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An LMI Optimization Approach for Structured Linear Controllers

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***Abstract*— This paper presents a new algorithm for the design of linear controllers with special structural constraints imposed on the control gain matrix. This so called SLC (Structured Linear Control) problem can be formulated with linear matrix inequalities (LMI's) with a nonconvex equality constraint. This class of problems includes fixed order output feedback control, multi-objective controller design, decentralized controller design, joint plant and controller design, and other interesting control problems.**

Our approach includes two main contributions. The first is that many design specifications are described by a similar matrix inequality. A new matrix variable is introduced to give more freedom to design the controller. Indeed this new variable helps to find the optimal fixed-order output feedback controller. The second contribution is to propose a linearization algorithm to search for a solution to the nonconvex SLC problems. This has the effect of adding a certain potential function to the nonconvex constraints to make them convex. The convexified matrix inequalities will not bring significant conservatism because they will ultimately go to zero, guaranteeing the feasibility of the original nonconvex problem. Numerical examples demonstrate the performance of the proposed algorithms and provide a comparison with some of the existing methods.

I. Introduction

Control problems are usually formulated as optimization problems. Unfortunately, most of them are not convex [1], and a few of them can be formulated as linear matrix inequalities (LMI's). In the LMI framework, one can solve several linear control problems in the form of $\min_{\mathcal{K}} f(\mathbf{T}(\zeta))$, where $\mathcal{K}$ is a controller gain matrix, $f(\cdot)$ is a suitably defined convex objective function and $\mathbf{T}(\zeta)$ is the transfer function from a given input to a given output of interest. For this problem, one can find a solution efficiently with the use of any LMI solver [2], [3]. However, the problem becomes difficult when one adds some constraints on the controller gain matrix $\mathcal{K}$.

Any linear control problem with structure imposed on the controller parameter $\mathcal{K}$ will be called a "Structured Linear Control (SLC)" problem. This SLC problem includes a large class of problems such as decentralized control, fixed-order output feedback, linear model reduction, linear fixed-order filtering, the simultaneous design of plant and controller, norm bounds on the control gain matrix, and multi-objective control problems. Among these problems, the fixed order output feedback problem is known to be NP-hard. There are many attempts to solve this problem [6], [7], [5], [1], [8], [12], [13], [15]. Most algorithms try to obtain a stable controller rather than find an optimal controller. Among those, the approach proposed in [15] is quite similar to our approach. There the author expanded the domain of the problem by introducting new extra variables and then applying a coordinate-descent method to compute local optimal solutions for the mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem via static output feedback. Unfortunately, this approach does not guarantee local convergence.

Multi-objective control problems also remain open. Indeed, these problems can also be formulated as an SLC problem, since this problem is equivalent to finding multiple controllers for multiple plants where we restrict all controllers to be identical. For the full-order output feedback case, authors have proposed to specify the closed-loop objectives in terms of a common Lyapunov function which can be efficiently solved by convex programming methods [4]. An extended approach has been proposed to relax the constraint on the Lyapunov matrix [9]. It is well known that these approaches are conservative and can not be applicable to the "fixed-order multi-objective controller synthesis problem".

Recently, a convexifying algorithm has been proposed [11] with interesting features. This algorithm solves convexified matrix inequalities iteratively. These convexified problems can be obtained by adding convexifying potential functions to the original nonconvex matrix inequalities at each iteration. Although the convexifying potential function is added, the convexified matrix inequalities will not bring significant conservatism because they will go to zero by resetting the convexifying potential function to zero at each iteration. Due to the lack of convexity, only local convergence can be guaranteed. However, this algorithm is easily implemented and can be used to improve available suboptimal solutions. Moreover, this algorithm is so general that it can be applicable to almost all SLC problems.

The main objective of this paper is to present the optimal controller for SLC problems using a linearization method. The second objective is to present new system performance analysis conditions which have several advantages over the original performance analysis conditions. Many design specifications such as general $\mathcal{H}_2$ performance including $\mathcal{H}_2$ performance, $\mathcal{H}_\infty$ performance, ℓ_∞ performance, and the upper covariance bounding controllers can be written in a very similar matrix inequality. We introduce a new matrix variable for these system performance analysis conditions. As a result, we have more freedom to find the optimal controller.

The paper is organized as follows. Section II describes a framework for SLC problems and then we derive new system performance analysis conditions. Based on these, a new linearization algorithm is proposed in section III. Two numerical examples illustrate the performance of the proposed algorithms as compared with the existing results in section IV. Conclusions follow.

II. System Performance Analysis

A. Models for Control Design

For synthesis purposes, we consider the following discrete time linear system.

$$\mathbf{P}\begin{cases} \mathbf{x}_p(k+1) &= \mathbf{A}_p\mathbf{x}_p(k) + \mathbf{B}_p\mathbf{u}(k) + \mathbf{D}_p\mathbf{w}(k) \\ \mathbf{z}(k) &= \mathbf{C}_z\mathbf{x}_p(k) + \mathbf{B}_z\mathbf{u}(k) + \mathbf{D}_z\mathbf{w}(k) \\ \mathbf{y}(k) &= \mathbf{C}_y\mathbf{x}_p(k) + \mathbf{D}_y\mathbf{w}(k) \end{cases} \tag{1}$$

where $\mathbf{x}_p \in \Re^{n_p}$ is the plant state, $\mathbf{z} \in \Re^{n_z}$ is the controlled output, and $\mathbf{y} \in \Re^{n_y}$ is the measured output. We assume that all matrices have suitable dimensions. Our goal is to compute an output-feedback controller that meets various specifications on the closed-loop behavior,

$$\mathbf{K}\left\{\begin{array}{rcl}\mathbf{x}_c(k+1) & = & \mathbf{A}_c\mathbf{x}_c(k)+\mathbf{B}_c\mathbf{y}(k)\\ \mathbf{u}(k) & = & \mathbf{C}_c\mathbf{x}_c(k)+\mathbf{D}_c\mathbf{y}(k)\end{array}\right. \tag{2}$$

where $\mathbf{x}_c \in \Re^{n_c}$ is the controller state and $\mathbf{u} \in \Re^{n_u}$ is the control input. By assembling the plant $\mathbf{P}$ and the controller $\mathbf{K}$ defined as above, we have the compact closed-loop system

$$\left[\begin{array}{c}\mathbf{x}(k+1)\\ \mathbf{z}(k)\end{array}\right]=\left[\begin{array}{cc}\mathbf{A}_{cl}(\mathcal{K}) & \mathbf{B}_{cl}(\mathcal{K})\\ \mathbf{C}_{cl}(\mathcal{K}) & \mathbf{D}_{cl}(\mathcal{K})\end{array}\right]\left[\begin{array}{c}\mathbf{x}(k)\\ \mathbf{w}(k)\end{array}\right] \tag{3}$$

where the controller parameter $\mathcal{K}$ and the closed loop states $\mathbf{x}$ are

$$\mathcal{K} \triangleq \left[\begin{array}{cc}\mathbf{D}_c & \mathbf{C}_c\\ \mathbf{B}_c & \mathbf{A}_c\end{array}\right] ;\ \mathbf{x} \triangleq \left[\begin{array}{c}\mathbf{x}_p\\ \mathbf{x}_c\end{array}\right]$$

and the closed loop matrices

$$\begin{array}{rcl}\mathbf{A}_{cl}(\mathcal{K}) & \triangleq & \mathcal{A}+\mathcal{B}\mathcal{K}\mathcal{C}\ ;\ \mathbf{B}_{cl}(\mathcal{K}) \triangleq \mathcal{D}_p+\mathcal{B}\mathcal{K}\mathcal{D}_y\\ \mathbf{C}_{cl}(\mathcal{K}) & \triangleq & \mathcal{C}_z+\mathcal{B}_z\mathcal{K}\mathcal{C}\ ;\ \mathbf{D}_{cl}(\mathcal{K}) \triangleq \mathcal{D}_z+\mathcal{B}_z\mathcal{K}\mathcal{D}_y\end{array}$$

are all affine mappings on the variable $\mathcal{K}$, that is

$$\left[\begin{array}{cc}\mathbf{A}_{cl}(\mathcal{K}) & \mathbf{B}_{cl}(\mathcal{K})\\ \mathbf{C}_{cl}(\mathcal{K}) & \mathbf{D}_{cl}(\mathcal{K})\end{array}\right]=\mathbf{\Theta}+\mathbf{\Gamma}\mathcal{K}\mathbf{\Lambda}$$

where

$$\mathbf{\Theta} \triangleq \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p\\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right],\ \mathbf{\Gamma} \triangleq \left[\begin{array}{c}\mathcal{B}\\ \mathcal{B}_z\end{array}\right],\ \mathbf{\Lambda} \triangleq \left[\begin{array}{cc}\mathcal{C} & \mathcal{D}_y\end{array}\right]$$

and all matrices given by

$$\begin{array}{rcl}\mathcal{A} & \triangleq & \left[\begin{array}{cc}\mathbf{A}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_{\mathbf{n_c}}\end{array}\right],\ \mathcal{B} \triangleq \left[\begin{array}{cc}\mathbf{B}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n_c}\end{array}\right],\\ \mathcal{C} & \triangleq & \left[\begin{array}{cc}\mathbf{C}_y & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n_c}\end{array}\right],\ \mathcal{C}_z \triangleq \left[\begin{array}{cc}\mathbf{C}_z & \mathbf{0}\end{array}\right],\ \mathcal{D}_y \triangleq \left[\begin{array}{c}\mathbf{D}_y\\ \mathbf{0}\end{array}\right],\\ \mathcal{B}_z & \triangleq & \left[\begin{array}{cc}\mathbf{B}_z & \mathbf{0}\end{array}\right],\ \mathcal{D}_p \triangleq \left[\begin{array}{c}\mathbf{D}_p\\ \mathbf{0}\end{array}\right],\ \mathcal{D}_z \triangleq \mathbf{D}_z\end{array}$$

are constant matrices that depend only on the plant properties.

B. *Multi-objective Control*

The multi-objective control problem is defined as the problem of determining a controller that simultaneously meets several closed-loop design specifications. We assume that these design specifications are formulated with respect to the closed loop transfer functions of the form $\mathbf{T}_i(\zeta) \triangleq \mathbf{L}_i\mathbf{T}(\zeta)\mathbf{R}_i$ where the matrices $\mathbf{L}_i, \mathbf{R}_i$ select the appropriate input/output channels or channel combinations. From the dynamic matrices of system (1), a state-space realization of the closed loop system $\mathbf{T}_i(\zeta)$ is obtained by defining new matrices as follows

$$\begin{array}{l}(\mathbf{D}_p)_i \triangleq \mathbf{D}_p\mathbf{R}_i\ \ (\mathbf{D}_y)_i \triangleq \mathbf{D}_y\mathbf{R}_i\ \ (\mathbf{D}_z)_i \triangleq \mathbf{L}_i\mathbf{D}_z\mathbf{R}_i\\ (\mathbf{B}_z)_i \triangleq \mathbf{L}_i\mathbf{B}_z\ \ (\mathbf{C}_z)_i \triangleq \mathbf{L}_i\mathbf{C}_z\end{array}$$

in the closed-loop matrices (3). In this form, closed-loop system performance and robustness may be ensured by constraining the general $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the transfer functions associated to the pairs of signals $\mathbf{w}_i \triangleq \mathbf{R}_i\mathbf{w}$ and $\mathbf{z}_i \triangleq \mathbf{L}_i\mathbf{z}$.

C. *General $\mathcal{H}$ Control Synthesis*

System gains for the discrete-time system (3) can be defined as follows[1].

- *Energy-to-Peak Gain :* $\Upsilon_{ep} \triangleq \sup_{\|\mathbf{w}\|_{\ell_2}\le 1} \|\mathbf{z}\|_{\ell_\infty}$.
- *Energy-to-Energy Gain :* $\Upsilon_{ee} \triangleq \sup_{\|\mathbf{w}\|_{\ell_2}\le 1} \|\mathbf{z}\|_{\ell_2}$.
- *Pulse-to-Energy Gain :* $\Upsilon_{ie} \triangleq \sup_{\mathbf{w}(k)=\mathbf{w}_0\delta(k), \|\mathbf{w}_0\|\le 1} \|\mathbf{z}\|_{\ell_2}$.

where $\|\mathbf{z}\|_{\ell_2} \triangleq \left(\sum_{k=0}^{\infty}\|\mathbf{z}(k)\|^2\right)^{\frac{1}{2}}$, $\|\mathbf{z}\|_{\ell_\infty} \triangleq \sup_{k\ge 0}\|\mathbf{z}(k)\|$, and $\delta(\cdot)$ is the Kronecker delta : $\delta(k)=0$ for all $k \ne 0$. $\|\mathbf{A}\|$ is the spectral norm of a matrix $\mathbf{A}$. These system gains are characterized in terms of algebraic conditions. The following results are essential to derive a new system performance analysis.

Lemma 1: Consider the asymptotically stable system (3). Then the following statements are equivalent.

(i) There exist matrices $\mathcal{X}, \mathbf{\Upsilon}$ and $\mathcal{K}$ such that

$$\left.\begin{array}{rcl}\mathcal{X} & > & \mathbf{A}_{cl}(\mathcal{K})\mathcal{X}\mathbf{A}_{cl}^T(\mathcal{K})+\mathbf{B}_{cl}(\mathcal{K})\mathbf{B}_{cl}^T(\mathcal{K})\\ \mathbf{\Upsilon} & > & \mathbf{C}_{cl}(\mathcal{K})\mathcal{X}\mathbf{C}_{cl}^T(\mathcal{K})+\mathbf{D}_{cl}(\mathcal{K})\mathbf{D}_{cl}^T(\mathcal{K})\end{array}\right\} \tag{4}$$

(ii) There exist matrices $\mathcal{X}, \mathbf{\Upsilon}, \mathcal{Z}$ and $\mathcal{K}$ such that

$$\left[\begin{array}{cc}\mathcal{X} & \mathcal{Z}\\ \mathcal{Z}^T & \mathbf{\Upsilon}\end{array}\right] > \left[\begin{array}{cc}\mathbf{A}_{cl} & \mathbf{B}_{cl}\\ \mathbf{C}_{cl} & \mathbf{D}_{cl}\end{array}\right]\left[\begin{array}{cc}\mathcal{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{array}\right]\left[\begin{array}{cc}\mathbf{A}_{cl} & \mathbf{B}_{cl}\\ \mathbf{C}_{cl} & \mathbf{D}_{cl}\end{array}\right]^T \tag{5}$$

(4) is the existence condition of (5) for $\mathcal{Z}$. One can easily prove this lemma using the elimination lemma. Similarly, we can obtain the following results for the dual form of (5).

Corollary 1: Consider the asymptotically stable system (3). Then the following statements are equivalent.

(i) There exist matrices $\mathcal{X}, \mathbf{\Upsilon}$, and $\mathcal{K}$ such that

$$\left.\begin{array}{rcl}\mathcal{Y} & > & \mathbf{A}_{cl}^T(\mathcal{K})\mathcal{Y}\mathbf{A}_{cl}(\mathcal{K})+\mathbf{B}_{cl}^T(\mathcal{K})\mathbf{B}_{cl}(\mathcal{K})\\ \mathbf{\Upsilon} & > & \mathbf{C}_{cl}^T(\mathcal{K})\mathcal{Y}\mathbf{C}_{cl}(\mathcal{K})+\mathbf{D}_{cl}^T(\mathcal{K})\mathbf{D}_{cl}(\mathcal{K})\end{array}\right\} \tag{6}$$

(ii) There exist matrices $\mathcal{X}, \mathbf{\Upsilon}, \mathcal{Z}$, and $\mathcal{K}$ such that

$$\left[\begin{array}{cc}\mathcal{Y} & \mathcal{Z}\\ \mathcal{Z}^T & \mathbf{\Upsilon}\end{array}\right] > \left[\begin{array}{cc}\mathbf{A}_{cl} & \mathbf{B}_{cl}\\ \mathbf{C}_{cl} & \mathbf{D}_{cl}\end{array}\right]^T\left[\begin{array}{cc}\mathcal{Y} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{array}\right]\left[\begin{array}{cc}\mathbf{A}_{cl} & \mathbf{B}_{cl}\\ \mathbf{C}_{cl} & \mathbf{D}_{cl}\end{array}\right] \tag{7}$$

Note that (4) describes an upper bound to the observability Gramian $\mathcal{X}$ and (6) describes an upper bound to the controllability Gramian $\mathcal{Y}$. Using Lemma 1 and Corollary 1, we can establish new system performance analysis conditions as follows.

Theorem 1: Consider the asymptotically stable system (3). Suppose a positive scalar γ is given. Then the following statements are true.

(i) $\Upsilon_{ep} < \gamma$ if and only if there exist matrices $\mathcal{K}$, $\mathcal{Z}, \mathcal{X}$ and $\mathbf{\Upsilon}$ such that $\gamma\mathbf{I} > \mathbf{\Upsilon}$ and (5) holds.

(ii) $\Upsilon_{ie} < \gamma$ if and only if there exist matrices $\mathcal{Z}, \mathcal{X}$ and $\mathbf{\Upsilon}$ such that $\gamma\mathbf{I} > \mathbf{\Upsilon}$ and (7) holds.

(iii) $\Upsilon_{\mathcal{H}_2} = \left\|\mathbf{C}_{cl}(\mathcal{K})\left(\zeta\mathbf{I}-\mathbf{A}_{cl}(\mathcal{K})\right)^{-1}\mathbf{B}_{cl}(\mathcal{K})+\mathbf{D}_{cl}(\mathcal{K})\right\|_2 < \gamma$ if and only if there exist matrices $\mathcal{K}$, $\mathcal{Z}$, $\mathcal{X}$ and $\mathbf{\Upsilon}$ such that $trace[\mathbf{\Upsilon}] < \gamma^2$ and (5) hold.

(iv) $\Upsilon_{ee} = \left\|\mathbf{C}_{cl}(\mathcal{K})\left(\zeta\mathbf{I}-\mathbf{A}_{cl}(\mathcal{K})\right)^{-1}\mathbf{B}_{cl}(\mathcal{K})+\mathbf{D}_{cl}(\mathcal{K})\right\|_\infty < \gamma$ if and only if there exist matrices $\mathcal{K}$, $\mathcal{X}$ and $\mathbf{\Upsilon}$ such that $\gamma^2\mathbf{I} > \mathbf{\Upsilon}$ and (5) hold with $\mathcal{Z}=\mathbf{0}$.

The statement (i) is often called the general $\mathcal{H}_2$ control problem [4]. The statement (iii) characterizes the $\mathcal{H}_2$ control problem and the statement (iv) characterizes the $\mathcal{H}_\infty$ control problem. We can easily see that the $\mathcal{H}_2$ norm and the $\mathcal{H}_\infty$ norm is closely related. One of the interesting features of Theorem 1 is its compact form, and the fact that many performance specifications have similar forms. Indeed all

$$\mathcal{H}_o(\mathcal{X},\mathcal{Z}) \triangleq \left\{ (\mathcal{X},\mathcal{Z}) \left| \left[\begin{array}{c|c} \left[\begin{array}{cc}\mathcal{C} & \mathcal{D}_y\end{array}\right]_\perp^T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}\end{array}\right] \left[\begin{array}{cc|c} \mathcal{X}^{-1} & \mathbf{0} & (\star)^T \\ \mathbf{0} & \mathbf{I} & \\ \hline \mathcal{A} & \mathcal{D}_p & \mathcal{X} \quad \mathcal{Z} \\ \mathcal{C}_z & \mathcal{D}_z & \mathcal{Z}^T \quad \mathbf{\Upsilon} \end{array}\right] \left[\begin{array}{c|c} \left[\begin{array}{cc}\mathcal{C} & \mathcal{D}_y\end{array}\right]_\perp & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}\end{array}\right] > \mathbf{0} \right.,\right.$$
$$\left. \left[\begin{array}{cc}\mathcal{B}^T & \mathcal{B}_z^T\end{array}\right]_\perp^T \left(\left[\begin{array}{cc}\mathcal{X} & \mathcal{Z} \\ \mathcal{Z}^T & \mathbf{\Upsilon}\end{array}\right] - \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right] \left[\begin{array}{cc}\mathcal{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\end{array}\right] \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right]^T \right) \left[\begin{array}{cc}\mathcal{B}^T & \mathcal{B}_z^T\end{array}\right]_\perp > \mathbf{0}\,,\ \mathcal{X} > \mathbf{0} \right\} \tag{8}$$

$$\mathcal{H}_c(\mathcal{Y},\mathcal{Z}) \triangleq \left\{ (\mathcal{Y},\mathcal{Z}) \left| \left[\begin{array}{c|c} \left[\begin{array}{cc}\mathcal{B}^T & \mathcal{B}_z^T\end{array}\right]_\perp^T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}\end{array}\right] \left[\begin{array}{cc|c} \mathcal{Y}^{-1} & \mathbf{0} & (\star)^T \\ \mathbf{0} & \mathbf{I} & \\ \hline \mathcal{A} & \mathcal{D}_p & \mathcal{Y} \quad \mathcal{Z} \\ \mathcal{C}_z & \mathcal{D}_z & \mathcal{Z}^T \quad \mathbf{\Upsilon} \end{array}\right] \left[\begin{array}{c|c} \left[\begin{array}{cc}\mathcal{B}^T & \mathcal{B}_z^T\end{array}\right]_\perp & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}\end{array}\right] > \mathbf{0} \right.,\right.$$
$$\left. \left[\begin{array}{cc}\mathcal{C} & \mathcal{D}_y\end{array}\right]_\perp^T \left(\left[\begin{array}{cc}\mathcal{Y} & \mathcal{Z} \\ \mathcal{Z}^T & \mathbf{\Upsilon}\end{array}\right] - \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right] \left[\begin{array}{cc}\mathcal{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\end{array}\right] \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right]^T \right) \left[\begin{array}{cc}\mathcal{C} & \mathcal{D}_y\end{array}\right]_\perp > \mathbf{0} \right\} \tag{9}$$

$$\left.\begin{array}{l} \mathbf{\Phi}_\mathcal{X}(\mathcal{X}) \triangleq \left\{ \mathcal{X} \left| \mathcal{X} > \mathbf{0}\,,\ \left[\begin{array}{cc}\mathcal{B}^T & \mathcal{B}_z^T\end{array}\right]_\perp^\mathbf{T} \left(\left[\begin{array}{cc}\mathcal{X} & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I}\end{array}\right] - \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right] \left[\begin{array}{cc}\mathcal{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\end{array}\right] \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right]^\mathbf{T} \right) \left[\begin{array}{cc}\mathcal{B}^T & \mathcal{B}_z^T\end{array}\right]_\perp > \mathbf{0} \right.\right\} \\ \mathbf{\Phi}_\mathcal{Y}(\mathcal{Y}) \triangleq \left\{ \mathcal{Y} \left| \mathcal{Y} > \mathbf{0}\,,\ \left[\begin{array}{cc}\mathcal{C} & \mathcal{D}_y\end{array}\right]_\perp^\mathbf{T} \left(\left[\begin{array}{cc}\mathcal{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\end{array}\right] - \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right]^\mathbf{T} \left[\begin{array}{cc}\mathcal{Y} & \mathbf{0} \\ \mathbf{0} & \gamma^{-2}\mathbf{I}\end{array}\right] \left[\begin{array}{cc}\mathcal{A} & \mathcal{D}_p \\ \mathcal{C}_z & \mathcal{D}_z\end{array}\right] \right) \left[\begin{array}{cc}\mathcal{C} & \mathcal{D}_y\end{array}\right]_\perp > \mathbf{0} \right.\right\} \end{array}\right\} \tag{10}$$

matrix inequalities given in Theorem 1 can be parametrized by the matrix inequality

$$(\mathbf{\Theta} + \mathbf{\Gamma}\mathcal{K}\mathbf{\Lambda})\,\mathbf{R}\,(\mathbf{\Theta} + \mathbf{\Gamma}\mathcal{K}\mathbf{\Lambda})^T < \mathbf{Q}. \tag{11}$$

The analysis of this important matrix inequality is available in [1]. It is important that we have introduced a new matrix variable $\mathcal{Z}$ in Lemma 1 and Corollary 1. This new variable may help to find the optimal solution since we enlarge the domain of the problem. It is well known in a variety of mathematical problems that enlarging the domain in which the problem is posed can often simpify the mathematical treatment. Many nonlinear problems admit solutions using linear techniques by enlaring the domain of the problem. The most important feature in Theorem 1 is that we have only one matrix inequality which involves the control gain matrix $\mathcal{K}$. Hence we can eliminate the control gain matrix $\mathcal{K}$ using the elimination lemma. Usually, the performance of LMI solvers is greatly affected by the problem size (the size of matrix inequalities) and the number of variables. So eliminating the control variables may have advantages. We shall see the effect of eliminating the control variables later. Note that all problems described in Theorem 1 are bilinear matrix inequalities (BMI). When we eliminate the variable $\mathcal{K}$, all problems in Theorem 1 are functions of a matrix pair $(\mathcal{X},\mathcal{Z})$. Once we obtain a matrix pair $(\mathcal{X},\mathcal{Z})$, our problems are convex with respect to $\mathcal{K}$. Applying the elimination lemma to Lemma 1 yields the following results.

Theorem 2: Let a matrix $\mathbf{\Upsilon} > \mathbf{0}$ be given and consider the linear-time-invariant discrete-time system (3). Then the following statements are equivalent.

(i) There exists a stabilizing dynamic output feedback controller $\mathcal{K}$ of order n_c, matrices $\mathcal{X}$ and $\mathcal{Z}$ satisfying (5).

(ii) There exists a matrix pair $(\mathcal{X},\mathcal{Z}) \in \mathcal{H}_o$ where $\mathcal{H}_o$ is given in (8).

The statement (ii) is the existence condition for a stabilizing controller $\mathcal{K}$ of order n_c. Note that we omitted the controller formula in this theorem for brevity. One can easily prove this theorem using the elimination lemma and obtain the controller formula from [1]. Similarly, we can obtain the dual form of Theorem 2 for the pulse-to-energy gain control problem.

Corollary 2: Let a matrix $\mathbf{\Upsilon} > \mathbf{0}$ be given and consider the linear-time-invariant discrete-time system (3). Then the following statements are equivalent.

(i) There exist a stabilizing dynamic output feedback controller $\mathcal{K}$ of order n_c, matrices $\mathcal{Y}$ and $\mathcal{Z}$ satisfying (7).

(ii) There exists a matrix pair $(\mathcal{Y},\mathcal{Z}) \in \mathcal{H}_c$ where $\mathcal{H}_c$ is given in (9).

III. Linearization Algorithm

All matrix inequalities given in the previous sections are nonconvex since all matrix inequalities have a term $\mathcal{X}^{-1}$. In this section, we propose a new class of algorithms to handle this nonvex term. Consider the following optimization problem :

Problem 1: Let $\mathbf{\Psi}$ be a convex set, a scalar convex function $f(\mathbf{X})$, a matrix function $\mathcal{J}(\mathbf{X})$ and $\mathcal{H}(\mathbf{X})$ be given and consider the nonconvex optimization problem :

$$\min_{\mathbf{X}\in\mathbf{\Psi}} f(\mathbf{X})\ ,\ \ \mathbf{\Psi} \triangleq \{\mathbf{X}|\ \mathcal{J}(\mathbf{X}) + \mathcal{H}(\mathbf{X}) < \mathbf{0}\} \tag{12}$$

Suppose $\mathcal{J}(\mathbf{X})$ is convex, $\mathcal{H}(\mathbf{X})$ is not convex, and $f(\mathbf{X})$ is a first order differentiable convex function bounded from below on the convex set $\mathbf{\Psi}$.

One of possible approaches to solve this nonconvex problem is linearization of a nonconvex term. Now, we establish the linearization algorithm as following.

Theorem 3: The problem 1 can be solved (locally) by iterating a sequence of convex sub-problems if there exists a matrix function $\mathcal{G}(\mathbf{X},\mathbf{X}_k)$ such that

$$\begin{aligned} \mathbf{X}_{k+1} &= \arg\min_{\mathbf{X}\in\mathbf{\Psi}_k} f(\mathbf{X}) \\ \mathbf{\Psi}_k &\triangleq \{\mathbf{X} \mid \mathcal{J}(\mathbf{X}) + \mathrm{LIN}\,(\mathcal{H}(\mathbf{X}),\mathbf{X}_k) + \mathcal{G}(\mathbf{X},\mathbf{X}_k) < \mathbf{0} \\ &\qquad \mathcal{H}(\mathbf{X}) \le \mathcal{G}(\mathbf{X},\mathbf{X}_k) + \mathrm{LIN}\,(\mathcal{H}(\mathbf{X}),\mathbf{X}_k)\} \end{aligned} \tag{13}$$

where $\mathrm{LIN}\,(\star,\mathbf{X}_k)$ is the linearization operator at given $\mathbf{X}_k$.

Proof : First note that every point $\mathbf{X}_{k+1} \in \mathbf{\Psi}_k$ is also in $\mathbf{\Psi}$ since

$$\mathcal{J}(\mathbf{X}) + \mathcal{H}(\mathbf{X}) \le \mathcal{J}(\mathbf{X}) + \mathrm{LIN}\,(\mathcal{H}(\mathbf{X}),\mathbf{X}_k) + \mathcal{G}(\mathbf{X},\mathbf{X}_k) < \mathbf{0}.$$

As long as $\mathbf{X}_k \in \mathbf{\Psi}_k$, $f(\mathbf{X}_{k+1}) < f(\mathbf{X}_k)$ holds strictly until $\mathbf{X}_{k+1} = \mathbf{X}_k$. The fact that $f(\mathbf{X})$ is bounded from below ensures that this strictly decreasing sequence converges to a stationary point. ∎

The linearization algorithm is to solve a sufficient condition. This approach is conservative, but the conservatism will be minimized since we shall solve the problem iteratively. Due to the lack of convexity, only local optimality is guaranteed. It should be mentioned that the linearization algorithm is a convexifying algorithm, in the spirit of [11]. A convexifying algorithm must find a convexifying potential function. There might exist many candidates for convexifying potential functions for a given nonconvex matrix inequality, and some convexifying potentials may yield too much conservatism. Finding a nice convexifying function is generally difficult. Our linearization approach may provide such a nice convexifying potential function.

All matrix inequalities given in the previous sections are convex except for the term $\mathcal{X}^{-1}$. One can ask "How can we linearize this nonconvex term $\mathcal{X}^{-1}$ at given $\mathcal{X}_k > \mathbf{0}$?". Since our variables are matrices, we need to develop the taylor series expansion for matrix variables. The following lemma provides the linearization of $\mathbf{X}^{-1}$ and $\mathbf{XWX}$.

Lemma 2: Let a matrix $\mathbf{W} \in \Re^{n\times n} > 0$ be given. Then the following statements are true.

(i) The linearization of $\mathbf{X}^{-1} \in \Re^{n\times n}$ about the value $\mathbf{X}_k > 0$ is

$$\mathrm{LIN}\left(\mathbf{X}^{-1}, \mathbf{X}_k\right) = \mathbf{X}_k^{-1} - \mathbf{X}_k^{-1}\left(\mathbf{X} - \mathbf{X}_k\right)\mathbf{X}_k^{-1} \tag{14}$$

(ii) The linearization of $\mathbf{XWX} \in \Re^{n\times n}$ about the value $\mathbf{X}_k$ is

$$\mathrm{LIN}\left(\mathbf{XWX}, \mathbf{X}_k\right) = -\mathbf{X}_k\mathbf{W}\mathbf{X}_k + \mathbf{XW}\mathbf{X}_k + \mathbf{X}_k\mathbf{WX} \tag{15}$$

where $\mathrm{LIN}\left(\star, \mathbf{X}_k\right)$ is the linearization operator at given $\mathbf{X}_k$.
One can easily show that $-\mathbf{X}^{-1} - \mathrm{LIN}\left(-\mathbf{X}^{-1}, \mathbf{X}_k\right) \leq \mathbf{0}$ and $-\mathbf{XWX} - \mathrm{LIN}\left(-\mathbf{XWX}, \mathbf{X}_k\right) \leq \mathbf{0}$ in order to use Theorem 3. Thus we can set a matrix function $\mathcal{G}(\mathbf{X}, \mathbf{X}_k) = \mathbf{0}$ for this nonconvex term and the equality is attained when $\mathbf{X} = \mathbf{X}_k$. Note that this provides the updating rules. Using the linearization algorithm, we can establish two main algorithms. One is for a feasibility problem and the other is for an optimization problem. We first propose a new algorithm for the optimal fixed-order output feedback control problem and then propose another algorithm for a general SLC problem. In both cases, we propose new feasibility algorithms using the same linearization approach.

A. Optimal Fixed-Order Output Feedback Control Problem

The following algorithm is suitable for solving (i),(iii), and (iv) in Theorem 1.

Algorithm 1: **Optimal General $\mathcal{H}$ Control Problem**

1) Set $\epsilon > 0$ and $k = 0$.
2) Solve the following convex optimization problem.

$$\begin{aligned} \mathcal{X}_{k+1} \quad &= \quad \arg\min_{\mathcal{X},\mathcal{Z},\mathbf{\Upsilon}} \ \|\mathbf{\Upsilon}\| \\ \text{subject to} \quad & \quad \left\{(\mathcal{X},\mathcal{Z},\mathbf{\Upsilon}) \in \mathrm{LIN}\left(\mathcal{H}_o(\mathcal{X},\mathcal{Z},\mathbf{\Upsilon}), \mathcal{X}_k\right)\right\} \end{aligned}$$

where $\mathcal{H}_o(\mathcal{X},\mathcal{Z},\mathbf{\Upsilon})$ is given by (8).
3) If $\|\mathbf{\Upsilon}\| < \epsilon$, go to step 4. Otherwise, set $k \leftarrow k+1$ and go back to Step 2.
4) Calculate the controller by solving the following convex optimization problem after fixing $\mathcal{X} = \mathcal{X}_{k+1}$.

$$\mathcal{K} = \arg\min_{\mathcal{K},\mathcal{Z},\mathbf{\Upsilon}} \ \|\mathbf{\Upsilon}\| \ , \ \text{subject to } (5)$$

Similarly, one can easily build an algorithm for (ii) in Theorem 1. It is worthwhile to comment that we can immediately use the controller formula given by [1] instead of solving the step (iv). This is the basic feature of the BMI problem as we explained before.

Algorithm 1 may find an initial feasible solution. If they fail, we need to find an initial feasible solution. Several algorithms for feasibility problems for fixed-order output feedback control problems are already available [1], [5], [8], [7]. Here, we also propose a new feasibility algorithm for the completeness of the proposed algorithm using the linearization approach. $\mathcal{H}_\infty$ control problem is suitable for feasibility problem. If there is no γ satisfying $\mathcal{H}_\infty$ constraint, then there is neither $\mathcal{H}_\infty$ control nor $\mathcal{H}_2$ control. Let's consider two nonempty constraint sets $\mathbf{\Phi}_\mathcal{X}(\mathcal{X})$ and $\mathbf{\Phi}_\mathcal{Y}(\mathcal{Y})$ given by (10). Note that we have the constraint $\mathcal{XY} = \mathbf{I}$ in the above matrix inequalities. The feasibility problem is to find a $\mathcal{X}$ in the set $\mathbf{\Phi}_\mathcal{X}(\mathcal{X})$ which is closest to the set $\mathbf{\Phi}_\mathcal{Y}(\mathcal{Y})$. This problem can be relaxed and solved by the following optimization problem.

Algorithm 2: **Feasibility**

1) Set $\gamma > 0, \epsilon > 0$ and $k = 0$.
2) Solve the following convex optimization problem.

$$\begin{aligned} \mathcal{X}_{k+1} \ &= \ \arg\min_{\mathbf{X},\mathbf{Y}} trace[\mathbf{\Upsilon}] \\ \text{subject to} \ & \left\{ \begin{array}{l} -\mathbf{\Upsilon} + \mathcal{Y} - \mathrm{LIN}\left(\mathcal{X}^{-1}, \mathcal{X}_k\right) < \mathbf{0}, \\ \begin{bmatrix} \mathcal{X} & \mathbf{I} \\ \mathbf{I} & \mathcal{Y} \end{bmatrix} \geq \mathbf{0}, \\ \mathbf{\Upsilon} \geq \mathbf{0} \ , \ \mathcal{X} \in \mathbf{\Phi}_\mathcal{X}(\mathcal{X}) \ , \ \mathcal{Y} \in \mathbf{\Phi}_\mathcal{Y}(\mathcal{Y}) \end{array} \right. \end{aligned} \tag{16}$$

3) If $trace[\mathbf{\Upsilon}] < \epsilon$, stop. Otherwise, set $k \leftarrow k+1$ and go back to Step 2.

The feasibility problem is not convex either, however, this problem has the same nature as the previous optimization problem and we can linearize this term. Notice that the proposed algorithm is very similar to the one proposed in [5], which adopts cone-complementarity linearization algorithm. The new proposed algorithm minimizes $trace[\mathcal{Y} + \mathcal{X}_k^{-1}\mathcal{X}\mathcal{X}_k^{-1}]$, while the cone complementarity linearization algorithm minimizes $trace[\mathcal{Y}\mathcal{X}_k + \mathcal{X}\mathcal{Y}_k + \mathcal{Y}_k\mathcal{X} + \mathcal{X}_k\mathcal{Y}]$. It is clear that the cone complementarity algorithm linearizes at a matrix pair $(\mathcal{X}_k, \mathcal{Y}_k)$ and our algorithm linearizes only at $\mathcal{X}_k$. Also we minimize $\left\|\mathcal{Y} - \mathcal{X}^{-1}\right\|$ and the cone complementarity linearization algorithm minimizes $\|\mathcal{XY} + \mathcal{YX}\|$. Clearly we minimize the controllability Gramian $\mathcal{Y}$ and maximize the observability Gramian $\mathcal{X}$. This implies that our algorithm is suitable for initializing the optimal $\mathcal{H}_2$ control problem, while the cone complementarity linearization algorithm is suitable for initializing optimal $\mathcal{H}_\infty$ control problem since there always exists a positive scalar γ such that $\mathcal{XY} \leq \gamma\mathbf{I}$ [1]. Note that we can establish another feasibility algorithm since

$$\mathcal{XY} + \mathcal{YX} = (\mathcal{X} + \mathcal{Y})(\mathcal{X} + \mathcal{Y}) - \mathcal{X}^2 - \mathcal{Y}^2.$$

Hence we can just replace the first matrix inequality in (16) with the matrix inequality

$$\begin{bmatrix} -\mathbf{\Upsilon} - \mathrm{LIN}\left(\mathcal{X}^2, \mathcal{X}_k\right) - \mathrm{LIN}\left(\mathcal{Y}^2, \mathcal{Y}_k\right) & \mathcal{X} + \mathcal{Y} \\ \mathcal{X} + \mathcal{Y} & -\mathbf{I} \end{bmatrix} < \mathbf{0}.$$

This approach also linearizes at a matrix pair $(\mathcal{X}_k, \mathcal{Y}_k)$.

B. Structured Linear Control

Whenever a controller has some given structural constraints, we can not use Algorithm 1 and 2, since Theorem 2 and Corollary 2 are no longer the existence conditions of $\mathcal{K}$ for SLC problems. For

SLC problems, we should apply a linearization algorithm directly to (5) or (7).

Algorithm 3: **Structured Linear Control**

1) Set $\epsilon > 0$ and $k = 0$.
2) Solve the following convex optimization problem.

$$\mathcal{X}_{k+1} = \arg \min_{\mathcal{X},\mathcal{Z},\mathbf{\Upsilon},\mathcal{K}} \|\mathbf{\Upsilon}\|$$
$$\text{subject to} \left[\begin{array}{cc|cc} \mathcal{X} & \mathcal{Z} & \mathbf{A}_{cl}(\mathcal{K}) & \mathbf{B}_{cl}(\mathcal{K}) \\ \mathcal{Z}^T & \mathbf{\Upsilon} & \mathbf{C}_{cl}(\mathcal{K}) & \mathbf{D}_{cl}(\mathcal{K}) \\ \hline \multicolumn{2}{c|}{(\star)^T} & \mathrm{LIN}\left(\mathcal{X}^{-1}, \mathcal{X}_k\right) & \mathbf{0} \\ & & \mathbf{0} & \mathbf{I} \end{array}\right] > \mathbf{0}$$

3) If $\|\mathbf{\Upsilon}\| < \epsilon$, stop. Otherwise, set $k \leftarrow k + 1$ and go back to Step 2.

Alternatively, we can apply a linearization algorithm directly to (4) or (6), in which the newly introduced variable $\mathcal{Z}$ is eliminated. In this case, our algorithm is the same as one in [11]. Since the step 1 and 3 are the same as those in Algorithm 3, we describe the step 2 only.

Algorithm 4: **Elimination of** Z

2. Solve the following convex optimization problem.

$$\mathcal{Y}_{k+1} = \arg \min_{\mathbf{\Upsilon},\mathcal{K},\mathcal{Y}} \|\mathbf{\Upsilon}\|$$
$$\text{subject to} \left\{ \begin{array}{l} \begin{bmatrix} \mathrm{LIN}\left(\mathcal{Y}^{-1}, \mathcal{Y}_k\right) & \mathbf{A}_{cl}(\mathcal{K}) & \mathbf{B}_{cl}(\mathcal{K}) \\ \mathbf{A}_{cl}^T(\mathcal{K}) & \mathcal{Y} & \mathbf{0} \\ \mathbf{B}_{cl}^T(\mathcal{K}) & \mathbf{0} & \mathbf{I} \end{bmatrix} > \mathbf{0} \\ \begin{bmatrix} \mathbf{\Upsilon} & \mathbf{C}_{cl}(\mathcal{K}) & \mathbf{D}_{cl}(\mathcal{K}) \\ \mathbf{C}_{cl}^T(\mathcal{K}) & \mathcal{Y} & \mathbf{0} \\ \mathbf{D}_{cl}^T(\mathcal{K}) & \mathbf{0} & \mathbf{I} \end{bmatrix} > \mathbf{0} \end{array} \right.$$

Similarly, we can make a feasibility algorithm for SLC problems. We describe the step 2 only.

Algorithm 5: **Feasibility for SLC**

2. Solve the following convex optimization problem.

$$\mathcal{X}_{k+1} = \arg \min_{\mathbf{\Upsilon},\mathcal{K},\mathcal{X},\mathcal{Y},\mathcal{Z}} \|\mathbf{\Upsilon}\|$$
$$\text{subject to} \left\{ \begin{array}{l} \begin{bmatrix} \mathcal{X} & \mathbf{I} \\ \mathbf{I} & \mathcal{Y} \end{bmatrix} \geq \mathbf{0} \\ -\mathbf{\Upsilon} + \mathcal{Y} - \mathrm{LIN}\left(\mathcal{X}^{-1}, \mathcal{X}_k\right) < \mathbf{0} \\ \left[\begin{array}{cc|cc} \mathcal{X} & \mathcal{Z} & \mathbf{A}_{cl}(\mathcal{K}) & \mathbf{B}_{cl}(\mathcal{K}) \\ \mathcal{Z}^T & \mathbf{\Upsilon} & \mathbf{C}_{cl}(\mathcal{K}) & \mathbf{D}_{cl}(\mathcal{K}) \\ \hline \multicolumn{2}{c|}{(\star)^T} & \mathcal{Y} & \mathbf{0} \\ & & \mathbf{0} & \mathbf{I} \end{array}\right] > \mathbf{0} \end{array} \right.$$

IV. Illustrative Examples

A. Fixed-order Optimal $\mathcal{H}_2$ Output Feedback Control

Consider the following discrete-time plant [11].

$$\mathbf{A}_p = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix}$$

$$\mathbf{B}_p = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix}, \; \mathbf{B}_z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; \mathbf{D}_y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C}_z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \mathbf{D}_p = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix}$$

Our goal is to minimize $\mathcal{H}_2$ norm of the transfer function $\mathbf{T}_{wz}(\zeta)$ using a fixed order output feedback controller. By calculating the full-order optimal $\mathcal{H}_2$ controller which provides the lower bound, we obtain the minimum achievable values for this norm $\min \|\mathbf{T}_{wz}(\zeta)\|_{\mathcal{H}_2} = 0.3509$. In order to use initialization algorithm 2, we set $\mathcal{X}_0 = \mathbf{I} + \mathbf{R}\mathbf{R}^T$ and $\mathcal{Y}_0 = \mathcal{X}_0^{-1}$, where $\mathbf{R}$ is a random matrix. After using the initialization algorithm, we have run the algorithm 1, 3, and 4 with the controller order $n_c = 0$ and $n_c = 1$. The precision ϵ has been set to 10^{-3}.

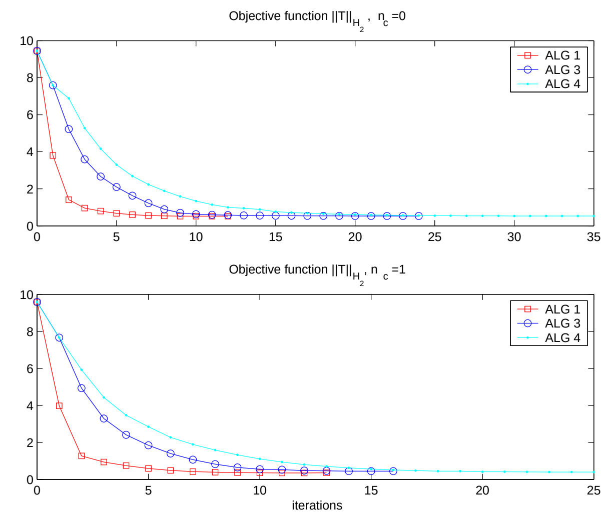

Fig. 1. $\mathcal{H}_2$ performance of the optimal fixed-order output feedback controller

Figure 1 shows the performance of three algorithms. One can easily see that the behaviors of Algorithm 1 and 3 are better than that of Algorithm 4. Specially, Algorithm 1 quickly converges compared with Algorithm 3 and 4. Moreover, the Algorithm 1 and 3 converged uniformly in most cases. The cost $\|\mathbf{T}_{wz}(\zeta)\|_{\mathcal{H}_2}$ are $0.5178, 0.5261,$ and 0.52 with $n_c = 0$ and $0.3513, 0.3738,$ and 0.3741 with $n_c = 1$ for Algorithm 1, 3, and 4 respectively. Notice that the performance of the output feedback controller with $n_c = 1$ is just 0.1 % worse than that of the full-order output feedback controller. This example shows that controller reduction is possible, without sacrificing much performance.

B. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control

Design of feedback controllers satisfying both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ specifications is important because it offers robust stability and nominal performance, and it is not always possible to have full access to the state vector. In this problem, we look for a unique static output feedback controller that minimizes an $\mathcal{H}_2$ performance cost while satisfying some $\mathcal{H}_\infty$ constraint. Consider the following simple discrete-time unstable plant [11].

$$\begin{aligned} \mathbf{x}_p(k+1) &= \begin{bmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \mathbf{x}_p(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{w}(k) \\ \mathbf{z}_1(k) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_p(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ \mathbf{z}_2(k) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_p(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_p(k) + \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{w}(k) \end{aligned}$$

By calculating the dynamic output feedback optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers, we obtain the following minimum achievable values for these norms

$$\min \|\mathbf{T}_{wz_1}(\zeta)\|_{\mathcal{H}_2} = 4.0957 \ , \ \min \|\mathbf{T}_{wz_2}(\zeta)\|_{\mathcal{H}_\infty} = 6.3409.$$

Our objective is to design a static output feedback controller that minimizes $\|\mathbf{T}_{wz_1}(\zeta)\|_{\mathcal{H}_2}$ while keeping $\|\mathbf{T}_{wz_2}(\zeta)\|_{\mathcal{H}_\infty}$ below a certain level γ. Let's set $\gamma = 7$. Note that Algorithm 1 can be applicable with the constraint $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$. Our algorithm can be composed of three sub algorithms which are

1) Run Algorithm 2 (initialization).
2) Run Algorithm 1 with the constraint $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$.
3) From this sub-optimal solution, Run Algorithm 3 or 4.

Or alternatively,

1) Run Algorithm 5 (initialization).
2) From this sub-optimal solution, Run Algorithm 3 or 4.

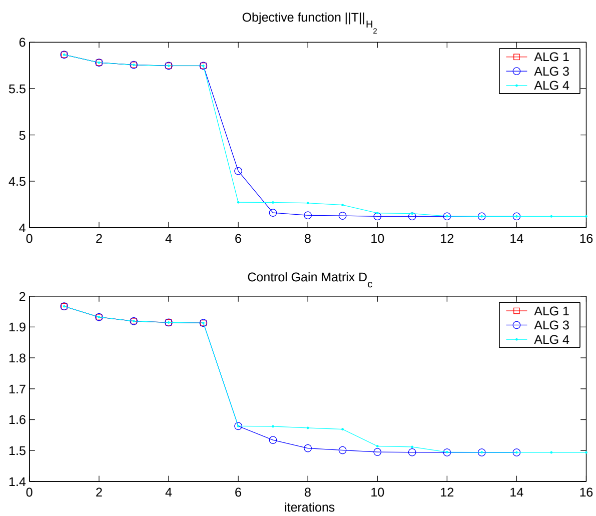

Fig. 2. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance of the optimal static output feedback controller

Figure 2 shows the performance of those algorithms. The specified precision ϵ is 10^{-4}. We can easily see that the performance of Algorithm 1 with the constraint $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$ is conservative as expected. For this problem, the behavior of Algorithm 3 is better than 4 for this problem. Note that we achieved $\|\mathbf{T}_{wz_1}(\zeta)\|_{\mathcal{H}_2} = 4.1196$, $\|\mathbf{T}_{wz_2}(\zeta)\|_{\mathcal{H}_\infty} < 7$. This is just 0.5% worse than the $\mathcal{H}_2$ optimal dynamic output feedback controller.

V. CONCLUSION

We have addressed the SLC (Structured Linear Control) problem for linear discrete-time systems. New system performance analysis conditions have been derived. These new results introduce an augmented matrix variable $\mathcal{Z}$. It turns out that the new system performance analysis conditions are better than the original ones, since we could derive the equivalent conditions using the elimination lemma for a fixed order output feedback control problem.

In the SLC framework, these objectives are characterized as a set of LMI's with an additional nonconvex equality constraint. To overcome this nonconvex constraint, we proposed a linearization method. At each iteration, a certain potential function is added to the nonconvex constraints to enforce convexity. Although we solved those sufficient conditions iteratively, this approach will not bring significant conservatism because the added conditions will converge to zero. Local optimality is guaranteed. The results given in this paper can also be applied to linear continuous-time systems with no difficulties. Moreover, our approach can be applied to other linear synthesis problems as long as the dependence on the augmented plant on the synthesis parameters are affine.

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