \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \tag{45}$$

Proof : From (42), (43) , we have $\mathbf{t} = \mathbf{V}_2\mathbf{z}_t$ where $\mathbf{z}_t$ is the free solution of (36). Then from (35)

$$\mathbf{V}_2^T(\mathbf{s} - \mathbf{S}_1^T\mathbf{b}) = \mathbf{V}_2^T(\mathbf{\Gamma}^{-1}\mathbf{V}_2\mathbf{z}_t - \mathbf{S}_1^T\mathbf{b})$$

Since $\mathbf{V}_2$ has full column rank, the matrix $\mathbf{M} = \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2$ is invertible if it exists (that is, if $\gamma_n \neq 0,\ n = 1, \cdots, n_s$). Hence (43) is satisfied when $\mathbf{z}_t = \mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b}$, and this gives (45). ■

We now consider the solution of (37), (38) when $\mathbf{w} = \mathbf{0}$.

Lemma 6 *When $\mathbf{w} = \mathbf{0}$, a necessary and sufficient condition for (37), (38) to have a solution $\mathbf{b} \in \mathcal{R}^{3n_b}$ is given by*

$$(\mathbf{X} - \mathbf{\Lambda})\mathbf{b} = \mathbf{0} \tag{46}$$

where

$$\mathbf{X} \stackrel{\Delta}{=} (\mathbf{S}_1\mathbf{V}_2)\mathbf{M}^{-1}(\mathbf{S}_1\mathbf{V}_2)^T \ ;\ \mathbf{M} \stackrel{\Delta}{=} \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \tag{47}$$

In particular, define

$$r_b \stackrel{\Delta}{=} \rho(\mathbf{X} - \mathbf{\Lambda}) \tag{48}$$

Then:

(i) When $r_b = 3n_b$, then $\mathbf{b} = \mathbf{0}$ is the only solution of (46),
(ii) When $r_b = 0$, any $\mathbf{b} \in \mathcal{R}^{3n_b}$ is a solution of (46), and
(iii) When $0 < r_b < 3n_b$, all solutions $\mathbf{b}$ satisfy the equation

$$[\mathbf{I} - \mathbf{V}_{X1}\mathbf{\Sigma}_{X11}^{-1}\mathbf{U}_{X1}^T\mathbf{\Lambda}]\mathbf{b} = \mathbf{V}_{X2}\mathbf{z}_b \tag{49}$$

where $\mathbf{z}_b \in \mathcal{R}^{3n_b - r_b}$ is free, and where $\{\mathbf{U}_X, \mathbf{\Sigma}_X, \mathbf{V}_X\}$ is the singular value decomposition of the matrix $\mathbf{X} \in \mathcal{R}^{3n_b \times 3n_b}$; that is

$$\mathbf{U}_X = [\mathbf{U}_{X1}, \mathbf{U}_{X2}] \ ;\ \mathbf{\Sigma}_X = \begin{bmatrix} \mathbf{\Sigma}_{X11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ ;\ \mathbf{V}_X = [\mathbf{V}_{X1}, \mathbf{V}_{X2}] \ ;\ r_X = rank(\mathbf{X})$$

with $\mathbf{U}_{X1} \in \mathcal{R}^{3n_b \times r_X}$, $\mathbf{V}_{X2} \in \mathcal{R}^{3n_b \times (3n_b - r_X)}$.

Proof : From (45), (38) and (37)

$$\mathbf{S}_1\mathbf{t} - \mathbf{f} = \mathbf{S}_1\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b} - \mathbf{\Lambda}\mathbf{b} = (\mathbf{X} - \mathbf{\Lambda})\mathbf{b}$$

The result then follows from the singular value decomposition of $\mathbf{X}$ after writing (46) in the form $\mathbf{X}\mathbf{b} = \mathbf{f},\ \mathbf{f} = \mathbf{\Lambda}\mathbf{b}$. ■

For the establishment of a non-trivial tensegrity structure, it is necessary that the node component vectors p_k are not required to lie on any line for a planar tensegrity structure and it is necessary that the node component vectors p_k are not required to lie on any plane for a spatial tensegrity structure. Hence we establish the following Corollary.

Corollary 1 *When $\mathbf{w} = \mathbf{0}$, a necessary and sufficient condition for (37), (38) to have a nontrivial solution $\mathbf{b} \in \mathcal{R}^{3n_b}$ is given by*

$$(\mathbf{X} - \mathbf{\Lambda})\mathbf{b} = \mathbf{0} \quad \mathbf{r_b} \stackrel{\Delta}{=} \rho(\mathbf{X} - \mathbf{\Lambda}) \leq (\mathbf{n_b} - \mathbf{2})\mathbf{d}$$

where dimension $\mathbf{d}$ is 2 for planar tensegrity structure, and 3 for spatial tensegrity structure.

Proof : When $r_b = n_b \times d$, $\mathbf{b} = \mathbf{0}$ is the only solution. When $r_b = (n_b - 1) \times d$, all bar vectors are parallel.

Corollary 2 *If $\mathbf{\Gamma} = \gamma\mathbf{I}$, and $\mathbf{\Lambda} = \lambda\mathbf{I}$, then all unforced tensegrity equilibria are characterized by the modal data of the matrix $\bar{\mathbf{X}} = \mathbf{S}_1\mathbf{V}_2\mathbf{V}_2^T\mathbf{S}_1^T$. That is, all admissible values of $\frac{\lambda}{\gamma}$ are eigenvalues of $\bar{\mathbf{X}}$, and all bar vectors are eigenvectors of $\bar{\mathbf{X}}$.*

Proof : From (46) and $\mathbf{\Gamma} = \gamma\mathbf{I}, \mathbf{\Lambda} = \lambda\mathbf{I}$, $\mathbf{X} = \gamma\mathbf{S}_1\mathbf{V}_2\mathbf{V}_2^T\mathbf{S}_1^T = \gamma\bar{\mathbf{X}}$. Then $\left(\gamma\bar{\mathbf{X}} - \lambda\mathbf{I}\right)\mathbf{b} = \mathbf{0}$ or $\left(\bar{\mathbf{X}} - \rho\mathbf{I}\right)\mathbf{b} = \mathbf{0}$, where $\rho = \frac{\lambda}{\gamma}$. Hence ρ and $\mathbf{b}$ are the eigenvalues and eigenvectors of $\bar{\mathbf{X}}$. ∎

Lemma 7 *Consider the partitioned matrix*

$$\mathcal{A} = \left[\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array}\right].$$

Then

$$\begin{array}{rcl} (i)\ \ \det\mathcal{A} & = & \det\left[\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array}\right] = \det[\mathbf{A}]\det[\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}] \ \text{ if } \ \det[\mathbf{A}] \neq 0. \\ (ii)\ \ \det\mathcal{A} & = & \det\left[\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array}\right] = \det[\mathbf{D}]\det[\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}] \ \text{ if } \ \det[\mathbf{D}] \neq 0. \end{array}$$

Now $\mathbf{\Gamma}$ positive definite implies $\mathbf{M}$ in (47) is positive definite. Then $\mathbf{M}$ positive definite implies $\mathbf{X}$ in (47) is positive semi-definite with at least one non-zero eigenvalue. Since (46) has a nonzero solution $\mathbf{b}$ if and only if $\det(\mathbf{X} - \mathbf{\Lambda}) = 0$, the following Lemma is important.

Lemma 8 *Given a nonzero symmetric positive semi-definite matrix* $\mathbf{X}$ *and a positive definite diagonal matrix* $\mathbf{\Lambda}$*, the following statements are equivalent.*

(i) $\det(\mathbf{X} - \mathbf{\Lambda}) = 0$*, where* $\mathbf{X} = \mathbf{S}_1\mathbf{V}_2\left(\mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2\right)^{-1}\mathbf{V}_2^T\mathbf{S}_1^T$.

(ii) $\det\left\{\left[\begin{array}{cc} \mathbf{\Lambda} & \mathbf{S}_1\mathbf{V}_2 \\ \mathbf{V}_2^T\mathbf{S}_1^T & \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \end{array}\right]\right\} = 0.$

(iii) $\det\left\{\left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^T \end{array}\right]\left[\begin{array}{cc} \mathbf{\Lambda} & \mathbf{S}_1 \\ \mathbf{S}_1^T & \mathbf{\Gamma}^{-1} \end{array}\right]\left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 \end{array}\right]\right\} = 0.$

(iv) $\det\left\{\mathbf{V}_2^T\left(\mathbf{\Gamma}^{-1} - \mathbf{S}_1^T\mathbf{\Lambda}^{-1}\mathbf{S}_1\right)\mathbf{V}_2\right\} = 0.$

Proof : Since $\mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2$ is a positive definite matrix, the solutions of $\det\{\mathbf{X} - \mathbf{\Lambda}\} = 0$ are not affected by multiplying by $\det\{\mathbf{\Lambda}\}$. This yields

$$\det\left\{\mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2\right\}\det\{\mathbf{X} - \mathbf{\Lambda}\} = 0.$$

Now we can apply (ii) in Lemma 7 by setting $\mathbf{A} \triangleq \mathbf{\Lambda}$, $\mathbf{B} \triangleq \mathbf{S}_1\mathbf{V}_2$, $\mathbf{C} = \mathbf{V}_2^T\mathbf{S}_1^T$, and $\mathbf{D} \triangleq \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2$, which proves part (ii). Part (iii) can be easily proved by the equality given by

$$\mathcal{A} \triangleq \left[\begin{array}{cc} \mathbf{\Lambda} & \mathbf{S}_1\mathbf{V}_2 \\ \mathbf{V}_2^T\mathbf{S}_1^T & \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \end{array}\right] = \left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^T \end{array}\right]\left[\begin{array}{cc} \mathbf{\Lambda} & \mathbf{S}_1 \\ \mathbf{S}_1^T & \mathbf{\Gamma}^{-1} \end{array}\right]\left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 \end{array}\right].$$

Since $\mathbf{\Lambda}$ is a positive definite matrix, applying (i) in Lemma 7 to $\mathcal{A}$ yields (iv). ∎

Remark 1 *For* $0 < r_b < 3n_b$*,* $(\mathbf{X} - \mathbf{\Lambda})\,\mathbf{b} = \mathbf{0}$ *always has at least one solution* $\mathbf{b} \neq \mathbf{0}$ *of the form (49) where the freedom in the choice of* $\mathbf{b}$ *is available in the choice of a free vector* $\mathbf{z}_b$*.*

Theorem 1 *Consider a class I tensegrity structure as defined by the geometry and force equations in the absence of external load as described by the geometric conditions*

$$\mathbf{B}^T\mathbf{p} = \mathbf{b}\ ;\ \mathbf{S}^T\mathbf{p} = \mathbf{s}\ ;\ \mathbf{p} \in \mathcal{R}^{3n_p},\ \mathbf{b} \in \mathcal{R}^{3n_b},\ \mathbf{s} \in \mathcal{R}^{3n_s}$$

with $n_p = 2n_b$*, and the equilibrium conditions*

$$\mathbf{S}\mathbf{t} = \mathbf{B}\mathbf{f}\ ;\ \mathbf{t} = \mathbf{\Gamma}\mathbf{s},\ \mathbf{f} = \mathbf{\Lambda}\mathbf{b}$$

where for $\{\gamma_m \geq 0,\ \lambda_n \geq 0\}$,

$$\mathbf{\Gamma} = \mathrm{diag}\{\gamma_1\mathbf{I}_3, \gamma_2\mathbf{I}_3,\ \ldots\ , \gamma_{n_s}\mathbf{I}_3\}\ ;\ \mathbf{\Lambda} = \mathrm{diag}\{\lambda_1\mathbf{I}_3, \lambda_2\mathbf{I}_3,\ \ldots\ , \lambda_{n_b}\mathbf{I}_3\}.$$

Then given any tensile force coefficients $\{\gamma_m \geq 0,\ 1 \leq m \leq n_s\}$, *there exits compressive force coefficients* $\{\lambda_n \geq 0,\ 1 \leq m \leq n_b\}$ *and* $\mathbf{b}$, *which define an equilibrium structure, satisfy the condition:*

$$(\mathbf{X} - \mathbf{\Lambda})\,\mathbf{b} = \mathbf{0}\ ;\ \mathbf{X} \triangleq (\mathbf{S_1 V_2})\mathbf{M^{-1}}(\mathbf{S_1 V_2})^{\mathbf{T}},\ \mathbf{M} \triangleq \mathbf{V_2^T \Gamma^{-1} V_2}, \tag{50}$$

where $\mathbf{S}_1$ *is given by (17), and* $\mathbf{V}_2$ *is given by (43).*

Moreover, for any $\mathbf{b}$ *and* $\mathbf{\Gamma}$ *satisfying (50), the nodal vector* $\mathbf{p}$ *is of the form*

$$\mathbf{p} = \mathbf{PQ}[\mathbf{b}^T, \mathbf{z}_e^T]^T \tag{51}$$

where $\mathbf{z}_e$ *may be arbitrarily chosen and*

$$\mathbf{Q} \triangleq \begin{bmatrix} \mathbf{I}_{3n_b} & \mathbf{0} \\ \mathbf{L} & \mathbf{U}_2 \end{bmatrix}\ ;\ \mathbf{L} \triangleq \mathbf{U}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{V}_1^T[\mathbf{\Gamma}^{-1}\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T - \mathbf{I}_{3n_s}]\mathbf{S}_1^T. \tag{52}$$

The corresponding tensegrity tension vector $\mathbf{t}$, *string vector* $\mathbf{s}$, *and compression vector* $\mathbf{f}$ *are given in terms of the bar vector* $\mathbf{b}$ *by*

$$\mathbf{t} = \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b}\ ;\ \mathbf{s} = \mathbf{\Gamma}^{-1}\mathbf{t}\ ;\ \mathbf{f} = \mathbf{\Lambda}\mathbf{b} \tag{53}$$

Externally Loaded Structures

Under the action of an external force vector $\mathbf{w}$ with component vectors $\{\mathbf{w}_j\}$ given by

$$\mathbf{w}^T = [\mathbf{w}_1^T, \mathbf{w}_2^T,\ \ldots\ , \mathbf{w}_{n_p}^T] \tag{54}$$

suppose that the new equilibrium structure is assumed to be given by node vector $\mathbf{p}$, bar vector $\mathbf{b}$, string vector $\mathbf{s}$, compressive force vector $\mathbf{f}$, tensile vector $\mathbf{t}$, compressive force coefficient matrix $\mathbf{\Lambda}$ and tensile force matrix $\mathbf{\Gamma}$ as described by (34)-(38). Note that all force coefficients together with all node geometry will normally change. We now seek necessary and sufficient conditions for the externally loaded structure to be in equilibrium. An extension of Lemma 5 and Lemma 6 gives us the following result.

Theorem 2 *(i) All solutions* $\mathbf{t}$ *of (36) which guarantee (43) are of the form*

$$\begin{aligned} \mathbf{t} &= \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\mathbf{b} + \mathbf{D}_1\mathbf{w}\ ;\ \mathbf{M} \triangleq \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \\ \mathbf{D}_1 &= (\mathbf{I}_{3n_s} - \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^T\mathbf{\Gamma}^{-1})\mathbf{V}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{U}_1^T\mathbf{P}_2 \end{aligned} \tag{55}$$

(ii) A necessary and sufficient condition for (37), (38) to have a solution $\mathbf{b} \in \mathcal{R}^{3n_b}$ *is given by*

$$\begin{aligned} (\mathbf{X} - \mathbf{\Lambda})\mathbf{b} &= \mathbf{D}\mathbf{w} \\ \mathbf{U}_2^T\mathbf{P}_2\mathbf{w} &= \mathbf{0} \end{aligned} \tag{56}$$

where $\mathbf{X}$ *is given by (47), and*

$$\mathbf{D} \triangleq \mathbf{P}_1 - \mathbf{S}_1\mathbf{D}_1 \tag{57}$$

Proof : Since (34) is not directly dependent on $\mathbf{w}$, Lemma 4 applies for $\mathbf{w} \neq \mathbf{0}$. Now consider the solution of (36) for $\mathbf{w} \neq \mathbf{0}$. A necessary condition for the existence of a solution $\mathbf{t}$ is $\mathbf{U}_2^T \mathbf{P}_2 \mathbf{w} = \mathbf{0}$, and in this case, all solutions $\mathbf{t}$ are of the form

$$\mathbf{t} = \mathbf{V}_2 \mathbf{z}_t + \mathbf{V}_1 \mathbf{\Sigma}_{11}^{-1} \mathbf{U}_1^T \mathbf{P}_2 \mathbf{w}$$

for any $\mathbf{z}_t \in \mathcal{R}^{3n_b - r}$ where as in (41), r is the rank of $\mathbf{S}_2$. Now in order that condition (43) is satisfied, $\mathbf{z}_t$ must be selected such that

$$\mathbf{V}_2^T \mathbf{\Gamma}^{-1} (\mathbf{V}_2 \mathbf{z}_t + \mathbf{V}_1 \mathbf{\Sigma}_{11}^{-1} \mathbf{U}_1^T \mathbf{P}_2 \mathbf{w}) - \mathbf{V}_2^T \mathbf{S}_1^T \mathbf{b} = \mathbf{0}$$

That is

$$\mathbf{z}_t = \mathbf{M}^{-1} \mathbf{V}_2^T \mathbf{S}_1^T \mathbf{b} - \mathbf{M}^{-1} \mathbf{V}_2^T \mathbf{\Gamma}^{-1} \mathbf{V}_1 \mathbf{\Sigma}_{11}^{-1} \mathbf{U}_1^T \mathbf{P}_2 \mathbf{w}$$

which gives (55). From (55), (38) and (36)

$$\mathbf{0} = \mathbf{S_1 t} - \mathbf{P_1 w} - \mathbf{f} = \mathbf{S_1 V_2 M^{-1} V_2^T S_1^T b} + \mathbf{S_1 D_1 w} - \mathbf{P_1 w} - \mathbf{\Lambda b}$$

which gives (56). ∎

The first condition in (56) is a nonhomogeneous equivalent of condition (46). However it is unlikely (although not impossible) that $\mathbf{\Lambda} = \mathbf{X}$ for $\mathbf{w} \neq \mathbf{0}$. Instead, is more likely that $r_b \triangleq \rho(\mathbf{X} - \mathbf{\Lambda})$ satisfies $0 < r_b \leq 3n_b$. If $r_b = 3n_b$, then $\mathbf{b} = (\mathbf{X} - \mathbf{\Lambda})^{-1} \mathbf{D} \mathbf{w}$ is unique.

When $0 < r_b < 3n_b$, suppose $\{\mathbf{U}_b, \mathbf{\Sigma}_b, \mathbf{V}_b\}$ is the singular value decomposition of the matrix $\mathbf{X} - \mathbf{\Lambda} \in \mathcal{R}^{3n_b \times 3n_b}$; that is

$$\mathbf{U}_b = [\mathbf{U}_{b1}, \mathbf{U}_{b2}] \ ; \ \mathbf{\Sigma}_b = \begin{bmatrix} \mathbf{\Sigma}_{b11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ ; \ \mathbf{V}_b = [\mathbf{V}_{b1}, \mathbf{V}_{b2}]$$

with $\mathbf{U}_{b1} \in \mathcal{R}^{3n_b \times r_b}$, $\mathbf{V}_{b2} \in \mathcal{R}^{3n_b \times (3n_b - r_b)}$. Then when $\mathbf{U}_2^T \mathbf{D} \mathbf{w} = \mathbf{0}$, the solution $\mathbf{b}$ is of the form

$$\mathbf{b} = \mathbf{V}_{b2} \mathbf{z}_b + \mathbf{V}_{b1} \mathbf{\Sigma}_{b11}^{-1} \mathbf{U}_{b1}^T \mathbf{D} \mathbf{w} \ ; \ \mathbf{U}_2^T \mathbf{D} \mathbf{w} = \mathbf{0}$$

where $\mathbf{z}_b \in \mathcal{R}^{3n_b - r_b}$ is arbitrary. The particular equilibrium obtained will depend on the way in which the external load $\mathbf{w}$ is introduced. The structural implications of the null space condition $\mathbf{U}_2^T \mathbf{D} \mathbf{w} = \mathbf{0}$ on the external load $\mathbf{w}$ would then also require a physical interpretation.

The *existence* of an equilibrium solution however requires the *second condition* in (56) on the external force $\mathbf{w}$ to be satisfied. In this regard, we have the following result.

Lemma 9 *For all structures* $\{\mathbf{S}, \mathbf{B}\}$, *the* $(3n_b - r) \times 3n_s$ *matrix product* $\mathbf{U}_2^T \mathbf{P}_2$ *is of the form*

$$\mathbf{U}_2^T = \mathbf{E}[\mathbf{I}_3, \mathbf{I}_3, \ \dots \ , \mathbf{I}_3] \tag{58}$$

for some nonsingular matrix $\mathbf{E}$. *Hence* $\mathbf{U}_2^T \mathbf{P}_2 \mathbf{w} = \mathbf{0}$ *if and only if*

$$\sum_{k=1}^{n_p} \mathbf{w}_k = \mathbf{0} \tag{59}$$

Proof : It follows from svd($\mathbf{S}_2$) in Lemma 4 that $\mathbf{U}_2^T \mathbf{S}_2 = \mathbf{0}$ which from (17) implies $\mathbf{U}_2^T \mathbf{P}_2 \mathbf{S} = \mathbf{0}$. Now from Lemma 1, each column of $\mathbf{S}$ has exactly one block $\mathbf{I}_3$ and exactly one block $-\mathbf{I}_3$ with all other column elements $\mathbf{0}$. Furthermore, for every ith row of $\mathbf{S}$, there exists a column j such that the ijth component of $\mathbf{S}$ is $\pm \mathbf{I}_3$. These properties of $\mathbf{S}$ then imply that $\mathbf{U}_2^T \mathbf{P}_2$ is of the form (58) for some (but not necessarily nonsingular) matrix $\mathbf{E}$, and so

$$\mathbf{U}_2^T\mathbf{P}_2\mathbf{w} = \mathbf{E}\sum_{k=1}^{n_p}\mathbf{w}_k$$

Now if the full row rank matrix $\mathbf{U}_2$ is partitioned into the block form $\mathbf{U}_2^T = [\mathbf{Z}_1, \mathbf{Z}_2, \ \dots \ , \mathbf{Z}_{n_b}]$ it follows from (23) that

$$\mathbf{U}_2^T\mathbf{P}_2 = [\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_2 \ \dots \ , \mathbf{Z}_{n_b}, \mathbf{Z}_{n_b}]$$

which then guarantees that $\mathbf{U}_2^T\mathbf{P}_2$ also has full row rank, and consequently that the matrix $\mathbf{E}$ is invertible. ∎

Condition (59) expresses the requirement that for an externally loaded tensegrity structure to be in equilibrium, it is *necessary* (but not sufficient) that the sum of the external forces is zero.

COMPUTATIONAL ALGORITHM FOR EQUILIBRIA

One suggested procedure for construction of a class I tensegrity structure in equilibrium is provided as follows.

Design Algorithm A :

step 1. Given the connectivity matrices $\mathbf{S}$ and $\mathbf{B}$ from the network topology, find a nonsingular matrix $\mathbf{P} = [\mathbf{P}_1^T, \mathbf{P}_2^T]$ such that $\mathbf{B}_q^T = \mathbf{B}^T\mathbf{P} = [\mathbf{I}_{n_b}, \mathbf{0}_{n_b}]$, and calculate $\{\mathbf{S}_1 = \mathbf{P}_1\mathbf{S}, \ \mathbf{S}_2 = \mathbf{P}_2\mathbf{S}\}$. (Note that if $\mathbf{B}$ is defined as in (19), then $\{\mathbf{P}_1, \mathbf{P}_2\}$ are given by (23) ,(27), or (30).

step 2. Choose $\{\gamma_m > 0\}$ and $\{\lambda_n > 0\}$ such that $\det(\mathbf{X} - \mathbf{\Lambda}) = 0$.

step 3. Select suitable $\mathbf{b}_z$ and compute $\mathbf{b} = (\mathbf{X} - \mathbf{\Lambda})^{\perp}\mathbf{b}_z$ based on a desired bar vector.

step 4. Choose the free vector $\mathbf{z}_e$ in (51), (52) to give a suitable node vector $\mathbf{p}$ based on a desired shape.

step 5. calculate $\{\mathbf{t}, \mathbf{s}, \mathbf{f}\}$ from (53).

Design Algorithm for Given Bar Vector b

In the design of tensegrity structures, an alternative approach for the selection of the node vector $\mathbf{p}$ and bar vector $\mathbf{b}$ may be useful. From (50), it is easy to eliminate the variable $\mathbf{\Lambda}$ if $\mathbf{b}$ is known, since

$$\mathbf{\Lambda}\mathbf{b} = \tilde{\mathbf{b}}\lambda$$

where $\tilde{\mathbf{b}} \triangleq \mathrm{blockdiag}\,[\mathbf{b}_1, \cdots, \mathbf{b}_{n_b}]$; $\lambda \triangleq [\lambda_1, \cdots, \lambda_{n_b}]^T$. Hence, we can establish the following Lemma.

Lemma 10 *For any tensegrity equilibrium, λ is given uniquely by*

$$\lambda = \tilde{\mathbf{b}}^{+}\mathbf{X}\mathbf{b}, \tag{60}$$

where $\tilde{\mathbf{b}}^{+} = \left(\tilde{\mathbf{b}}^T\tilde{\mathbf{b}}\right)^{-1}\tilde{\mathbf{b}}^T$ and $\mathbf{b}$ and $\mathbf{\Gamma}$ for all tensegrity structures must satisfy

$$\tilde{\mathbf{b}}_L\mathbf{X}\mathbf{b} = \mathbf{0}. \tag{61}$$

where $\tilde{\mathbf{b}}_L$ is the left null space basis of $\tilde{\mathbf{b}}$ and satisfies

$$\tilde{\mathbf{b}}_L\tilde{\mathbf{b}} = \mathbf{0} \ ; \quad \tilde{\mathbf{b}}_{\mathbf{L}}\tilde{\mathbf{b}}_{\mathbf{L}}^{\mathbf{T}} > \mathbf{0}.$$

Proof : From (50), note that

$$\mathbf{X}\mathbf{b} = \mathbf{\Lambda}\mathbf{b} = \tilde{\mathbf{b}}\lambda$$

has a solution λ if and only if (61) holds. In this case, the unique solution is given by (60) assuming $\tilde{\mathbf{b}}$ has full column rank. This assumption is guaranteed for $\mathbf{b}_i \neq \mathbf{0}$, $i = 1, \cdots, n_b$. ∎

Note that (61) is equivalent to

$$\left(\mathbf{I} - \tilde{\mathbf{b}}\tilde{\mathbf{b}}^{+}\right)\mathbf{X}\mathbf{b} = \mathbf{0} \tag{62}$$

since $\left(\mathbf{I} - \tilde{\mathbf{b}}\tilde{\mathbf{b}}^{+}\right)$ forms the left null space of $\tilde{\mathbf{b}}$.

Design Algorithm B :

step 1. Same as in Design Algorithm A.
step 2. Choose $\mathbf{b}$ and then compute $\mathbf{\Gamma}$ to satisfy (61).
step 3. Compute λ from (60).
step 4. Choose the free vector $\mathbf{z}_e$ in (51), (52) to give a suitable node vector $\mathbf{p}$ based on a desired shape.
step 5. calculate $\{\mathbf{t}, \mathbf{s}, \mathbf{f}\}$ from (53).

Design Algorithm for Given Nodal Vector p

One can first select the node vector $\mathbf{p}$ as in (51), (52) subject (if necessary) to other design constraints. The corresponding bar vector $\mathbf{b}$ is then determined from the first $3n_b$ components of the vector $\mathbf{P}^{-1}\mathbf{p}$. When the node position vector $\mathbf{p}$ is given, we can establish following lemma.

Lemma 11 *A given node position vector* $\mathbf{p}$ *corresponds to a class I tensegrity equilibrium if and only if there exists a nonzero vector* $\mathbf{z}$ *such that*

$$\begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{S}\tilde{\mathbf{s}} & -\mathbf{B}\tilde{\mathbf{b}} \end{bmatrix}^{\perp} \mathbf{z} > \mathbf{0}, \tag{63}$$

where

$$\tilde{\mathbf{b}} \triangleq \text{blockdiag}\left[\mathbf{b}_1, \cdots, \mathbf{b}_{n_b}\right] \ ; \quad \tilde{\mathbf{s}} \triangleq \text{blockdiag}\left[\mathbf{s}_1, \cdots, \mathbf{s}_{n_s}\right].$$

Proof : Using $\mathbf{f} = \mathbf{\Lambda}\mathbf{b} = \tilde{\mathbf{b}}\lambda$ and $\mathbf{t} = \mathbf{\Gamma}\mathbf{s} = \tilde{\mathbf{s}}\gamma$ leads to the equilibrium condition

$$\mathbf{S}\mathbf{t} - \mathbf{B}\mathbf{f} = \begin{bmatrix} \mathbf{S}\tilde{\mathbf{s}} & -\mathbf{B}\tilde{\mathbf{b}} \end{bmatrix} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = \mathbf{0}. \tag{64}$$

The solutions of (64) for positive γ and λ yields (63). ∎

When we premultiply (64) by the coordinate transformation matrix $\mathbf{P}^T$ given by (13), we can further simplify (64). Hence we establish the following Corollary.

Corollary 3 *A given node position vector* $\mathbf{p}$ *corresponds to a class I tensegrity equilibrium if and only if there exists a nonzero vector* $\mathbf{z}$ *such that*

$$\begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \tilde{\mathbf{b}}^{+}\mathbf{S}_1\tilde{\mathbf{s}} \end{bmatrix} \left(\mathbf{S}_2\tilde{\mathbf{s}}\right)^{\perp} \left[\tilde{\mathbf{b}}_L \mathbf{S}_1 \tilde{\mathbf{s}} \left(\mathbf{S}_2\tilde{\mathbf{s}}\right)^{\perp}\right]^{\perp} \mathbf{z} > \mathbf{0}. \tag{65}$$

Proof :

Premultiplying by the coordinate transformation matrix $\mathbf{P}^T$ given by (23) to (64) and using (20) yields

$$\mathbf{P}^T(\mathbf{St}-\mathbf{Bf}) = \mathbf{S}_q\mathbf{t}-\mathbf{B}_q\mathbf{f} = \begin{bmatrix} \mathbf{S}_q\tilde{\mathbf{s}} & -\mathbf{B}_q\tilde{\mathbf{b}} \end{bmatrix}\begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1\tilde{\mathbf{s}} & -\tilde{\mathbf{b}} \\ \mathbf{S}_2\tilde{\mathbf{s}} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = \mathbf{0}.$$

Hence the existence condition for γ and λ require

$$(\mathbf{S}_2\tilde{\mathbf{s}})^{\perp} \neq \mathbf{0}\ ;\quad \tilde{\mathbf{b}}_{\mathbf{L}}\mathbf{S}_1\tilde{\mathbf{s}}\gamma = \mathbf{0}. \tag{66}$$

Then, all admissible force coefficients are

$$\begin{aligned} \gamma &= (\mathbf{S}_2\tilde{\mathbf{s}})^{\perp}\mathbf{z}_r \\ \lambda &= \tilde{\mathbf{b}}^{+}\mathbf{S}_1\tilde{\mathbf{s}}\gamma = \tilde{\mathbf{b}}^{+}\mathbf{S}_1\tilde{\mathbf{s}}(\mathbf{S}_2\tilde{\mathbf{s}})^{\perp}\mathbf{z}_r. \end{aligned} \tag{67}$$

Substituting γ in (67) into the existence condition (66) yields

$$\tilde{\mathbf{b}}_L\mathbf{S}_1\tilde{\mathbf{s}}(\mathbf{S}_2\tilde{\mathbf{s}})^{\perp}\mathbf{z}_r = \mathbf{0}.$$

Hence $\mathbf{z}_r$ is $\mathbf{z}_r = \left[\tilde{\mathbf{b}}_L\mathbf{S}_1\tilde{\mathbf{s}}(\mathbf{S}_2\tilde{\mathbf{s}})^{\perp}\right]^{\perp}\mathbf{z}$, which completes the proof. ∎

Design Algorithm P :

step 1. Same as in Design Algorithm A.
step 2. Choose nodal vector $\mathbf{p}$.
step 3. Choose the free vector $\mathbf{z}$ in (63) and Compute λ and γ.
step 4. calculate $\{\mathbf{t},\mathbf{s},\mathbf{f}\}$ from (53).

ILLUSTRATED EXAMPLES

We now illustrate the construction procedure for a simple tensegrity structure.

2-bar 4-string planar tensegrity structure

A general force configuration for the class I tensegrity structure in Example 1 and 2 will be investigated in this section. Suppose the force coefficient matrices are given by $\mathbf{\Lambda} = \text{diag}\{\lambda_1,\lambda_2\}$ and $\mathbf{\Gamma} = \text{diag}\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$. We will demonstrate two procedures using Design Algorithms A and B.

Design Algorithm A :

step 1. The connectivity matrices $\mathbf{S}$, $\mathbf{B}$ and the coordinate transformation $\mathbf{P} = [\mathbf{P}_1^T,\mathbf{P}_2^T]$ are given in Example 1 and 2. Since $\mathbf{V}_2$ spans the null space of $\mathbf{S}_2$, we compute

$$\mathbf{V}_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\bigotimes \mathbf{I}_2.$$

step 2. Choose $\{\gamma_m > 0\}$ and $\{\lambda_n > 0\}$ such that $\det(\mathbf{X}-\mathbf{\Lambda}) = 0$, where

$$\begin{aligned} \mathbf{X} &= \mathbf{S}_1\mathbf{V}_2\left(\mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2\right)^{-1}\mathbf{V}_2^T\mathbf{S}_1^T \\ &= \frac{1}{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\begin{bmatrix} (\gamma_2+\gamma_3)(\gamma_4+\gamma_1) & (-\gamma_3\gamma_1+\gamma_2\gamma_4) \\ (-\gamma_3\gamma_1+\gamma_2\gamma_4) & (\gamma_3+\gamma_4)(\gamma_2+\gamma_1) \end{bmatrix}\bigotimes \mathbf{I}_2. \end{aligned}$$

In all choices for $\{\lambda_n\}$ that led to the rank of $(\mathbf{X}-\mathbf{\Lambda})$ having rank 1, the 4×2 matrix $\mathbf{V}_{X2}$ is of the form $\mathbf{V}_{X2}^T = [\mathbf{A}_1^T, \pm\mathbf{A}_1^T]$; that is, in (49), the two bar vectors $\{\mathbf{b}_1, \mathbf{b}_2\}$ are always parallel, so the equilibrium structure is one dimensional with $\{\mathbf{p}_1 = \mathbf{p}_3, \mathbf{p}_2 = \mathbf{p}_4\}$. Hence for a two-dimensional structure, $\mathbf{X}-\mathbf{\Lambda}$ must have rank zero. This requires

$$\begin{aligned}
\gamma_4 &= \frac{\gamma_3\gamma_1}{\gamma_2} \\
\lambda_1 &= \frac{(\gamma_2+\gamma_3)(\gamma_4+\gamma_1)}{\gamma_2+\gamma_3+\gamma_4+\gamma_1} \\
\lambda_2 &= \frac{(\gamma_3+\gamma_4)(\gamma_2+\gamma_1)}{\gamma_2+\gamma_3+\gamma_4+\gamma_1},
\end{aligned} \tag{68}$$

where γ_1, γ_2, and γ_3 are free positive constants. If we choose $\{\gamma_k = 1; k = 1,2,3\}$, then $\mathbf{\Gamma} = \mathbf{I}_4$. It follows that $\mathbf{X} = \mathbf{I}_4$ and $\mathbf{\Lambda} = \mathbf{I}_2$ satisfy condition (50) in Theorem 1.

step 3. Select suitable $\mathbf{b}_z$ and compute $\mathbf{b} = (\mathbf{X}-\mathbf{\Lambda})^{\perp}\mathbf{b}_z$ based on a desired bar vector.

Since we choose γ and λ such that $\mathbf{X}-\mathbf{\Lambda} = \mathbf{0}$, the bar vector is arbitrary. Let's choose $\mathbf{b}_1 = [2,0]^T, \mathbf{b}_2 = [0,2]^T$.

step 4. Choose the free vector $\mathbf{z}_e$ in (51), (52) to give a suitable node vector $\mathbf{p}$ based on a desired shape.

The nodes $\{\mathbf{p}_1 = [-1,0]^T, \mathbf{p}_2 = [1,0]^T, \mathbf{p}_3 = [0,-1]^T, \mathbf{p}_4 = [0,1]^T\}$ define an equilibrium solution from (51) setting $\mathbf{z}_e$ is zero. When $\mathbf{z}_e = [1,1]^T$, we get

$$\mathbf{p} = \begin{bmatrix} -0.6464 & 0.3536 & 1.3536 & 0.3536 & 0.3536 & -0.6464 & 0.3536 & 1.3536 \end{bmatrix}^T.$$

This choices of $\mathbf{z}_e$ only translate the geometric center of the structure from $[0,0]^T$ to $0.3536[1,1]^T$, since the force coefficients and $\mathbf{b}$ have been specified.

Design Algorithm B :

step 1. Connectivity matrices and the coordinate transformation matrix is the same as the step 1 in design algorithm A.

step 2. Let's choose $\mathbf{b}_1 = [2,0], \mathbf{b}_2 = [0,2]$. From (61),

$$\tilde{\mathbf{b}}_L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \;;\; \tilde{\mathbf{b}}_L\mathbf{X}\mathbf{b} = \frac{2}{\sum_{i=1}^4 \gamma_i}\begin{bmatrix} -\gamma_1\gamma_3+\gamma_2\gamma_4 \\ -\gamma_1\gamma_3+\gamma_2\gamma_4 \end{bmatrix} = \mathbf{0}.$$

Hence the equilibrium condition is reduced to

$$\gamma_4 = \frac{\gamma_3\gamma_1}{\gamma_2}. \tag{69}$$

Let's choose $\gamma_1 = \gamma_2 = \gamma_3 = 1$, and then $\gamma_4 = 1$ given by (69).

step 3. Compute λ from (60).

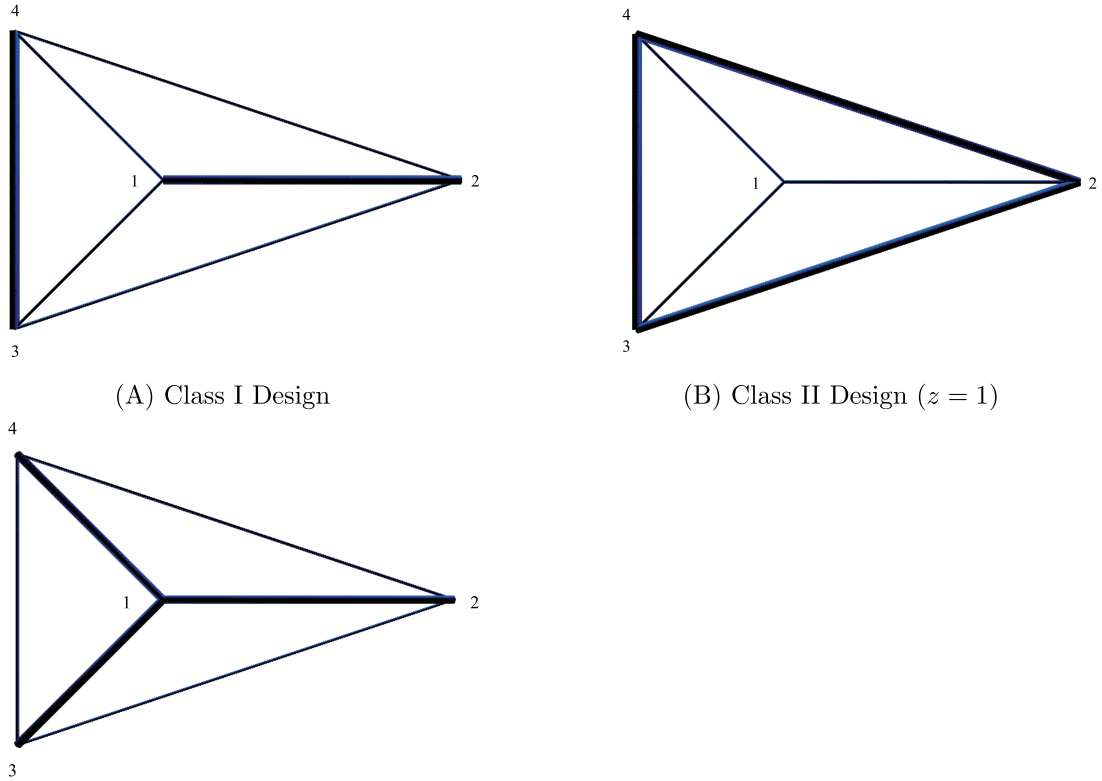
$$\lambda = \tilde{\mathbf{b}}^{+}\mathbf{X}\mathbf{b} = \frac{1}{\gamma_2+\gamma_3+\gamma_4+\gamma_1}\begin{bmatrix} (\gamma_2+\gamma_3)(\gamma_4+\gamma_1) \\ (\gamma_3+\gamma_4)(\gamma_2+\gamma_1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{70}$$

step 4. When we choose the free vector $\mathbf{z}_e = [1,-1]^T$ in (51), we get the node vector

$$\mathbf{p} = \begin{bmatrix} -0.6464 & -0.3536 & 1.3536 & -0.3536 & 0.3536 & -1.3536 & 0.3536 & 0.6464 \end{bmatrix}^T.$$

Note that the existence conditions (70),(69) for λ and γ give the same results as (68).

Design Algorithm P :



(A) Class I Design

(B) Class II Design ($z = 1$)

(C) Class II Design ($z = -1$)

FIG. 2. 2-Bar 4-String Tensegrity Structure

step 1. Same as in design algorithm A.
step 2. We choose $\mathbf{p}_1 = [1, 0]^T, \mathbf{p}_2 = [3, 0]^T, \mathbf{p}_3 = [0, -1]^T, \mathbf{p}_4 = [0, 1]^T$.(See Figure 2.(A))
step 3. There is no choice of $\mathbf{z}_e$ which satisfies (63). The choice of $z = 1$ yields

$$\gamma = \begin{bmatrix} 3 & -1 & -1 & 3 \end{bmatrix}^T \; ; \; \lambda = \begin{bmatrix} -3 & 1 \end{bmatrix}^T .$$

The choice of $\mathbf{z}_e = -1$ obviously reverses all signs of this γ and λ.

This proves that the configuration in Figure 2A can only be stabilized with class II tensegrity with negative γ_2, γ_3 and λ_1. Negative γ_2, γ_3 means that compression is required in these members, hence bars must replace these strings and negative λ_1 means that tension is required in this member. This yields the structure in Figure 2B, where thick lines are bars and thin lines are strings. In conclusion, the only class I tensegrity that exists for the topology of Figure 1 requires overlap of the bars. For the nodal configurations in Figure 2A, the bars do not overlap and no class I tensegrity exists. Two configurations of class II tensegrities are possible as shown in Figure 2B and 2C.

Symmetrical Force Configuration for 3-bars 9-strings Tensegrity Structure

A $(3; 9; 3)$ class I tensegrity [9] consisting of one stage of $n_b = 3$ bars (so $n_p = 6$ nodes) with $n_s = 9$ strings has topology as illustrated in Figure 3

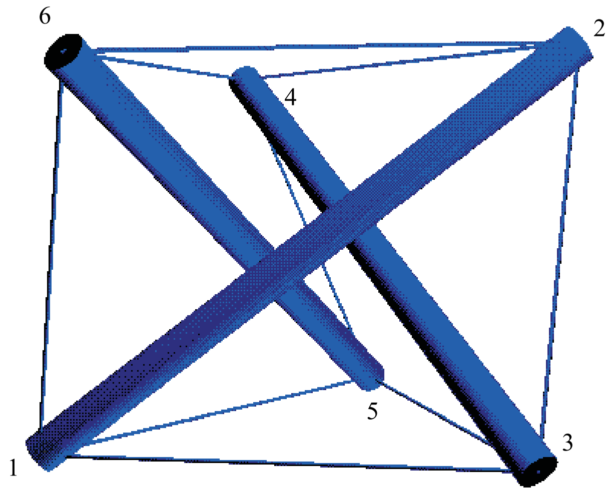


FIG. 3. 3-Bar 9-String $(3;9;3)$ Class I Tensegrity

The corresponding geometric matrices $\{\mathbf{S},\mathbf{B}\}$ are given by

$$\mathbf{S}=\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix};\quad \mathbf{B}=\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

A symmetrical force configuration will be investigated with equal bar force coefficients $\{\lambda_1=\lambda_2=\lambda_3=\lambda\}$, equal 'base' string force coefficients $\{\gamma_1=\gamma_2=\gamma_3\triangleq\gamma_b\}$, equal 'top' string coefficients $\{\gamma_4=\gamma_5=\gamma_6\triangleq\gamma_t\}$, and equal vertical string coefficients $\{\gamma_7=\gamma_8=\gamma_9\triangleq\gamma_v\}$. Then

$$\begin{aligned}\mathbf{X} &= \mathbf{S}_1\mathbf{V}_2\left(\mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2\right)^{-1}\mathbf{V}_2^T\mathbf{S}_1^T\\ &= \frac{1}{(2\gamma_b{}^2+8\gamma_b\gamma_t+6\gamma_b\gamma_v+6\gamma_v\gamma_t+2\gamma_t{}^2+3\gamma_v{}^2)\gamma_t^3\gamma_b^3\gamma_v^3}\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3\end{bmatrix}\bigotimes\mathbf{I}_3,\end{aligned}$$

where

$$\mathbf{X}_1=\begin{bmatrix}(\gamma_t+\gamma_v+\gamma_b)\left(\gamma_v{}^2+3\gamma_v\gamma_t+3\gamma_b\gamma_v+4\gamma_b\gamma_t+\gamma_b{}^2+\gamma_t{}^2\right)\\ -\gamma_b\gamma_v\gamma_t-\gamma_b\gamma_t{}^2-4\gamma_b{}^2\gamma_t-3\gamma_b{}^2\gamma_v-\gamma_b{}^3+\gamma_v\gamma_t{}^2+2\gamma_t\gamma_v{}^2+\gamma_v{}^3\\ -4\gamma_b\gamma_t{}^2-\gamma_b\gamma_v\gamma_t-\gamma_b{}^2\gamma_t+2\gamma_b\gamma_v{}^2+\gamma_b{}^2\gamma_v-3\gamma_v\gamma_t{}^2-\gamma_t{}^3+\gamma_v{}^3\end{bmatrix}$$

$$\mathbf{X}_2=\begin{bmatrix}-\gamma_b\gamma_v\gamma_t-\gamma_b\gamma_t{}^2-4\gamma_b{}^2\gamma_t-3\gamma_b{}^2\gamma_v-\gamma_b{}^3+\gamma_v\gamma_t{}^2+2\gamma_t\gamma_v{}^2+\gamma_v{}^3\\ 10\gamma_b\gamma_v\gamma_t+4\gamma_b\gamma_t{}^2+7\gamma_b{}^2\gamma_t+5\gamma_b\gamma_v{}^2+6\gamma_b{}^2\gamma_v+\gamma_b{}^3+2\gamma_v\gamma_t{}^2+3\gamma_t\gamma_v{}^2+\gamma_v{}^3\\ -3\gamma_b\gamma_t{}^2-\gamma_b\gamma_v\gamma_t-\gamma_v\gamma_t{}^2+\gamma_t\gamma_v{}^2-3\gamma_b{}^2\gamma_t+\gamma_b\gamma_v{}^2-\gamma_b{}^2\gamma_v+\gamma_v{}^3\end{bmatrix}$$

$$\mathbf{X}_3=\begin{bmatrix}-4\gamma_b\gamma_t{}^2-\gamma_b\gamma_v\gamma_t-\gamma_b{}^2\gamma_t+2\gamma_b\gamma_v{}^2+\gamma_b{}^2\gamma_v-3\gamma_v\gamma_t{}^2-\gamma_t{}^3+\gamma_v{}^3\\ -3\gamma_b\gamma_t{}^2-\gamma_b\gamma_v\gamma_t-\gamma_v\gamma_t{}^2+\gamma_t\gamma_v{}^2-3\gamma_b{}^2\gamma_t+\gamma_b\gamma_v{}^2-\gamma_b{}^2\gamma_v+\gamma_v{}^3\\ 6\gamma_v\gamma_t{}^2+5\gamma_t\gamma_v{}^2+10\gamma_b\gamma_v\gamma_t+7\gamma_b\gamma_t{}^2+4\gamma_b{}^2\gamma_t+\gamma_t{}^3+\gamma_v{}^3+3\gamma_b\gamma_v{}^2+2\gamma_b{}^2\gamma_v\end{bmatrix}$$

Now we have to solve the force coefficients based on the equilibrium conditions given by Lemma 8. If we apply (i) condition in Lemma 8,

$$\det(\mathbf{X}-\mathbf{\Lambda}) \;=\; -\frac{(\lambda-\gamma_v)}{2\,{\gamma_b}^2+8\,\gamma_b\,\gamma_t+6\,\gamma_b\,\gamma_v+6\,\gamma_v\,\gamma_t+2\,{\gamma_t}^2+3\,{\gamma_v}^2}\lambda_{second}=0,$$

where

$$\begin{aligned}\lambda_{second} \triangleq (& - & 2\,{\gamma_t}^3\lambda+2\,{\gamma_v}^2{\gamma_t}^2+2\,\gamma_v\,{\gamma_t}^3+6\,{\gamma_b}^3\gamma_t+15\,{\gamma_b}^2{\gamma_t}^2+2\,{\gamma_b}^2{\gamma_v}^2\\ & + & 6\,\gamma_b\,{\gamma_t}^3+2\,{\gamma_b}^3\gamma_v-6\,{\gamma_v}^2\gamma_t\,\lambda-10\,\gamma_v\,{\gamma_t}^2\lambda-16\,{\gamma_b}^2\gamma_t\,\lambda\\ & + & 2\,{\gamma_b}^2\lambda^2-2\,{\gamma_b}^3\lambda-10\,{\gamma_b}^2\gamma_v\,\lambda+6\,\gamma_b\,\lambda^2\gamma_v-6\,\gamma_b\,\lambda{\gamma_v}^2\\ & + & 8\,\gamma_b\,\lambda^2\gamma_t-16\,\gamma_b\,\lambda{\gamma_t}^2-22\,\gamma_b\,\lambda\gamma_v\,\gamma_t+2\,\lambda^2{\gamma_t}^2+3\,\lambda^2{\gamma_v}^2\\ & + & 6\,\lambda^2\gamma_v\,\gamma_t+16\,\gamma_b\,{\gamma_t}^2\gamma_v+8\,\gamma_b\,\gamma_t\,{\gamma_v}^2+16\,{\gamma_b}^2\gamma_v\,\gamma_t\quad).\end{aligned}$$

Since smaller rank of $(\mathbf{X}-\mathbf{\Lambda})$ yields more freedom for the choice of $\mathbf{b}$, we choose $\gamma_v=\lambda$. Next evaluating second term when $\gamma_v=\lambda$, we get

$$\begin{aligned}\lambda_{second}|_{\gamma_v=\lambda} &= -6\,\lambda^2{\gamma_t}^2-6\,{\gamma_b}^2\lambda^2+6\,\gamma_b\,{\gamma_t}^3+6\,{\gamma_b}^3\gamma_t+15\,{\gamma_b}^2{\gamma_t}^2-6\,\gamma_b\,\lambda^2\gamma_t+3\,\lambda^4\\ &= \left(\lambda^2-\gamma_t\gamma_b-2\gamma_b^2\right)\left(\lambda^2-\gamma_t\gamma_b-2\gamma_t^2\right).\end{aligned}$$

We conclude $\lambda=\sqrt{\gamma_t\gamma_b+2\gamma_b^2}$ or $\lambda=\sqrt{\gamma_t\gamma_b+2\gamma_t^2}$, since $\lambda>0$.

When we apply $\lambda=\sqrt{\gamma_t\gamma_b+2\gamma_t^2}$, we get

$$\mathbf{X}-\mathbf{\Lambda}=\frac{(\gamma_b-\gamma_t)\,(\gamma_b+\gamma_t)}{2\,{\gamma_b}^2+6\,\gamma_b\,\gamma_{bt}+11\,\gamma_b\,\gamma_t+8\,{\gamma_t}^2+6\,\gamma_t\,\gamma_{bt}}\bar{\mathbf{X}}$$

where $\gamma_{bt}\triangleq\sqrt{\gamma_t\,(\gamma_b+2\,\gamma_t)}$ and

$$\bar{\mathbf{X}} \;=\; \left[\begin{array}{ccc}\gamma_b+2\,\gamma_{bt}+3\,\gamma_t & -\gamma_b-4\,\gamma_t-3\,\gamma_{bt} & \gamma_t+\gamma_{bt}\\ -\gamma_b-4\,\gamma_t-3\,\gamma_{bt} & \gamma_b+4\,\gamma_{bt}+6\,\gamma_t & -2\,\gamma_t-\gamma_{bt}\\ \gamma_t+\gamma_{bt} & -2\,\gamma_t-\gamma_{bt} & \gamma_t\end{array}\right]\bigotimes\mathbf{I}_3$$

Note that the rank of the matrix $\bar{\mathbf{X}}$ is 1. An interesting case is when $\gamma_t=\gamma_b$. In this case, an equilibrium solution with $\mathbf{\Lambda}=\mathbf{X}$ is provided by $\lambda=\gamma_v=\sqrt{3}\gamma_t$ for all choices of $\{\gamma_t\}$. When $\gamma_t=\gamma_b=1$ and $\lambda=\gamma_v=\sqrt{3}$, the structural shape is the prism given in Figure 3.

Symmetrical Force Configuration for 6-bars 24-strings Tensegrity Structure

A (6;24;3;3) class I tensegrity structure consisting of two stage of $n_b = 6$ bars (so $n_p = 12$ nodes) with $n_s = 24$ strings has topology as illustrated in Figure 4. Since the connectivity matrix for this example is quite big, we introduced another notation 'Bar' and 'Tendon' which is defined by Bar (Bar Id#, { Node Id #1, Node Id #2}) and Tendon (Tendon Id#, { Node Id #1, Node Id #2}). For simplicity, a symmetrical force configuration will be investigated in this example. Bars are defined by

```
Bar(1, {1, 2}); Bar(2, {3, 4}); Bar(3, {5, 6});
Bar(4, {7, 8}); Bar(5, {9, 10}); Bar(6, {11, 12});
```

with equal bar force coefficients $\{\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=\lambda_6\triangleq\lambda\}$. Similariliy base and top strings are defined by

```
Tendon(7, {1, 3}); Tendon(9, {3, 5}); Tendon(11, {5, 1});
Tendon(8, {8, 10}); Tendon(10, {10, 12}); Tendon(12, {12, 8});
```

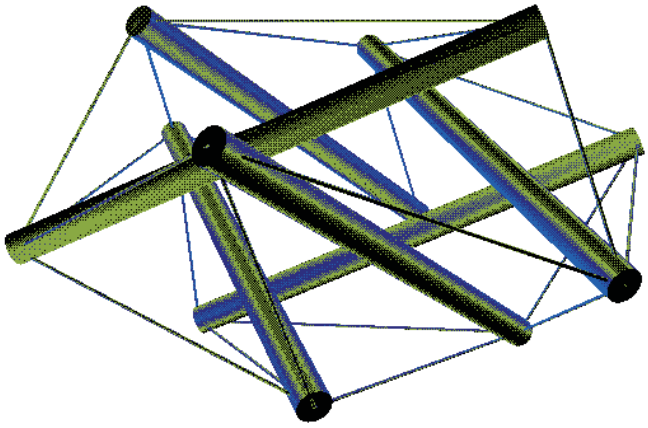


FIG. 4. 6-Bar 24-String Class I Tensegrity

with equal 'base' and 'top' string force coefficients $\{\gamma_i \triangleq \gamma_t\}$ for all $i = 7 \cdots 12$. Saddle strings are defined by

```
Tendon(13, {2, 7}); Tendon(14, {2, 11}); Tendon(15, {4, 9});
Tendon(16, {4, 7}); Tendon(17, {6, 11}); Tendon(18, {6, 9});
```

with equal 'saddle' string force coefficients $\{\gamma_i \triangleq \gamma_s\}$ for all $i = 13 \cdots 18$. Diagonal strings are defined by

```
Tendon(19, {1, 11}); Tendon(20, {3, 7}); Tendon(21, {5, 9});
Tendon(22, {2, 8}); Tendon(23, {4, 10}); Tendon(24, {6, 12});
```

with equal 'diagonal' string force coefficients $\{\gamma_i \triangleq \gamma_d\}$ for all $i = 19 \cdots 24$. Vertical strings are defined by

```
Tendon(25, {1, 6}); Tendon(26, {3, 2}); Tendon(27, {5, 4});
Tendon(28, {7, 10}); Tendon(29, {9, 12}); Tendon(30, {11, 8});
```

with equal 'vertical' string force coefficients $\{\gamma_i \triangleq \gamma_d\}$ for all $i = 25 \cdots 30$.

For simplicity of calculation, we introduce scaled variables as following

$$\bar{\gamma}_s \triangleq \frac{\gamma_s}{\gamma_t} \;;\;\; \bar{\gamma}_v \triangleq \frac{\gamma_v}{\gamma_t} \;;\;\; \bar{\gamma}_d \triangleq \frac{\gamma_d}{\gamma_t} \;;\;\; \bar{\lambda} \triangleq \frac{\lambda}{\gamma_t}.$$

Then the equilibrium condition (ii) in Lemma 8 is given by

$$\det \begin{bmatrix} \mathbf{\Lambda} & \mathbf{S}_1\mathbf{V}_2 \\ \mathbf{V}_2^T\mathbf{S}_1^T & \mathbf{V}_2^T\mathbf{\Gamma}^{-1}\mathbf{V}_2 \end{bmatrix} = 0.$$

This expression yields by using our scaled variables

$$\begin{aligned} &\frac{-6}{\bar{\gamma}_v^6\bar{\gamma}_s^6\bar{\gamma}_d^6}\left(-\bar{\gamma}_d + \bar{\lambda} - \bar{\gamma}_v\right)\left(\bar{\lambda}\,\bar{\gamma}_d - \bar{\gamma}_s\,\bar{\gamma}_d + \bar{\gamma}_s\,\bar{\lambda} - \bar{\gamma}_v\,\bar{\gamma}_d - \bar{\gamma}_s\,\bar{\gamma}_v\right) \\ &\left(\bar{\lambda}\,\bar{\gamma}_d - 3\,\bar{\gamma}_d - \bar{\gamma}_s\,\bar{\gamma}_d - \bar{\gamma}_v\,\bar{\gamma}_d - 3\,\bar{\gamma}_v + 3\,\bar{\lambda} + 3\,\bar{\gamma}_v\,\bar{\lambda} - 3\,\bar{\gamma}_s - \bar{\gamma}_s\,\bar{\gamma}_v + \bar{\gamma}_s\,\bar{\lambda}\right)^2 \\ &\left(\bar{\lambda}\,\bar{\gamma}_d - \bar{\gamma}_d - \bar{\gamma}_s\,\bar{\gamma}_d - \bar{\gamma}_v\,\bar{\gamma}_d - \bar{\gamma}_v + \bar{\lambda} + \bar{\gamma}_v\,\bar{\lambda} - 3\,\bar{\gamma}_s - \bar{\gamma}_s\,\bar{\gamma}_v + \bar{\gamma}_s\,\bar{\lambda}\right)^2 = 0 \end{aligned} \tag{71}$$

There are many solutions satisfying this condition. Since we are interested in three dimensional stable equilbria, we seek those combinations that generate stable equilibria. After simulation to verify the stability, the following combinations were found to generate stable equilibria.

$$\begin{aligned}
\bar{\lambda}\,\bar{\gamma}_d - \bar{\gamma}_s\,\bar{\gamma}_d + \bar{\gamma}_s\,\bar{\lambda} - \bar{\gamma}_d\,\bar{\gamma}_v - \bar{\gamma}_s\,\bar{\gamma}_v &= 0 \qquad (72)\\
\bar{\lambda}\,\bar{\gamma}_d - \bar{\gamma}_d - \bar{\gamma}_s\,\bar{\gamma}_d - \bar{\gamma}_d\,\bar{\gamma}_v - \bar{\gamma}_v + \bar{\lambda} + \bar{\gamma}_v\,\bar{\lambda} - 3\,\bar{\gamma}_s - \bar{\gamma}_s\,\bar{\gamma}_v + \bar{\gamma}_s\,\bar{\lambda} &= 0 \qquad (73)
\end{aligned}$$

Now we analyze the equilibria (72) and (73). We solve (72) for $\bar{\lambda}$ to obtain

$$\bar{\lambda} = \frac{\bar{\gamma}_s\,\bar{\gamma}_d + \bar{\gamma}_d\,\bar{\gamma}_v + \bar{\gamma}_s\,\bar{\gamma}_v}{\bar{\gamma}_s + \bar{\gamma}_d}. \qquad (74)$$

Since every variable is positive, this equation yields positive $\bar{\lambda}$ as required. Substituting (74) into (73) yields

$$-\bar{\gamma}_d\,\bar{\gamma}_s\,\bar{\gamma}_v + \bar{\gamma}_s\,\bar{\gamma}_d + \bar{\gamma}_d^2 - \bar{\gamma}_d\,\bar{\gamma}_v^2 - \bar{\gamma}_s\,\bar{\gamma}_v^2 + \bar{\gamma}_s^2 = 0. \qquad (75)$$

Requiring positiveness of $\bar{\gamma}_v$, (75) can be solved by

$$\bar{\gamma}_v = \frac{1}{2}\,\frac{-\bar{\gamma}_s\,\bar{\gamma}_d + \sqrt{\bar{\gamma}_d^2\bar{\gamma}_s^2 + 8\,\bar{\gamma}_s^2\bar{\gamma}_d + 8\,\bar{\gamma}_s\,\bar{\gamma}_d^2 + 4\,\bar{\gamma}_s^3 + 4\,\bar{\gamma}_d^3}}{\bar{\gamma}_s + \bar{\gamma}_d}. \qquad (76)$$

With the use of original variables, (74) and (76) can be expressed as following

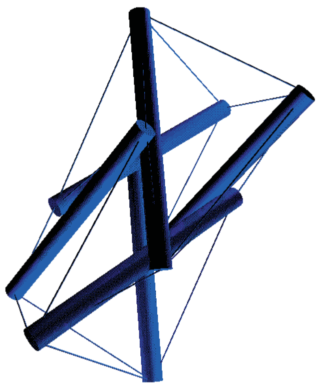
$$\begin{aligned}
\lambda &= \frac{\gamma_s\,\gamma_d + \gamma_d\,\gamma_v + \gamma_s\,\gamma_v}{\gamma_s + \gamma_d} \qquad (77)\\
\gamma_v &= \frac{1}{2}\frac{-\gamma_s\gamma_d + \sqrt{\gamma_d^2\gamma_s^2 + 4\gamma_t\left(\gamma_s + \gamma_v\right)\left(\gamma_s^2 + \gamma_v^2 + \gamma_s\gamma_v\right)}}{\gamma_s + \gamma_d}. \qquad (78)
\end{aligned}$$

CONCLUSION

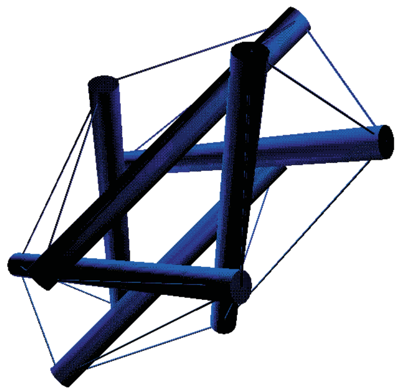
This paper characterizes the static equilibria of a class I tensegrity structure. Analytical expressions are derived for the equilibrium condition of a tensegrity structure in terms of member force coefficients and string and bar connectivity information. We use vectors to describe each element (bars and tendons), eliminating the need to use direction cosines and the subsequent transcendental functions that follow their use. By enlarging the vector space in which we characterize the problem, the mathematical structure of the equations admit treatment by linear algebra methods, for the most part. This reduces the study of a significant portion of the tensegrity equilibria to a series of linear algebra problems. Our results characterize the equilibria conditions of tensegrity structures in terms of a very small number of variables since the necessary and sufficient conditions of the linear algebra treatment has eliminated several of the original variables. This formulation offers insight and identifies the free parameters that may be used to achieve desired structural shapes. Since all conditions are necessary and sufficient, these results can be used in the design of any tensegrity structure. Special insightful properties are available in the special case when one designs a tensegrity structure so that all strings have the same force per unit length(γ), and all bars have the same force per unit length(λ). In this case, all admissible values of $\frac{\lambda}{\gamma}$ are the discrete set of eigenvalues of a matrix given in terms of only the string connectivity matrix. Furthermore the only bar vectors which can be assigned are eigenvectors of the same matrix. Future papers will integrate these algorithms into software to make these designs more efficient.

ACKNOWLEDGEMENTS

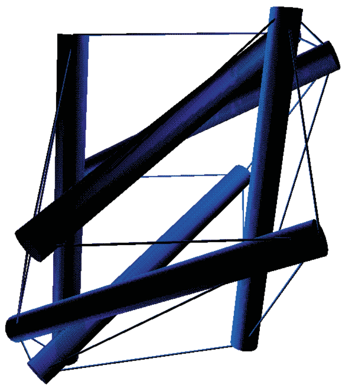
This research were conducted by Dynamic Systems Research Inc. under the sponsorship of the Jet Propulsion Laboratory, technical monitors Dr. Marco Quadrelli and Fred Hadaegh.



(i) Equilibrium Configuration when $\gamma_t = 0.1$



(ii) Equilibrium Configuration when $\gamma_t = 1$



(iii) Equilibrium Configuration when $\gamma_t = 10$

FIG. 5. Two-stage Shell Class Tensegrity Structure when $\gamma_s = \gamma_d = 1$

References

[1] R. Connelly, Rigidity and Energy, *Inventiones Mathematicae*, 66 (1), pp 11-33, 1982.

[2] R. Connelly, Rigidity in: P.M. Gruber and J.M. Wills, ed., *Handbook of Convex Geometry*, Elsevier Publishers Ltd, pp 223-271, 1993.

[3] R. Connelly, Tensegrity Structures: Why are they stable?, in: M.F. Thorpe and P.M. Duxbury, ed., *Rigity Theory and Applications* , Plenum Press, pp47-54, New York, 1999.

[4] R. Connelly and A. Bach, *Mathematics and Tensegrity*, American Scientist, March, 1998.

[5] D. E. Ingber, *Tensegrity: The Architectural Basis of Cellular Mechanotransduction*, Ann. Rev. of Physiology, 59, pp575-599, 1997.

[6] D. E. Ingber, *Architecture of Life*, Scientific American, Jan., pp48-57, 1998.

[7] R.E. Skelton and C. Sultan, *Controllable Tensegrity*, Proc. SPIE 4th Ann. Symp. Smart Str. & Mats., San Diego, March 1997.

[8] R.E. Skelton, W.J. Helton and R. Adhikari, *Mechanics of Tensegrity Beams*, UCSD, Structural Systems & Contr. Lab., Rep. No. 1998-1, 1998.

[9] D. Williamson and R.E. Skelton, *A General Class of Tensegrity Systems: Geometric Definition*, Engineering Mechanics for the 21st Century, ASCE Conference, La Jolla, California, May, 1998.

[10] D. Williamson and R.E. Skelton, *A General Class of Tensegrity Systems: Equilibrium Analysis*, Engineering Mechanics for the 21st Century, ASCE Conference, La Jolla, California, May, 1998.

[11] R. Motro, *Tensegrity Systems: The State of the Art*, Int. J. Space Structures, 7(2):75-83, 1992.

[12] Tibert and S. Pellegrino, *Review of Form-Finding Methods for Tensegrity Strucutres*, Tensegrity Workshop, Rome, May, 2001

[13] Motro, R., *Foldable Tensegrities*, in: S. Pellegrino, ed. Deployable Structures, Springer Verlag Wien-New York, 2001.

[14] Ingber, D.E., *Cellular tensegrity: defining new rules of biological design that govern the cytoskeleton*, Journal of Cell Science, 104 , 1993, 613-627.

[15] Pugh, A., *An Introduction to Tensegrity*, University of California Press, Berkeley, California, 1976.

[16] Connelly, R. and Terrell, M., *Globally rigid symmetric tensegrities, Structural Topology*, 21 , 1995, 59-78.

[17] Vassart, N. and Motro, R., *Multiparametered form.nding method: application to tensegrity systems*, International Journal of Space Structures, 14 (2), 1999, 147-154.

[18] Sultan, C., Corless, M., and Skelton, R.E., *Reduced prestressability conditions for tensegrity structures*, Proceedings of 40th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, 12-15 April 1999, St Louis, MO, AIAA, 1999.

[19] Barnes, M.R., *Form-finding and analysis of prestressed nets and membranes*, Computers & Structures, 30 (3), 1988, 685-695.

[20] Motro, R., *Forms and forces in tensegrity systems*, in: H. Nooshin, ed., Proceedings of Third International Conference on Space Structures, Elsevier, Amsterdam, 1984, 180-185.

[21] Motro, R., *Tensegrity systems and geodesic domes*, International Journal of Space Structures, 5 (3-4), 1990, 341-351.

[22] Motro, R., Belkacem, S., and Vassart, N., *Form finding numerical methods for tensegrity systems*, in: J.F. Abel, J.W. Leonard, and C.U. Penalba, ed., Proceedings of IASS-ASCE International Symposium on Spatial, Lattice and Tension Structures, Atlanta, USA, ASCE, 1994, 707-713.

[23] Kenner, H., *Geodesic Math and How to Use It*, University of California Press, Berkeley, California, 1976.

[24] Schek, H.-J., *The force density method for form finding and computation of general networks, Computer Methods in Applied Mechanics and Engineering*, 3, 1974, 115-134.

[25] Linkwitz, K., *Form finding by the direct approach and pertinent strategies for the conceptual design of prestressed and hanging structures*, International Journal of Space Structures, 14 (2), 1999, 73-87.